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LINEAR DIFFERENTIAL EQUATIONS WITH DELAYS:

Admissibility and Conditional Stability

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C. V. Coffman and J. J. Schäffer

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# Linear Differential Equations with Delays: Admissibility and Conditional Stability

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## 1. Introduction

In Perron's classical paper on stability [3], a central concern is the relationship, for linear differential equations, between the condition that the non-homogeneous equation have some bounded solution for every bounded "second member" on the one hand, and a certain form of conditional stability of the solutions of the homogeneous equation on the other. This idea was later extensively developed, by Massera and Schäffer among others; their work is collected in a monograph [2]. In a previous paper [1], the present authors examined linear difference equations and provided for them the analogues of the central results for differential equations in [2]. The important new difficulty encountered was, of course, the irreversibility of the process described by a difference equation, and new conceptual tools were developed to overcome it.

The present paper is an initial attempt to apply the methods in [1], [2] to the type of linear systems "next in order of complexity", viz., linear functional-differential equations or linear differential equations with delay. We lean heavily on [1] and, less heavily, on [2] for motivation and techniques; we must assume familiarity with [1] at least.

Specifically, we consider an equation of the form

$$\dot{u} + Lu + Mu = r \tag{1.1}$$

where $u$ and $r$ take values in a Banach space $E$ (which will often be finite-dimensional), $L$ has operators in $E$ as its values, and $M$, the "memory functional", takes a function $u$ linearly into a function $Mu$ in such a way that the value of $Mu$ at any given value of the argument $t$ depends on the values of $u$ at <u>preceding</u> values of $t$. The detailed definitions and assumptions are discussed in Sections 4 and 5.

In this initial attempt, some rather severe restrictions have been imposed on the "scope" of the memory functional. Roughly speaking, $M$ "remembers" only values of $u$ at arguments that lag behind $t$ by at least $1$ (this "gap" has been normalized) and at most a fixed bound, taken for convenience to be an integer $p$. Among many other cases covered despite this restriction is the case of a finite number of fixed delays. The technical form adopted for this restriction in Section 4 actually gives each "slice" of length 1 with integral endpoints of $Mu$ in terms of the preceding "slice" of length $p$ of $u$; it is thus slightly less severe. This technical form avoids a statement on how the dependence of $Mu(t)$ on $u$ varies locally with $t$; it thus allows the theory to cover such cases as that of a single continuously varying delay, which would otherwise be excluded by measurability complications.

Rather than apply the previously developed methods afresh to equation (1.1), we prefer transforming the equation into an equivalent difference equation in a function space. Values of solutions of the difference equation correspond to slices of solutions of (1.1), and the properties of (1.1) to be investigated are reflected in corresponding properties of the difference equation. It will be seen that, so far as the behaviour of slices of solutions of (1.1) is concerned, the previously established theory for difference equations, with its built-in irreversibility, is sufficient to yield analogous results for (1.1). It is true that only slices with integral endpoints are primarily accounted for; this blemish is implicit in the assumptions on $M$; we feel it is minor, and it can in fact be removed in most respects with some additional effort.

The technical core of this paper is Section 6, where we show how knowledge about (1.1) can be translated into analogous knowledge about the difference equation. The difficulty in this translation, and the method used to overcome it, were foreshadowed, in a simpler form, in [4]; that paper was in fact first written to give a technique for a cruder approach to our present problem. In Section 6 we have tried to keep the discussion at a high level of generality in such matters as the function and sequence spaces involved, so as to make the results available for future developments.

In Sections 7 and 8 we sketch the main results obtained by our method; these are the desired theorems that relate "admissibility" properties of (1.1) with the behaviour of the solutions of the homogeneous equation. Here it turns out that the results are very clear-cut for finite-dimensional $E$, and we impose this restriction for the pertinent results. In contrast to the generality aimed at in Section 6, the function spaces appearing in this sketch have been kept within a small class of well-known ones; there is no difficulty in extending this class to include Orlicz spaces and many others with the appropriate translation invariance properties.

Throughout the paper, we deal simultaneously with the "Carathéodory case", in which Mu and r are measurable and locally integrable, and the "continuous case", in which they are presupposed to be continuous. We should properly speak of "continuous case" only when L is also continuous-- so that solutions are continuously differentiable and (1.1) is satisfied pointwise, not merely locally in $\underset{\sim}{L}^1$ --, but these additional restrictions do not make as significant a distinction of cases as the one we have adopted.

Future efforts will be directed to removing the assumption of the short-range "gap" in the memory, and to formulating an adequate concept of "dichotomy" for our functional-differential equations, perhaps in the framework of a more inclusive concept of "transition systems".

## 2. Spaces

Throughout this paper, E will denote a real or complex Banach space. The norm in E, as in all normed spaces except the scalar fields and the sequence and function spaces mentioned below, is denoted by $\|\cdot\|$. If X,Y are Banach spaces, $[X \to Y]$ denotes the Banach space of operators (bounded linear mappings) from $X$ to $Y$, and we set $\tilde{X} = [X \to X]$.

In this paper, spaces of sequences occur together with spaces of functions on certain intervals of the real line. For the former, we adopt without further elaboration the terminology and notation described in [1; Sections 2,3], with which we assume the reader to be familiar.

The intervals that serve as domains for measurable functions will be of two types: firstly, intervals of the form $[-m, 0]$ for some (here usually integral) $m > 0$ and, secondly, intervals of the type $[m, \infty)$ for some (here usually integral) real number $m$. We shall generally adopt the notations and terminology in [2; Chapter 2] for functions on these intervals and for spaces consisting of them; some exceptions will be noted here.

Whenever the fact that the interval $[-m, 0]$ is the domain of the functions is to be recorded, a subscript $m$ to the left will be added. Thus, e.g., ${}_m\underset{\sim}{L}^1(E)$ is the Banach space of (equivalence classes modulo null sets of) Bochner-integrable functions $f : [-m, 0] \to E$, with the norm $\int_{-m}^{0} \|f(t)\| dt$ ; ${}_m\underset{\sim}{C}(E)$ is the Banach space of continuous functions $f : [-m, 0] \to E$ with the norm $\sup\{\|f(t)\| : t \in [-m, 0]\}$. We further agree to denote the norms in such spaces with thick hollow bars and the appropriate subscript (or none if the (essential) supremum norm is involved), as $\llbracket f \rrbracket_1$, $\llbracket f \rrbracket$ in the precedingly specified examples. This convention permits the following arrangement: suppose that, e.g., $g \in \underset{\sim}{f}({}_m\underset{\sim}{L}^q(E))$, where $1 \leqq q \leqq \infty$, and $\underset{\sim}{f}$ is a sequence space, say $\underset{\sim}{f} \in b\ell$. Then $\|g\|$ is the element of $\underset{\sim}{f}({}_m\underset{\sim}{L}^q)$ (argument $R$ omitted, as usual) given by $\|g\|(n) = \|g(n)\|$, $n = 0,1,\ldots$ (where $\|g(n)\|(t) = \|g(n)(t)\|$--the latter being the norm in $E$--for all $t \in [-m,0]$); $\llbracket g \rrbracket_q$ is the element of $\underset{\sim}{f}$ given by $\llbracket g \rrbracket_q(n) = \llbracket g(n) \rrbracket_q = \llbracket \|g(n)\| \rrbracket_q$, $n = 0,1,\ldots$; thus $\llbracket g \rrbracket_q = \llbracket \|g\| \rrbracket_q$; and $\mathbf{|} g \mathbf{|}_{\underset{\sim}{f}} = \mathbf{|} \llbracket g \rrbracket_q \mathbf{|}_{\underset{\sim}{f}}$ is the norm of $g$ as an element of $\underset{\sim}{f}({}_m\underset{\sim}{L}^q(E))$.

We recall that (with our present convention) ${}_m b\mathfrak{F}$ is the class of all Banach spaces $\underset{\sim}{F}$ of (equivalence classes of) measurable functions $\varphi : [-m,0] \to R$ such that

(a): $\underset{\sim}{F}$ is stronger than ${}_m\underset{\sim}{L}^1$, i.e., $\underset{\sim}{F}$ is algebraically contained in ${}_m\underset{\sim}{L}^1$ and there exists a number $\alpha_{\underset{\sim}{F}} > 0$ such that $\llbracket \varphi \rrbracket_1 \leqq \alpha_{\underset{\sim}{F}} \llbracket \varphi \rrbracket_{\underset{\sim}{F}}$ for all $\varphi \in \underset{\sim}{F}$;

(b): if $\varphi \in \underset{\sim}{F}$ and $\psi : [-m,0] \to R$ is measurable, with $|\psi| \leqq |\varphi|$, then $\psi \in \underset{\sim}{F}$ and $\llbracket \psi \rrbracket_{\underset{\sim}{F}} \leqq \llbracket \varphi \rrbracket_{\underset{\sim}{F}}$.

Similarly, ${}_m b\mathfrak{F}\mathfrak{C}$ is the class of all Banach spaces $\underset{\sim}{F}$ of continuous functions $[-m,0] \to R$ such that (ac): $\underset{\sim}{F}$ is stronger than ${}_m\underset{\sim}{C}$, i.e., there exists a number $\gamma_{\underset{\sim}{F}} > 0$ such that $\|\varphi\| \leqq \gamma_{\underset{\sim}{F}} \|\varphi\|_{\underset{\sim}{F}}$ for all $\varphi \in \underset{\sim}{F}$;

(bc): if $\varphi \in \underset{\sim}{F}$ and $\psi : [-m,0] \to R$ is continuous, with $|\psi| \leqq |\varphi|$, then $\psi \in \underset{\sim}{F}$ and $\|\psi\|_F \leqq \|\varphi\|_F$.

For further details, see [2 ; Sections 22,24; esp. 24.D].

We observe that $\varphi \in \underset{\sim}{C}$ implies $\varphi \in {}_m\underset{\sim}{L}^1$ with $\|\varphi\|_1 \leqq m\,\|\varphi\|$; therefore, if $\underset{\sim}{F} \in {}_m b\mathfrak{F}C$, we have $\|\varphi\|_1 \leqq m\,\gamma_{\underset{\sim}{F}}\|\varphi\|_{\underset{\sim}{F}}$ for all $\varphi \in \underset{\sim}{F}$; so that there exists $\alpha_{\underset{\sim}{F}} > 0$ such that $\|\varphi\|_1 \leqq \alpha_{\underset{\sim}{F}}\|\varphi\|_{\underset{\sim}{F}}$ for all $\varphi \in \underset{\sim}{F}$ in this case too. We shall in every case specify $\alpha_{\underset{\sim}{F}}$, $\gamma_{\underset{\sim}{F}}$ to be the least numbers having the stated properties.

If $\underset{\sim}{F} \in {}_m b\mathfrak{F}$ [if $\underset{\sim}{F} \in {}_m b\mathfrak{F}C$] then $\underset{\sim}{F}(E)$ denotes, as usual, the Banach space of all measurable [continuous] functions $f : [-m,0] \to E$ such that $\|f\| \in \underset{\sim}{F}$, with $\|f\|_{\underset{\sim}{F}} = \| \|f\| \|_{\underset{\sim}{F}}$.

In considering functions defined on intervals of the type $[m, \infty)$, we borrow a notational device from [1]: the subscript $[m]$ will denote, in some sense, "restriction to $[m, \infty)$". Specifically, suppose $m, m'$ are real numbers and $f$ is some function defined on $[m, \infty)$. Then $f_{[m']}$, defined on $[m', \infty)$, is given by

$$f_{[m']}(t) = \begin{cases} f(t) & t \geqq m \\ 0 & m' \leqq t < m \end{cases}$$

and is thus an actual restriction if $m' \geqq m$.

The subscript $[m]$ is also used when the fact that $[m, \infty)$ is the domain has to be recorded (these usages are compatible). Thus $\underset{\sim}{L}_{[m]}(E)$ denotes the space of all (equivalence classes of) measurable functions $f : [m, \infty) \to E$ that are Bochner-integrable on each compact interval (the "full" notation in [2] would be ${}_{[m, \infty)}\underset{\sim}{L}(E)$), and similarly for, say, $\underset{\sim}{L}^{q}_{[m]}(E)$, $1 \leqq q \leqq \infty$, $\underset{\sim}{M}_{[m]}(E)$, $\underset{\sim}{C}_{[m]}(E)$, etc. A new notation is $\underset{\sim}{K}_{[m]}(E)$, which stands for the space of all continuous functions $f : [m, \infty) \to E$; this is a vector space; it may be thought of as provided with the topology of uniform convergence on compact intervals, thus becoming a Fréchet space, but this aspect will not be used in this paper. As usual, the argument is omitted if $E = R$.

In all these uses, the subscript $[m]$ is retained even when $m = 0$.

Norms in Banach spaces of functions defined on $[m, \infty)$ will be indicated, as in [2], by thick bars with the appropriate subscript as, e.g., $\| f \|_{\underset{\sim}{F}}$.

## 3. Slicing maps

From now on and throughout the paper, $p$ will always denote a fixed positive integer and $m, m'$ will be used only to denote non-negative integers.

For given $m$, let $f$ be a function defined on $[m, \infty)$. For each integer $n > m$ we define the function $\varpi(n) f$ on $[-1,0]$ by

$$(3.1) \qquad (\varpi(n) f)(t) = f(t+n), \qquad t \in [-1,0].$$

Thus $\varpi(n)$ maps $f$ into the "slice" of $f$ between $n-1$ and $n$, transplanted to $[-1,0]$ for convenience. Then $\varpi f$ denotes the sequence $(\varpi(n) f)$, i.e., the function on $\omega_{[m+1]}$ whose values are the slices of $f$: $(\varpi f)(n) = \varpi(n) f$, $n = m+1, m+2, \ldots$. In particular, for the given Banach space $E$, $\varpi(n)$ maps $\underset{\sim}{L}_{[m]}(E)$ onto ${}_1\underset{\sim}{L}^1(E)$ and $\underset{\sim}{K}_{[m]}(E)$ onto ${}_1\underset{\sim}{C}(E)$; and $\varpi$ maps $\underset{\sim}{L}_{[m]}(E)$ onto $\underset{\sim}{s}_{[m+1]}({}_1\underset{\sim}{L}^1(E))$ and $\underset{\sim}{K}_{[m]}(E)$ into (but not onto) $\underset{\sim}{s}_{[m+1]}({}_1\underset{\sim}{C}(E))$. These and similar restrictions of $\varpi(n)$, $\varpi$ shall be denoted by the same symbols. In particular, the mapping $\varpi: \underset{\sim}{L}_{[m]}(E) \to \underset{\sim}{s}_{[m+1]}({}_1\underset{\sim}{L}^1(E))$ is a Fréchet-space isomorphism.

$\varpi$ reduces to "natural" mappings between certain normed function spaces: e.g., $\varpi: \underset{\sim}{L}^q_{[m]}(E) \to \underset{\sim}{\ell}^q_{[m+1]}({}_1\underset{\sim}{L}^q(E))$ is a congruence (linear isometry) for $1 \leqq q \leqq \infty$; $\varpi: \underset{\sim}{M}_{[m]}(E) \to \underset{\sim}{\ell}^\infty_{[m+1]}({}_1\underset{\sim}{L}^1(E))$ is an isomorphism (with norm 1; the norm of the inverse is 2); $\varpi : \underset{\sim}{T}_{[m]}(E) \to \underset{\sim}{\ell}^{1}_{[m+1]}({}_1\underset{\sim}{L}^{\infty}(E))$ another (with norm 2; the norm of the inverse is 1). Indeed, we might thus define new normed spaces on $[m, \infty)$, but we shall not do this here.

We require another "slicing map", which we only define in a more restricted setting. For the given $p$ and $E$, and for any integers $m, n$, $n \geqq m \geqq 0$, we define $\Pi(n) : \underset{\sim}{K}_{[m-p]}(E) \to {}_p\underset{\sim}{C}(E)$ by

$$(3.2) \qquad (\Pi(n)f)(t) = f(t+n), \qquad t \in [-p, 0], \qquad f \in \underset{\sim}{K}_{[m-p]}(E),$$

and $\Pi : \underset{\sim}{K}_{[m-p]}(E) \to \underset{\sim}{s}_{[m]}({}_p\underset{\sim}{C}(E))$ by $(\Pi f)(n) = \Pi(n) f$, $n = m, m+1, \ldots$ . Thus, if $p = 1$, $m > 0$, $\Pi$ is just (the restriction to $\underset{\sim}{K}_{[m-1]}$ of) $\varpi$ itself.

## 4. The memory functional

We now make precise the assumptions on the "memory functional" $M$ that appears in (1.1). It must in any case be defined on $\underset{\sim}{K}_{[-p]}(E)$, and the crux of the conditions on $M$ as described in the Introduction is that, for any $u \in \underset{\sim}{K}_{[-p]}(E)$, the "slice" of $Mu$ between $n-1$ and $n$ depends only on the slice of $u$ between $n-1-p$ and $n-1$.

We must distinguish the "Carathéodory case" and the "continuous case" in further specifying the assumptions. Throughout the paper, the deviations pertaining to the continuous case are stated in angular brackets $\langle\ \rangle$.

We assume given a space $\underset{\sim}{F} \in {}_1b\mathfrak{F}$ $\langle \underset{\sim}{F} \in {}_1b\mathfrak{F}\mathfrak{C}\rangle$, $\underset{\sim}{F} \neq \{0\}$. We then assume that $M$ is a linear mapping from $\underset{\sim}{K}_{[-p]}(E)$ to $\underset{\sim}{L}_{[0]}(E)$ $\langle$ to $\underset{\sim}{K}_{[0]}(E)\rangle$ and that there exists a sequence $(\hat{M}(n))$ of mappings $\hat{M}(n) \in [{}_p\underset{\sim}{C}(E) \to \underset{\sim}{F}(E)]$, $n = 0,1,\ldots$--i.e., a function $\hat{M} \in \underset{\sim}{s}([{}_p\underset{\sim}{C}(E) \to \underset{\sim}{F}(E)])$--such that

$$(4.1)\quad (\varpi Mu)(n) = \varpi(n)(Mu) = \hat{M}(n-1)\ \Pi(n-1)u = \hat{M}\,\Pi\, u(n-1),$$
$$n = 1,2,\ldots;\ u \in \underset{\sim}{K}_{[-p]}(E).$$

Obviously, (4.1) permits, for every $m$, the restriction of $M$ to a mapping $M_{[m]}$ from $\underset{\sim}{K}_{[m-p]}(E)$ to $\underset{\sim}{L}_{[m]}(E)$ $\langle$ to $\underset{\sim}{K}_{[m]}(E)\rangle$, by means of

$$(4.2)\quad \varpi(n)(M_{[m]}u) = \hat{M}(n-1)\Pi(n-1)u\ ,\qquad n = m+1,m+2,\ldots;$$
$$u \in \underset{\sim}{K}_{[m-p]}(E)\ ,$$

so that, for $m' \geqq m \geqq 0$, we indeed have

$$(4.3)\qquad M_{[m']}u_{[m']} = (M_{[m]}u)_{[m']}\,, \quad u \in \underset{\sim}{K}_{[m-p]}(E)\,.$$

⟨ Let us determine the significance of the continuity requirement on $Mu$ in the continuous case; since $\underset{\sim}{F}(E)$ consists of continuous functions, (4.1) already accounts for the fact that the slices of $Mu$ are continuous; beyond this, all that is required is, clearly,

$$(4.4)\qquad \varpi(n)(Mu)(0) = (Mu)(n) = \varpi(n+1)(Mu)(-1)\,, \quad n = 1,2,\ldots$$

Combining (4.1) and (4.4) we obtain the following statement.

4.1. Lemma. If $v_1, v_2 \in {}_p\underset{\sim}{C}(E)$ and $v_1(t+1) = v_2(t)$, $-p \leqq t \leqq -1$, then $(\hat{M}(n)v_2)(-1) = (\hat{M}(n-1)v_1)(0)$, $n = 1,2,\ldots$ .

Proof. For a fixed $n$, there exists a function $u \in \underset{\sim}{K}_{[-p]}(E)$ such that $\Pi(n-1)u = v_1$, $\Pi(n)u = v_2$. Then (4.1), (4.4) yield $(\hat{M}(n)v_2)(-1) = (\hat{M}(n)\Pi(n)u)(-1) = \varpi(n+1)(Mu)(-1) = \varpi(n)(Mu)(0) = (\hat{M}(n-1)\Pi(n-1)u)(0) = (\hat{M}(n-1)v_1)(0)$.⟩

## 5. Solutions

We recall [2; Section 30] that a function $f \in \underset{\sim}{K}_{[m]}(E)$ is a primitive (function) if there exists $g \in \underset{\sim}{L}_{[m]}(E)$ such that $f(t) - f(m) = \int_m^t g(s)\,ds$ for all $t \in [m, \infty)$; then $g$ is unique, is denoted by $\dot{f}$, and is the derivative of $f$.

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Assume that we are given the space $\underset{\sim}{F}$ and the memory functional $\tilde{M}$ as specified in Section 4, and, in addition, $L \in \underset{\sim}{L}_{[0]}(E)$ and $r \in \underset{\sim}{L}_{[0]}(E)$ $\langle r \in \underset{\sim}{K}_{[0]}(E) \rangle$, with $\varpi(n) r \in \underset{\sim}{F}(E)$, $n = 1,2,\ldots$, i.e., $\varpi r \in \underset{\sim}{s}_{[1]}(\underset{\sim}{F}(E))$. A solution of the "differential equation with delay"

$$(5.1) \qquad \dot{u} + Lu + Mu = r$$

is a function $u \in \underset{\sim}{K}_{[-p]}(E)$ whose restriction $u_{[0]}$ to $[0, \infty)$ is a primitive that, together with its derivative $\dot{u}_{[0]}$, satisfies $\dot{u}_{[0]} + Lu_{[0]} + Mu = r$ in $\underset{\sim}{L}_{[0]}(E)$ (strictly speaking, this is the way the equation should be written). More generally, for every $m$, a solution of $(5.1)_{[m]}$ is a function $u \in K_{[m-p]}(E)$ whose restriction $u_{[m]}$ to $[m, \infty)$ is a primitive that satisfies $\dot{u}_{[m]} + L_{[m]}u_{[m]} + M_{[m]}u = r_{[m]}$ in $\underset{\sim}{L}_{[m]}(E)$. In particular, if $m' \geqq m \geqq 0$ and $u$ is a solution of $(5.1)_{[m]}$, then $u_{[m']}$ is a solution of $(5.1)_{[m']}$, on account of (4.3). These definitions and statements of course also apply to the homogeneous equation

$$(5.2) \qquad \dot{u} + Lu + Mu = 0 .$$

We define $V \in \underset{\sim}{K}_{[0]}(\tilde{E})$ as the solution of the operator equation $\dot{V} + LV = 0$ (in $\underset{\sim}{L}_{[0]}(\tilde{E})$) that satisfies $V(0) = I$ ($I$ is the identity on $E$). We refer to [2; Section 31] for details. $V$ is invertible-valued, and as usual we write $V^{-1} \in \underset{\sim}{K}_{[0]}(\tilde{E})$ for the function such that $V^{-1}(t) = (V(t))^{-1}$ $t \geqq 0$. We also have

$$(5.3) \qquad \|V(t)V^{-1}(s)\| < \exp\left(\left|\int_s^t \|L(\sigma)\|d\sigma\right|\right), \qquad s,t \geqq 0 .$$

With this notation and the use of (4.1), every solution of $(5.1)_{[m]}$ satisfies (cf. [2; Section 31])

$$(5.4) \quad (\Pi\, u(n))(t) = u(t+n) =$$

$$= \begin{cases} u(t+1+n-1) = (\Pi\, u(n-1))(t+1) & -p \leqq t \leqq -1 \\ V(t+n)V^{-1}(n-1)u(n-1) - \displaystyle\int_{-1}^{t} V(t+n)V^{-1}(s+n)\,(Mu(s+n)-r(s+n))\,ds = & \end{cases}$$

$$= V(t+n)V^{-1}(n-1)\,(\Pi\, u(n-1))(0) -$$

$$- \int_{-1}^{t} V(t+n)V^{-1}(s+n)\,(\hat{M}\,\Pi\, u(n-1) - \varpi r(n))(s)\,ds \qquad -1 \leqq t \leqq 0 ,$$

$$n = m+1, m+2, \ldots ;$$

and, conversely, every function $u \in \underset{\sim}{K}_{[m-p]}(E)$ such that $\Pi u$ satisfies (5.4)--more precisely, the equality between leftmost and rightmost sides--is a solution of $(5.1)_{[m]}$.

## 6. The associated difference equation

The relation (5.4) is a difference equation for $\Pi u$; we proceed to make explicit the form of this equation. For this purpose, we define $A \in \underset{\sim}{s}_{[1]}({}_{p}\underset{\sim}{C}(E)^{\sim})$ and $B \in \underset{\sim}{s}_{[1]}([\underset{\sim}{F}(E) \rightarrow {}_{p}\underset{\sim}{C}(E)])$ as follows.

(6.1) $(A(n)v)(t) =$

$$= \begin{cases} -v(t+1) & -p \leqq t \leqq -1 \\ -V(t+n)V^{-1}(n-1)v(0) + \int_{-1}^{t} V(t+n)V^{-1}(s+n)(\hat{M}(n-1)v)(s)\,ds & -1 \leqq t \leqq 0 \end{cases}$$

$$n = 1,2,\ldots; \quad v \in {}_{p}\underset{\sim}{C}(E),$$

(6.2) $$(B(n)g)(t) = \begin{cases} 0 & -p \leqq t \leqq -1 \\ \int_{-1}^{t} V(t+n)V^{-1}(s+n)g(s)\,ds & -1 \leqq t \leqq 0 \end{cases}$$

$$n = 1,2,\ldots; \qquad g \in \underset{\sim}{F}(E).$$

We note that the functions $A(n)v$, $B(n)g$ thus defined are indeed continuous, even at $t = -1$. We observe that $|||\hat{M}(n-1)v|||_1 \leqq \alpha_{\underset{\sim}{F}} |||\hat{M}(n-1)v|||_F \leqq \alpha_{\underset{\sim}{F}} \|\hat{M}(n-1)\| \, |||v|||$, and we therefore find, using (5.3),

$$(6.3) \quad |||A(n)v||| \leqq \max\{|||v|||, (|||v||| + |||\hat{M}(n-1)v|||_1)\exp(\int_{n-1}^{n}\|L(s)\|ds)\} \leqq$$

$$\leqq |||v|||(1 + \alpha_{\underset{\sim}{F}}\|\hat{M}(n-1)\| \exp(\int_{n-1}^{n}\|L(s)\|ds),$$

$$(6.4) \quad |||B(n)g||| \leqq |||g|||_1 \exp(\int_{n-1}^{n}\|L(s)\|ds) \leqq$$

$$\leqq \alpha_{\underset{\sim}{F}} |||g|||_{\underset{\sim}{F}} \exp(\int_{n-1}^{n}\|L(s)\|ds).$$

This shows that $A(n)$, $B(n)$ are indeed bounded linear mappings as claimed.

We consider the difference equations in ${}_{\underset{\sim}{p}}\underset{\sim}{C}(E)$

$$(6.5) \quad x(n) + A(n)x(n-1) = f(n) \qquad n = 1,2,\ldots$$

$$(6.6) \quad x(n) + A(n)x(n-1) = 0 \qquad n = 1,2,\ldots$$

and their restrictions $(6.5)_{[m]}$, $(6.6)_{[m]}$ to $n = m+1, m+2, \ldots$ . Here $f \in \underset{\sim}{s}_{[1]}({}_{\underset{\sim}{p}}\underset{\sim}{C}(E))$.

In what follows, $A$ and $B$ are defined by (6.1), (6.2).

6.1. *Lemma*. *Let* $m$ *be given*. *A function* $x \in \underset{\sim}{S}_{[m]}({}_p\underset{\sim}{C}(E))$ *is a solution of* $(6.5)_{[m]}$ *with* $f = B\varpi r$ *if and only if* $x = \Pi u$ *for some solution* $u$ *of* $(5.1)_{[m]}$. *In particular*, $x$ *is a solution of* $(6.6)_{[m]}$ *if and only if* $x = \Pi u$ *for some solution* $u$ *of* $(5.2)_{[m]}$.

*Proof*. If $u$ is a solution of $(5.1)_{[m]}$, then $\Pi u$ satisfies (5.4); together with (6.1), (6.2) this implies that $\Pi u(n) + A(n)\Pi u(n-1) = B(n)\varpi r(n)$, $n = m+1, m+2, \ldots$, i.e., that $\Pi u$ is a solution of $(6.5)_{[m]}$ with $f = B\varpi r$. Conversely, if $x$ is a solution of $(6.5)_{[m]}$ with $f = B\varpi r$, (6.2) implies $(f(n))(t) = 0$, $-p \leqq t \leqq -1$, $n = m+1, m+2, \ldots$, and this together with (6.1) implies that $(x(n))(t) = (x(n-1))(t+1)$ for all such $t, n$; therefore there exists a continuous $u$, i.e., $u \in \underset{\sim}{K}_{[m-p]}(E)$, such that $x = \Pi u$; and using again the fact that $x$ is a solution of $(6.5)_{[m]}$ with $f = B\varpi r$, we conclude that $\Pi u$ satisfies (5.4); thus $u$ is a solution of $(5.1)_{[m]}$.

It is clear that not every $f \in \underset{\sim}{S}_{[1]}({}_p\underset{\sim}{C}(E))$ is of the form $f = B\varpi r$; we intend to show, however, that we can still relate equation (6.5) with arbitrary $f$ to equation (5.1). This will be done here only under certain boundedness assumptions on $L, M$.

Specifically, we assume

$$(6.7) \qquad L \in \underset{\sim}{M}_{[0]}(\tilde{E}), \qquad \hat{M} \in \underset{\sim}{\ell}^{\infty}_{[1]}([{}_{p}\underset{\sim}{C}(E) \to \underset{\sim}{F}(E)]);$$

the latter condition is exactly $\| \hat{M} \| = \sup_{n} \|M(n)\| < \infty$.

(6.3) and (6.4) now yield

$$(6.8) \qquad \begin{array}{ll} A \in \underset{\sim}{\ell}^{\infty}_{[1]}({}_{p}\underset{\sim}{C}(E)^{\sim}), & B \in \underset{\sim}{\ell}^{\infty}_{[1]}([\underset{\sim}{F}(E) \to {}_{p}\underset{\sim}{C}(E)]) \\ \| A \| \leqq (1 + \alpha_{\underset{\sim}{F}} \| \hat{M} \|) \exp \| L \|_{\underset{\sim}{M}} & \| B \| \leqq \alpha_{\underset{\sim}{F}} \exp \| L \|_{\underset{\sim}{M}}. \end{array}$$

6.2. <u>Theorem</u>. <u>If</u> $L, M$ <u>satisfy</u> (6.7) <u>and if</u> $f \in \underset{\sim}{s}_1({}_{p}\underset{\sim}{C}(E))$, <u>there exists</u> $r \in \underset{\sim}{L}_{[0]}(E)$ $\langle r \in \underset{\sim}{K}_{[0]}(E) \rangle$ <u>with</u> $\varpi r \in \underset{\sim}{s}_{[1]}(\underset{\sim}{F}(E))$, <u>such that</u>

$$(6.9) \qquad \| (\omega r)_{[0]} \|_{\underset{\sim}{F}} \leqq k_1 \sum_{i=1}^{p+1} \| T^i f_{[0]} \|$$

<u>and such that the solution</u> $w$ <u>of</u>

$$(6.10) \qquad w(n) + A(n) w(n-1) = f(n) - B \varpi r(n) \qquad n = 1, 2, \ldots ,$$

<u>with</u> $w(0) = 0$ <u>satisfies</u>

$$(6.11) \qquad \|w\| \leqq k_2 \sum_{i=0}^{p} \|T^i f_{[0]}\| \quad ,$$

<u>where</u> $k_1, k_2 > 0$ <u>depend</u> <u>on</u> $\underset{\sim}{F}$, $\|L\|_{\underset{\sim}{M}}$, $\|\hat{M}\|$ <u>only</u>.

<u>Proof</u>. We can select $y \in \underset{\sim}{S}_{[1]}(\underset{\sim}{F}(E))$ in such a way that

$$(6.12) \qquad \int_{-1}^{0} (y(n))(s)ds = (f_{[0]}(n-1))(0) \qquad n = 1,2,\ldots$$

(recall that $f_{[0]}(0) = 0$) $\langle$and also

$$(6.13) \qquad (y(n))(-1) = (\hat{M} f_{[0]}(n-1))(-1), \qquad (y(n))(0) = 0 \qquad n = 1,2,\ldots \rangle,$$

and

$$(6.14) \qquad \|y_{[0]}\|_{\underset{\sim}{F}} \leqq k_0 \|T^+ f_{[0]}\|$$

for a number $k_0 > 0$ that depends only on $\underset{\sim}{F}$ $\langle$and on $\|\hat{M}\|\rangle$: a construction of $y$ will be given later.

We next define $w \in \underset{\sim}{s}({}_{p}\underset{\sim}{C}(E))$ by

$$(6.15)\quad (w(n))(t) = V(t+n)V^{-1}([t]+n)\int_{t-[t]-1}^{0}(T^{-[t]-1}y_{[0]}(n))(s)ds + $$

$$+\sum_{i=0}^{-[t]-1}(T^{i}f_{[0]}(n))(t+i), \qquad -p \leqq t \leqq 0\ ,$$

$$n = 0,1,\ldots$$

(here $[t]$ denotes the greatest integer $\leqq t$ ; the notation using translation operators is required to avoid the appearance of negative arguments for $y,f$).

(6.15) indeed yields a continuous function $w(n)$, for by (6.12) we have, for $k = 0,\ldots,p-1$,

$$(w(n))(-k) = V(n-k)V^{-1}(n-k)\int_{-1}^{0}(T^{k-1}y_{[0]}(n))(s)ds + \sum_{i=0}^{k-1}(T^{i}f_{[0]}(n))(i-k) =$$

$$= (T^{k}f_{[0]}(n))(0) + \sum_{i=0}^{k-1}(T^{i}f_{[0]}(n))(i-k) =$$

$$= \sum_{i=0}^{k}(T^{i}f_{[0]}(n))(i-k)$$

$$(w(n))(-k-0) = V(n-k)V^{-1}(n-k-1)\int_0^0 (T^k y_{[0]}(n))(s)\,ds +$$

$$+\sum_{i=0}^{k}(T^i f_{[0]}(n))(i-k) = \sum_{i=0}^{k}(T^i f_{[0]}(n))(i-k)\,.$$

This computation yields, in particular,

(6.16) $$(w(n))(0) = (f_{[0]}(n))(0)\,.$$

Also, since $T^j y_{[0]}(0) = 0$, $T^j f_{[0]}(0) = 0$ for $j = 0,1,\ldots$, (6.15) yields

(6.17) $$w(0) = 0\,.$$

Further, $[t+1] = [t]+1$, $-p \leqq t \leqq -1$, and therefore (6.15) also yields

(6.18) $$(w(n-1))(t+1) = V(t+n)V^{-1}([t]+n)\int_{t-[t]-1}^{0}(T^{-[t]-2}y_{[0]}(n-1))(s)\,ds +$$

$$+\sum_{i=0}^{-[t]-2}(T^i f_{[0]}(n-1))(t+1+i) =$$

$$= V(t+n)V^{-1}([t]+n)\int_{t-[t]-1}^{0}(T^{-[t]-1}y_{[0]}(n))(s)ds \; +$$

$$+\sum_{i=1}^{-[t]-1}(T^{i}f_{[0]}(n))(t+i) \;=\; (w(n)-f(n))(t)$$

$$-p \leqq t \leqq -1.$$

Finally, (6.15), (6.14), (5.3) yield

$$\|(w(n))(t)\| \leqq \| T^{-[t]-1}y_{[0]}(n)\|_1 \; \exp \|L\|_{\underset{\sim}{M}} + \sum_{i=0}^{-[t]-1} \|T^{i}f_{[0]}(n)\| \leqq$$

$$\leqq \alpha_{\underset{\sim}{F}} k_0 \; \|T^{-[t]}f_{[0]}(n)\| \; \exp \|L\|_{\underset{\sim}{M}} + \sum_{i=0}^{-[t]-1} \|T^{i}f_{[0]}\| \leqq$$

$$\leqq k_2 \sum_{i=0}^{-[t]} \|T^{i}f_{[0]}(n)\| \qquad -p \leqq t \leqq 0$$

$$n = 0,1,\ldots ,$$

with $k_2 = \max \{1 , \alpha_{\underset{\sim}{F}} k_0 \exp \|L\|_{\underset{\sim}{M}}\}$ , and (6.11) follows.

We now define $r \in \underset{\sim}{L}_{[0]}(E)$ $\langle r \in \underset{\sim}{K}_{[0]}(E)\rangle$ by

$$(6.19)\quad (\varpi r(n))(t) = -(\hat{M}w(n-1))(t) + V(t+n)V^{-1}(n-1)(y(n))(t)$$

$$-1 \leqq t \leqq 0\,,$$

$$n = 1,2,\ldots\,;$$

Since $V(\bullet + n)V^{-1}(n-1) \in {}_1\underset{\sim}{C}(\tilde{E})$ and $\hat{M}w(n-1)\,,\ y(n) \in \underset{\sim}{F}(E)$, we have $\varpi r \in \underset{\sim}{s}_{[1]}(\underset{\sim}{F}(E))$.

$\langle$The slices of $r$ are thus continuous. To show that (6.19) defines a continuous function, we use (6.13) and compute, for $n = 1,2,\ldots$,

$$r(n-0) = (\varpi r(n))(0) = -(\hat{M}w(n-1))(0) + V(n)V^{-1}(n-1)(y(n))(0) =$$

$$= -(\hat{M}(n-1)w(n-1))(0)$$

$$r(n+0) = (\varpi r(n+1))(-1) = -(\hat{M}w(n))(-1) + V(n)V^{-1}(n)(y(n+1))(-1) =$$

$$= -(\hat{M}(n)(w(n)-f(n)))(-1)\ ;$$

by (6.18), $v_1 = w(n-1)$ and $v_2 = w(n) - f(n)$ satisfy the assumption of Lemma 4.1; by that lemma, we find that $r(n-0) = r(n+0)$, so that $r$ is continuous, i.e., $r \in \underset{\sim}{K}_{[0]}(E)$.$\rangle$

From (6.19) and (5.3),

$$\| \varpi r(n) \|_{\underset{\sim}{F}} \leqq \| \hat{M}(n-1) w(n-1) \|_{\underset{\sim}{F}} + \| y(n) \|_{\underset{\sim}{F}} \exp \| L \|_{\underset{\sim}{M}} \leqq$$

$$\leqq \|\hat{M}(n-1)\| \; \| w(n-1) \| + \| y(n) \|_{\underset{\sim}{F}} \exp \| L \|_{\underset{\sim}{M}} ,$$

$$n = 1,2, \ldots ;$$

then (6.11), (6.14) yield

$$\| (\varpi r)_{[0]} \|_{\underset{\sim}{F}} \leqq \| \hat{M} \| \; \| T^{+} w \| + \| y \|_{\underset{\sim}{F}} \exp \| L \|_{\underset{\sim}{M}} \leqq k_1 \sum_{i=1}^{p+1} \| T^{i} f_{[0]} \| ,$$

i.e., (6.9), with $k_1 = k_2 \| \hat{M} \| + k_0 \exp \| L \|_{\underset{\sim}{M}}$.

In view of (6.9), (6.11), (6.17), it only remains to prove that $w$ is a solution of (6.10). Now (6.1), (6.2), (6.18) show that

$$(w(n) + A(n) w(n-1))(t) = (w(n))(t) - (w(n-1))(t+1) =$$

$$= (f(n))(t) = (f(n) - B \varpi r(n))(t) ,$$

$$-p \leqq t \leqq -1$$

$$n = 1,2, \ldots ;$$

so that (6.10) remains to be verified for $-1 \leqq t \leqq 0$ only. We use (6.15) (and the continuity of $w(n)$ at 0), (6.1), (6.16), (6.12) in turn to obtain, for $-1 \leqq t \leqq 0$, $n = 1,2,\ldots$,

(6.20) $(w(n) + A(n)w(n-1))(t) =$

$$= V(t+n)V^{-1}(n-1)\int_t^0 (y(n))(s)ds + (f(n))(t) -$$

$$- V(t+n)V^{-1}(n-1)(w(n-1))(0) + \int_{-1}^t V(t+n)V^{-1}(s+n)(\hat{M}w(n-1))(s)ds =$$

$$= (f(n))(t) + V(t+n)V^{-1}(n-1)\left(\int_t^0 (y(n))(s)ds - (f_{[0]}(n-1))(0)\right) +$$

$$+ \int_{-1}^t V(t+n)V^{-1}(s+n)(\hat{M}w(n-1))(s)\,ds =$$

$$= (f(n))(t) - V(t+n)V^{-1}(n-1)\int_{-1}^t (y(n))(s)\,ds +$$

$$+ \int_{-1}^t V(t+n)V^{-1}(s+n)(\hat{M}w(n-1))(s)\,ds$$

and (6.19), (6.2) yield

$$(6.21) \quad (f(n) - B\varpi r(n))(t) \; =$$

$$= \; (f(n))(t) + \int_{-1}^{t} V(t+n)V^{-1}(s+n)(\hat{M}\,w(n-1))(s)\,ds \; -$$

$$- \int_{-1}^{t} V(t+n)V^{-1}(s+n)V(s+n)V^{-1}(n-1)(y(n))(s)ds \; ,$$

and the rightmost sides of (6.20) and (6.21) are obviously equal.

We conclude the proof with a construction for $y$. In the Carathéodory case, let $\varphi \in \underset{\sim}{F}$ be such that $\varphi \geqq 0$ , $\|\varphi\|_1$ , and set $y(n) = \varphi \cdot (f_{[0]}(n-1))(0)$; then $y(n) \in \underset{\sim}{F}(E)$, (6.12) holds, and $\|y(n)\|_{\underset{\sim}{F}} \leqq \|\varphi\|_{\underset{\sim}{F}} \|f_{[0]}(n-1)\|$, so that (6.14) holds with $k_0 = \|\varphi\|_{\underset{\sim}{F}}$ .

⟨ In the continuous case, we have to distinguish two subcases. If $\psi(-1) = 0$ for all $\psi \in \underset{\sim}{F}$ , we proceed as in the Carathéodory case, merely requiring, as we may, that $\varphi(0) = 0$ . Then (6.13) also holds, since $\hat{M}\,f_{[0]}(n-1) \in \underset{\sim}{F}(E)$ , so that $(\hat{M}\,f_{[0]}(n-1))(-1) = 0$ ; and (6.12), (6.14) still hold, the latter with $k_0 = \|\varphi\|_{\underset{\sim}{F}}$ .

Otherwise, we may choose $\varphi \in \underset{\sim}{F}$ with $\varphi \geq 0$ , $\varphi(-1) = 1$ , $\varphi(0) = 0$, and then $\|\varphi\|_1 > 0$ . We set

$$y(n) = \varphi(\psi_1 \cdot (f_{[0]}(n-1))(0) + \psi_2 \cdot (\hat{M} f_{[0]}(n-1))(-1)),$$

where $\psi_1(t) = 2\|\varphi\|_1^{-2} \int_{-1}^{t} \varphi(s)ds$ , $\psi_2(t) = 1 - \|\varphi\|_1 \psi_1$ . Then $\psi_1(-1) = 0$ , $\psi_2(-1) = 1$ , $\int_{-1}^{0} \varphi(t)\psi_1(t)dt =$ $2\|\varphi\|_1^{-2} \int_{-1}^{0} \varphi(t)dt \int_{-1}^{t} \varphi(s)ds = 1$ , $\int_{-1}^{0} \varphi(t)\psi_2(t)dt =$ $\|\varphi\|_1 - \|\varphi\|_1 \cdot 1 = 0$ , $\|\psi_1\| = 2\|\varphi\|_1^{-1}$ , $\|\psi_2\| = 1$ . Therefore (6.12), (6.13) are satisfied, and

$$\|y(n)\|_{\underset{\sim}{F}} \leqq \|\varphi\|_{\underset{\sim}{F}} (2\|\varphi\|_1^{-1} \|f_{[0]}(n-1)\| + \|\hat{M} f_{[0]}(n-1)\|) \leqq$$

$$\leqq \|\varphi\|_{\underset{\sim}{F}} (2\|\varphi\|_1^{-1} + \gamma_{\underset{\sim}{F}} \|\hat{M}(n-1)\|) \|f_{[0]}(n-1)\| ,$$

so that (6.14) holds with $k_0 = \|\varphi\|_{\underset{\sim}{F}} (2\|\varphi\|_1^{-1} + \gamma_{\underset{\sim}{F}} \|\hat{M}\|)$.⟩

## 7. Admissibility

When we are studying equation (5.1) with given L,M (satisfying the boundedness conditions (6.7)), the discussion in the preceding section allows us to replace consideration of the differential equation with delay (5.1) by analysis of the associated difference equation (6.5). In this section we describe a significant instance of this.

*We assume the space* $\underset{\sim}{F} \in {}_1 b\,\mathfrak{F}$ $\langle \underset{\sim}{F} \in {}_1 b\,\mathfrak{F}C \rangle$ *given,* $M$ *a memory functional as described in Section 4, and* $L, M$ *satisfying* (6.7). $A, B$ are then defined by (6.1), (6.2).

For the concepts of $\ell$-*pairs* and $\ell^{\rightarrow}$-*pairs* of sequence spaces and their *admissibility* for difference equations, we refer to [1; Section 8].

7.1 *Theorem. For each given* $\ell^{\rightarrow}$-*pair* (*or, in particular,* $\ell$-*pair*) $(\underset{\sim}{b}, \underset{\sim}{d})$, *the following statements are equivalent*:

(a): $\underset{\sim}{b}$ *is stronger than* $\underset{\sim}{d}$; *and for every* $r \in \underset{\sim}{L}_{[0]}(E)$ $\langle r \in \underset{\sim}{K}_{[0]}(E) \rangle$ with $\varpi r \in \underset{\sim}{b}_{[1]}(F(E))$ equation (5.1) has a solution $u$ with $\Pi u \in \underset{\sim}{d}({}_p\underset{\sim}{C}(E))$;

(b): $(\underset{\sim}{b}, \underset{\sim}{d})$ *is admissible for* $A$; *i.e., for every* $f \in \underset{\sim}{b}_{[1]}({}_p\underset{\sim}{C}(E))$, *equation* (6.5) *has a solution* $x \in \underset{\sim}{d}({}_p\underset{\sim}{C}(E))$.

*Proof.* (a) *implies* (b). Let $f \in \underset{\sim}{b}_{[1]}({}_p\underset{\sim}{C}(E))$ be given. Since $\underset{\sim}{b} \in b\,\ell^{\rightarrow}$, we have $T^i f_{[0]} \in \underset{\sim}{b}({}_p\underset{\sim}{C}(E))$, $i = 0, 1, \ldots$ . Let $r, w$ be as provided by Theorem 6.2. Then $r \in \underset{\sim}{L}_{[0]}(E)$ $\langle r \in \underset{\sim}{K}_{[0]}(E) \rangle$, and (6.9) implies $\varpi r_{[0]} \in \underset{\sim}{b}({}_p\underset{\sim}{C}(E))$, i.e., $\varpi r \in \underset{\sim}{b}_{[1]}({}_p\underset{\sim}{C}(E))$. Further, (6.10) implies $w \in \underset{\sim}{b}({}_p\underset{\sim}{C}(E))$, and, since $\underset{\sim}{b}$ is stronger than $\underset{\sim}{d}$, $w \in \underset{\sim}{d}({}_p\underset{\sim}{C}(E))$.

By the assumption, (5.1) with this $r$ has a solution $u$ such that $\Pi u \in \underset{\sim}{d}({}_{p}\underset{\sim}{C}(E))$ ; but by Lemma 6.1 we have $\Pi u(n) + A(n)\Pi u(n-1) = B\varpi r(n)$, $n = 1,2,\ldots$ ; since $w$ is a solution of (6.10), we conclude that $x = \Pi u + w \in \underset{\sim}{d}({}_{p}\underset{\sim}{C}(E))$ is a solution of (6.5). Thus $(\underset{\sim}{b},\underset{\sim}{d})$ is admissible for $A$ .

(b) <u>implies</u> (a). The admissibility of $(\underset{\sim}{b},\underset{\sim}{d})$, together with (6.8), implies that $\underset{\sim}{b}$ is stronger than $\underset{\sim}{d}$ [4; Lemma 4.1]. Let now $r$ be given as specified in (a); then (6.8) implies $\|B\varpi r\| \leqq \|B\| \, \|\varpi r\|_{\underset{\sim}{F}}$ , so that $B\varpi r \in \underset{\sim}{b}_{[1]}({}_{p}\underset{\sim}{C}(E))$. By (b), there exists a solution $x \in \underset{\sim}{d}({}_{p}\underset{\sim}{C}(E))$ of $x(n) + A(n)x(n-1) = B\varpi r(n)$, $n = 1,2,\ldots$ , and by Lemma 6.1 there exists a solution $u$ of (5.1) with $\Pi u = x \in \underset{\sim}{d}({}_{p}\underset{\sim}{C}(E))$, so that (a) is verified.

If $\underset{\sim}{B}$ is a subset of $\underset{\sim}{L}_{[0]}(E)$ $\langle$ of $\underset{\sim}{K}_{[0]}(E)\rangle$ and $\underset{\sim}{D}$ is a subset of $\underset{\sim}{K}_{[-p]}(E)$, respectively, it is in keeping with earlier terminology to say that the pair $(\underset{\sim}{B},\underset{\sim}{D})$ <u>is admissible with respect to</u> L,M--more loosely, <u>with respect to</u> (5.1)--if for every $r \in \underset{\sim}{B}$ there exists a solution $u \in \underset{\sim}{D}$ of (5.1). $\underset{\sim}{B}$ will here be specified to be such a space as $\underset{\sim}{L}^{q}_{[0]}(E)$, $1 \leqq q \leqq \infty$ , $\underset{\sim}{M}_{[0]}(E)$, $\underset{\sim}{T}_{[0]}(E)$ , etc., and similarly for $\underset{\sim}{D}$ , but the choices can easily be extended in the spirit of [2; Chapter 2]. Following earlier practice, the name of the pair $(\underset{\sim}{L}^q_{[0]}(E), \underset{\sim}{C}_{[-p]}(E))$, etc., is abbreviated to $(\underset{\sim}{L}^q, \underset{\sim}{C})$ in this context, since there is no ambiguity.

With this terminology, Theorem 7.1 covers, among many others, the special cases we now record.

7.2. Corollary. *With* $\underset{\sim}{F}$, $(\underset{\sim}{B}, \underset{\sim}{D})$, $(\underset{\sim}{b}, \underset{\sim}{d})$ *as specified in the tables, the admissibility of* $(\underset{\sim}{B}, \underset{\sim}{D})$ *for* L,M *is equivalent to the admissibility of* $(\underset{\sim}{b}, \underset{\sim}{d})$ *for* A.

| $\underset{\sim}{F}$ | $\underset{\sim}{B}$ | $\underset{\sim}{b}$ | | $\underset{\sim}{D}$ | $\underset{\sim}{d}$ |
|---|---|---|---|---|---|
| $_1\underset{\sim}{L}^q$ | $\underset{\sim}{L}^q$ | $\underset{\sim}{\ell}^q$ | $1 \leqq q \leqq \infty$ | $\underset{\sim}{C}$ | $\underset{\sim}{\ell}^\infty$ |
| $_1\underset{\sim}{L}^1$ | $\underset{\sim}{M}$ | $\underset{\sim}{\ell}^\infty$ | | $\underset{\sim}{C}_0$ | $\underset{\sim}{\ell}_0^\infty$ |
| $_1\underset{\sim}{L}^\infty$ | $\underset{\sim}{T}$ | $\underset{\sim}{\ell}^1$ | | | |
| $\langle\ _1\underset{\sim}{C}$ | $\underset{\sim}{C}$ | $\underset{\sim}{\ell}^\infty\ \rangle$ | | | |
| $\langle\ _1\underset{\sim}{C}$ | $\underset{\sim}{C}_0$ | $\underset{\sim}{\ell}_0^\infty\ \rangle$ | | | |

Proof. Theorem 7.1 and the remarks in Section 4 on the "slicing operator" $\pi$. (Similarly, if $u \in \underset{\sim}{K}_{[-p]}(E)$, then $u \in \underset{\sim}{C}_{[-p]}(E)$ if and only if $\Pi u \in \underset{\sim}{\ell}^\infty({}_p\underset{\sim}{C}(E))$, etc.)

## 8. Admissibility and the solutions of the homogeneous equation

The results of the preceding section have enabled us to translate admissibility of certain pairs of function spaces with respect to the differential equation with delays (5.1) into admissibility of certain related pairs of sequence spaces with respect to the associated difference equation (6.5). This enables us to apply the theory developed in [1] to obtain conclusions about the solutions of the homogeneous difference equation (6.6) and, via Lemma 6.1, about those of the homogeneous equation (5.2).

The behaviour envisaged for the solutions of (6.6)--or, rather, of $(6.6)_{[m]}$, $m = 0,1,\ldots$ --is either an *ordinary* or an *exponential dichotomy*, as defined in [1; Section 7], types of conditional uniform stability, simple and asymptotic, respectively. In order to develop fully the programme outlined above, it would be necessary to translate the concept of a dichotomy, in the most general case, into a description of the behaviour of the solutions of (5.2); in particular, we should supply the analogue of a covariant sequence of subspaces. (Observe, at all events, that Lemma 6.1 ensures that information on *all* solutions of (5.2) corresponds to information on *all* solutions of (6.6).) Further, in order to take full advantage of the "direct theorems" in [1; Sections 9,10], we should require that any admissibility be "regular", and translate this in terms of (5.1); and we should consider, in addition, $(\underset{\sim}{b},\underset{\sim}{d})$-sequences for (6.5) and their translation in terms of (5.1).

All this can be done; at present, however, it would take us too far afield without involving or illustrating any new idea, and it fortunately turns out to be unnecessary in the fundamental case of finite-dimensional $E$ (note that the associated difference equations of course belong in the infinite-dimensional space ${}_{p}\underset{\sim}{C}(E)$). We therefore restrict our attention to this case here.

*We assume, then, in the remainder of this section, that* $E$ *is finite-dimensional, that the space* $\underset{\sim}{F} \in {}_1b\mathfrak{F}$ $\langle \underset{\sim}{F} \in {}_1b\mathfrak{F}C \rangle$ *is given,* $M$ *is a memory functional as described in Section* 4, *and* $L \in \underset{\sim}{L}_{[0]}(\tilde{E})$. $A$ and $B$ are defined by (6.1) and (6.2).

As regards equations (6.5), (6.6), we define, as usual, the transition operators $U(n,n_0) \in {}_{p}\underset{\sim}{C}(E)^{\sim}$ for integers $n \geqq n_0 \geqq 0$ by

$$(8.1)\quad U(n_0,n_0) = I\ , \qquad U(n,n_0) = (-1)^{n-n_0} A(n)A(n-1)\ldots A(n_0+1)\,,$$
$$n > n_0 \geqq 0\ .$$

The following propositions show why the assumption that $E$ is finite-dimensional is significant.

8.1. <u>Lemma</u>. $U(m+p,m)$ <u>is a compact operator</u> for $m = 0,1,\ldots$ .

<u>Proof</u>. By (6.1), (6.2), $A(n) = -J + K(n) + B(n)\hat{M}(n-1)$, $n = 1,2,\ldots$ , where $J, K(n) \in {}_{\underset{\sim}{p}}C(E)^{\sim}$ are given by

$$(Jv)(t) = \begin{cases} v(t+1) - v(0) & -p \leqq t \leqq -1 \\ 0 & -1 \leqq t \leqq 0 \end{cases}$$

$$v \in {}_{\underset{\sim}{p}}C(E)$$

$$(K(n)v)(t) = \begin{cases} -v(0) & -p \leqq t \leqq -1 \\ -V(t+n)V^{-1}(n-1)v(0) & -1 \leqq t \leqq 0 \end{cases}$$

$$v \in {}_{\underset{\sim}{p}}C(E)$$

Since $E$ is finite-dimensional, $K(n)$ has finite-dimensional range, hence is compact; and $B(n)$ is the restriction to $\mathbf{F}(E)$, with its stronger topology, if the linear mapping ${}_{\mathbf{1}}\mathbf{L}(E) \to {}_{\mathbf{p}}\mathbf{C}(E)$ given by (6.2), which is itself clearly compact for finite-dimensional $E$. We conclude that $A(n) + J$ is compact; it follows from (8.1) that $U(m + p,m) - J^{p}$ is compact. But induction shows that $(J^{k} v)(t) = 0$ for $-k \leqq t \leqq 0$, $k = 1,\ldots,p$, so that $J^{p} = 0$, and $U(m + p,m)$ is compact.

8.2. *Theorem. Assume that the* $\mathfrak{t}$*-pair or* $\mathfrak{t}^{\rightarrow}$*-pair* $(\mathbf{b},\mathbf{d})$ *is admissible for* $A$. *Then the covariant sequence* $({}_{\mathbf{p}}\mathbf{C}(E))_{O\mathbf{d}}$ *is regular, its terms have constant finite co-dimension in* ${}_{\mathbf{p}}\mathbf{C}(E)$, *and it induces a dichotomy for* $A$. *If* $(\mathbf{b},\mathbf{d})$ *is not weaker than* $(\boldsymbol{\ell}^{1},\boldsymbol{\ell}_{0}^{\infty})$, *then* $({}_{\mathbf{p}}\mathbf{C}(E))_{O\mathbf{d}}$ *induces an exponential dichotomy for* $A$.

*Proof*. By [1; Lemma 3.5], supp $(\mathbf{b})$ is an infinite set, so that statement (d) of [5; Lemma 4.2] holds for $(\mathbf{b},\mathbf{d})$ with respect to $A$. Since $U(p,0)$, say, is compact by Lemma 8.1, we conclude from [5; Theorem 4.3,(b)] that the covariant sequence $({}_{\mathbf{p}}\mathbf{C}(E))_{O\mathbf{d}}$ is regular and that its terms have constant finite co-dimension in ${}_{\mathbf{p}}\mathbf{C}(E)$. This sequence induces a dichotomy for $A$, by [1; Theorem 9.2]; and an exponential dichotomy if $(\mathbf{b},\mathbf{d})$ is not weaker than $(\boldsymbol{\ell}^{1}\boldsymbol{\ell}_{0}^{\infty})$, by [1; Theorem 10.2].

We are now able to prove a fundamental "direct" theorem by combining the preceding result with Theorem 7.1.

8.3. *Theorem*. *Assume that* L,M *satisfy* (6.7), *and that the* $t$-*pair or* $\vec{t}$-*pair* $(\mathbf{b},\mathbf{d})$ *is given. Assume further that for every* $r \in \mathbf{L}_{[0]}(E)$ $\langle r \in \mathbf{K}_{[0]}(E) \rangle$ *with* $\varpi r \in \mathbf{b}_{[1]}(F(E))$ *equation* (5.1) *has a solution* $u$ *with* $\Pi u \in \mathbf{d}({}_{\mathbf{p}}\mathbf{C}(E))$. *Then, with respect to the difference equation* (6.6), *the covariant sequence* $({}_{\mathbf{p}}\mathbf{C}(E))_{O\mathbf{d}}$ *is regular, its terms have constant finite co-dimension in* ${}_{\mathbf{p}}\mathbf{C}(E)$, *and it induces a dichotomy for* A. *If* $(\mathbf{b},\mathbf{d})$ *is not weaker than* $(\boldsymbol{\ell}^1, \boldsymbol{\ell}_0^{\infty})$, *then* $({}_{\mathbf{p}}\mathbf{C}(E))$ *induces an exponential dichotomy for* A.

*Proof*. Theorems 7.1 and 8.2.

The significance of the conclusion lies in the fact that, by Lemma 6.1, the solutions of (6.6) are exactly the sequences of slices of solutions of (5.2). We might now define a "dichotomy" (ordinary or exponential) of the solutions of (5.2) in such a way as to translate the corresponding behaviour of the solutions of (6.6). We do not wish, however, to explore all the implications of such a definition at this time, or to complicate future work by an inadequate definition. We shall therefore merely set down one description of the conclusion of Theorem 8.3 in terms of the solutions of (5.2). We shall consider only the cases $\underset{\sim}{d} = \underset{\sim}{\ell}^{\infty}$ and $\underset{\sim}{d} = \underset{\sim}{\ell}_{0}^{\infty}$. Since a complete proof would imply a purely technical reworking of a large part of the proof of [1; Theorem 7.1], we merely indicate the results used.

8.4. *Theorem. The following statements are equivalent:*

(a): *With respect to equation* (6.6), *the covariant sequence* $({}_{p}\underset{\sim}{C}(E))_{0}$ *is regular and its terms have constant finite co-dimension in* ${}_{p}\underset{\sim}{C}(E)$, *and it induces a dichotomy [an exponential dichotomy] for* $A$;

(b): *There exists [a number* $\nu > 0$ *and] a number* $N > 0$ *such that, for every* $m \in \omega$, *every bounded solution* $v$ *of* $(5.2)_{[m]}$ *satisfies*

(i): $\|v(n)\| \leqq N\|v(n_0)\|$ $\quad [\,\|v(n)\| \leqq Ne^{-\nu(n-n_0)}\|v(n_0)\|\,]$ $\qquad n \geqq n_0 \geqq m$;

*there further exists a finite-dimensional linear manifold* $\underset{\sim}{W}$ *of solutions of* (5.2), *and numbers* $[\nu' > 0,]\, N' > 0$, $\lambda_0 > 1$ *such that every solution* $u$ *of* $(5.2)_{[m]}$ *for any* $m \in \omega$ *satisfies* $u = v + w_{[m]}$ *with* $v$ *a bounded* *solution and* $w \in \underset{\sim}{W}$ , *and such that every solution* $w \in \underset{\sim}{W}$ *satisfies*

$$\text{(ii)}: \quad \|\Pi w(n)\| \geqq N'^{-1}\|\Pi w(n_0)\| \qquad [\, \|\Pi w(n)\| \geqq N'^{-1} e^{\nu'(n-n_0)} \|\Pi w(n_0)\| \,] \qquad n \geqq n_0 \geqq 0 ,$$

$$\text{(iii)}: \quad \|\Pi w(n)\| \leqq \lambda_0 \|\Pi w(n) - \Pi v(n)\| , \quad n \geqq m ,$$

*for any bounded solution* $v$ *of* $(5.2)_{[m]}$ , $m \in \omega$ .

*The equivalence of* (a), (b) *holds if* $({}_{\underset{\sim}{p}}C(E))_0$ *is replaced by* $({}_{\underset{\sim}{p}}C(E))_{00}$ *in* (a), *and* "*bounded*" *by* "*tending to* 0 *as* $n \to \infty$" *in* (b). [*If* L,M *satisfy* (6.7), *condition* (iii) *in* (b) *is redundant in the* "*exponential*" *case*.]

*Proof*. Lemma 6.1 is used to pass from solutions of $(5.2)_{[m]}$ to solutions of $(6.6)_{[m]}$ and vice versa. To establish that (a) implies (b), we observe that, in [1; Theorem 7.1], the finite co-dimensionality of the terms of the covariant sequence allows us to establish statement (c) with the splitting chosen to be a (linear) projection onto a finite-dimensional complement $Z$ of $({}_{\underset{\sim}{p}}C(E))_0(0)$ in ${}_{\underset{\sim}{p}}C(E)$. If $\underset{\sim}{W}$ is the set of solutions $w$ of (5.2) with $\Pi w(0) \in Z$ , the present statement (b) follows via Lemma 6.1 from [1; Theorem 7.1, (c)]. For the converse implication, Lemma 6.1 is combined with [1; Theorem 7.1]; condition (i) together with [1; Lemma 6.5] is used to show that the covariant sequence $({}_p\underset{\sim}{C}(E))_0$ is closed. The proof for the "tending to 0" case is the same. [The redundancy of (iii) when L,M satisfy (6.7) in the exponential case follows from [1; Lemma 7.2].]

8.5. Corollary. *Assume that* L,M *satisfy* (6.7). *If* $\underset{\sim}{F} = {}_1\underset{\sim}{L}^1$ *and* $(\underset{\sim}{L}^1,\underset{\sim}{C})$ or $(\underset{\sim}{L}^1,\underset{\sim}{C}_0)$ *is admissible for* L,M, *or if* $\underset{\sim}{F} = {}_1\underset{\sim}{L}^\infty$ *and* $(\underset{\sim}{T},\underset{\sim}{C})$ *or* $(\underset{\sim}{T},\underset{\sim}{C}_0)$ *is admissible for* L,M, *then the solutions of* (5.2) *have the "ordinary" behaviour described in Theorem 8.4*, (b). *If* $\underset{\sim}{F} = {}_1\underset{\sim}{L}^q$ *and* $(\underset{\sim}{L}^q,\underset{\sim}{C})$ *or* $(\underset{\sim}{L}^q,\underset{\sim}{C}_0)$ *is admissible for* L,M, $1 < q \leqq \infty$, *or if* $\underset{\sim}{F} = {}_1\underset{\sim}{L}^1$ *and* $(\underset{\sim}{M},\underset{\sim}{C})$ *or* $(\underset{\sim}{M},\underset{\sim}{C}_0)$ *is admissible for* L,M $\langle$ *or if* $\underset{\sim}{F} = {}_1\underset{\sim}{C}$ *and* $(\underset{\sim}{C},\underset{\sim}{C})$ *or* $(\underset{\sim}{C}_0,\underset{\sim}{C})$ *or* $(\underset{\sim}{C},\underset{\sim}{C}_0)$ *or* $(\underset{\sim}{C}_0,\underset{\sim}{C}_0)$ *is admissible for* L,M$\rangle$, *then the solutions of* (5.2) *have the "exponential" behaviour described in Theorem 8.4*, (b).

*Proof*. Corollary 7.2, Theorems 8.3, 8.4.

We conclude by stating, in a form corresponding to Theorem 8.3, a "converse" theorem. Theorem 8.4 and Corollary 7.2 provide some consequences in terms of solutions of (5.1), (5.2).

8.6. <u>Theorem</u>. <u>If some regular covariant sequence induces a dichotomy for</u> A, <u>then, for every</u> $r \in \underset{\sim}{L}_{[0]}(E)$ $\langle r \in \underset{\sim}{K}_{[0]}(E)\rangle$ <u>with</u> $\varpi r \in \underset{\sim}{\ell}^{1}_{[1]}(\underset{\sim}{F}(E))$ <u>equation</u> (5.1) <u>has a bounded solution</u>. <u>If some regular covariant sequence induces an exponential dichotomy for</u> A, <u>and the space</u> $\underset{\sim}{d} \in b\mathfrak{t}$ <u>is given, then for every</u> $r \in \underset{\sim}{L}_{[0]}(E)$ $\langle r \in \underset{\sim}{K}_{[0]}(E)\rangle$ <u>with</u> $\varpi r \in \underset{\sim}{d}_{[1]}(\underset{\sim}{F}(E))$ <u>equation</u> (5.1) <u>has a solution</u> u <u>with</u> $\Pi u \in \underset{\sim}{d}({}_{p}\underset{\sim}{C}(E))$.

<u>Proof</u>. [1; Theorems 9.2, 10.3], and the implication (b) $\Rightarrow$ (a) in Theorem 7.1. In the proof of that implication, the assumption that L,M satisfy (6.7) was used only to establish that $\underset{\sim}{b}$ is stronger than $\underset{\sim}{d}$; since here $\underset{\sim}{b}$ is assumed to be $\underset{\sim}{d}$ itself, the assumption on L,M can be dispensed with.

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