ON $\mathcal{L}_2$ - SOLUTIONS OF LINEAR

ORDINARY DIFFERENTIAL EQUATIONS

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1. Consider the second order self-adjoint linear differential equation:

$$(1) \qquad (p(t)x')' - q(t)x = 0, \qquad t \geq 0,$$

where $p(t)$ is absolutely continuous and positive, and $q(t)$ is locally integrable. We are here concerned with the existence of a non $\mathcal{L}_2[0, \infty)$ solution to equation (1), i.e. whenever equation (1) is not of limit circle type. When $p(t) \equiv 1$, two well known criteria due respectively to Weyl [12] and Hartman [6] state that if (i) $q(t) \geq 0$ or (ii) $q \in \mathcal{L}_2[0, \infty)$, then equation (1) is not of limit circle type. In fact, their results remain valid for general p which is absolutely continuous and positive, see Dunford and Schwartz [3]. The purpose of this note is to extend these results to the more general n^{th} order equation

$$(2) \qquad p_n(p_{n-1} \cdots \{p_1[p_o x]'\}' \cdots)' - q(t)\, x = 0, \quad t \geq 0,$$

where $p_o, p_1, \dots, p_n$ are continuous and sufficiently smooth so that equation (2) admits a solution for every choice of initial values. Analogously, we say equation (2) is not of limit circle type if not all solutions belong to $\mathcal{L}_2[0, \infty)$.

Our proposed extensions are the following two theorems:

THEOREM 1. *Let* p_i *be positive* $i = 0,1,2,\ldots,n$ *on* $[0, \infty)$. *If* $q(t) \geq 0$ *for* $t \geq 0$ *and* $p_0^{-1} \notin \mathcal{L}_2[0, \infty)$, *then equation* (2) *is not of limit circle type*.

THEOREM 2. *Let* $P_{n-i} = P_i$, $i = 0, 1,2,\ldots,n$. *If* $q(t) \in \mathcal{L}_2[0, \infty)$ *and* p_0 *positive non-increasing for* $t \geq 0$, *then equation* (2) *is not of limit circle type*.

For convenience, we introduce the differential operators D_i, $i = 0, 1,2,\ldots,n$, defined inductively by $D_0 x = p_0 x$, $D_i x = p_i (D_{i-1} x)'$, $i = 1,2,\ldots,n$. In this notation, equation (2) takes the simple form $D_n x = qx$.

PROOF OF THEOREM 1. Consider the solution $x(t)$ of (2) defined by the initial conditions $D_i x(0) = 1$, $i = 0, 1,2,\ldots,n-1$. Since $D_0 x(0) = 1$ and $(D_0 x)'(0) \geq 0$, hence $D_0 x(t) \geq 1$ in some right neighborhood of $t = 0$. We first prove that $D_0 x(t) \geq 1$ for all $t > 0$. Assume the contrary, then there must exist $T > 0$ such that $D_0 x(t) \geq 1$ for all $t \in (0,T]$ and $D_0 x(t) < 1$ to $t > T$. Denote the compact interval $[0,T]$ by I and let $\eta = \inf q(t)$ and $\rho_k = \sup_{t \in I} p_k(t)$, $k = 1,2,\ldots,n$. From equation (2), we obtain

$(D_{n-1}x(t))' \geq \frac{n}{\rho_n} \geq 0$, for all $t \in I$. Thus $D_{n-1}x(t) \geq 1$ on I. Using the definition of D_i's , we obtain inductively $(D_i x(t))' \geq \frac{1}{\rho_{i+1}}$, $i = n-2,\ldots,2,1,0$ for all $t \in I$. In particular $(D_o x(T))' \geq 0$ contradicting the fact that $D_o x(t) < 1$ for $t > T$. Thus, $D_o x(t) \geq 1$ for all $t \geq 0$, and $x(t) \geq p_o^{-1}$. By hypothesis $p_o^{-1} \notin \mathcal{L}_2[0, \infty)$, so does $x(t) \notin \mathcal{L}_2[0, \infty)$, proving the theorem.

PROOF OF THEOREM 2. Denote $\{x_1, x_2,\ldots,x_n\}$ the set of linearly independent solutions of (2) satisfying the initial conditions $D_{i-1}x_j(0) = \delta_{ij}$. Consider the generalized Wronskian function $\Phi(t) = \det (D_{i-1}x_j)$, $1 \leq i$, $j \leq n$. An easy computation using (2) shows that $\Phi(t) \equiv 1$. Consider the Laplace expansion of $\det (D_{i-1}x_j)$ with respect to the last row:

$$(3) \qquad 1 \equiv \Phi(t) = \sum_{j=1}^{n} (-1)^j (D_{n-1}x_j(t))\, X_j(t),$$

where X_j is the cofactor of $D_{n-1}x_j$, $j = 1,2,\ldots,n$. In view of the "adjointness" conditions that $p_{n-i} = p_i$, $i = 0, 1, \ldots, n$; we can prove by differentiating inductively that the functions $p_o^{-1} X_j$, $j = 1, 2, \ldots, n$, are again solutions of (2). Integrating (2), we obtain

$$(4) \qquad D_{n-1}x_j(t) = D_{n-1}x_j(0) + \int_o^t \frac{q(s)}{p_o(s)} x_j(s)\, ds .$$

Applying Schwarz's inequality to (4), we obtain

$$(5) \qquad |D_{n-1}x_j(t)| \leq 1 + \left(\int_o^t \frac{q^2(s)}{p_o^2(s)} ds\right)^{1/2} \left(\int_o^t x_j^2(s)\, ds\right)^{1/2}$$

Suppose that equation (2) is of limit circle type, hence there exists a constant $M_o > 0$ such that

$$(6) \qquad \left(\int_o^\infty x_j^2(s)\, ds\right)^{1/2} \leq M_o , \qquad j = 1, 2, \ldots, n.$$

Using (5) and (6), we can estimate $\Phi(t)$, given by (3), as follows:

$$|\Phi(t)| \leq \sum_{j=1}^{n} p_o(t)| D_{n-1}x_j(t)|| p_o^{-1} x_j(t)|$$

(7)

$$\leq \left[p_o(t) + M_o(p_o^2(t) \int_o^t \frac{q^2(s)}{p_o^2(s)} ds)^{1/2}\right] \sum_{j=1}^{n} | p_o^{-1} x_j(t)|$$

Recall that $p_o(t)$ is non-increasing and $q \in \mathcal{L}_2[0, \infty)$, hence from (7), we obtain

$$(8) \qquad |\Phi(t)| \leq M_1 \sum_{j=1}^{n} | p_o^{-1} x_j(t)| ,$$

with some appropriate positive constant M_1. Since $p_0^{-1} x_j$, $j = 1, 2, \ldots, n$ are again solutions of (2) and thus belong to $\mathcal{L}_2[0, \infty)$ by hypothesis. Hence, (8) produces the desired contradiction.

2. We next show how Theorems 1 and 2 may be used to obtain other results of this type. Consider the following n^{th} order equation:

$$(9) \qquad Ly = y^{(n)} + \sum_{k=0}^{n-1} a_k(t) y^{(k)} = b(t) y, \qquad t \geq 0.$$

Following Polya [10], the differential operator L is said to have property (W), if the differential equation:

$$(10) \qquad L z = 0, \qquad t \geq 0,$$

has n solutions $z_1, z_2, \ldots, z_n$ such that

$$W_k(t) \neq 0, \quad t \geq 0, \quad \text{for } k = 1, 2, \ldots, n,$$

where $W_k(t) = W(t; z_1, \ldots, z_k)$ is the Wronskian of the set of solutions $\{z_1, \ldots, z_k\}$ of (10). Define the adjoint operator of L by

$$L^+ y = (-1)^n y^{(n)} + \sum_{k=0}^{n-1} (-1)^k (a_k(t) y)^{(k)}$$

In the following we assume that the operator L have property (W) with respect to solutions $\{z_1, z_2, \ldots, z_n\}$ of (10).

THEOREM 3. *If* $z_1 \notin \mathcal{L}_2[0, \infty)$ *and* $b(t) \geq 0$, *then equation* (9) *is not of limit circle type*.

THEOREM 4. *Let* L *be a self-adjoint differential operator*, i.e. $L = L^+$. *If* $b \in \mathcal{L}_2[0, \infty)$, *and* $z_1(t)$ *positive and non-decreasing for* $t \geq 0$; *then equation* (9) *is not of limit circle type*.

Suppose that the linear differential operator L has Polya's property (W), then by the Frobenius factorization theorem ([7], p. 67), equation (9) can be written in the following form:

$$(11) \qquad r_n(r_{n-1} \cdots \{r_2[r_1(r_o y)']'\}' \cdots)' = by,$$

where $0 < r_j = \dfrac{W_j^2}{W_{j-1} W_{j+1}}$ with $W_o = W_{-1} = W_{n+1} = 1$. Since $r_o > 0$, so is $z_1(t) > 0$ for all $t \geq 0$. If $z_1 \notin \mathcal{L}_2[0, \infty)$, then Theorem 3 follows immediately from Theorem 1. On the other hand, if L is self-adjoint then $W_j = W_{n-j}$, $j = 0, 1, 2, \ldots, n$, (for a simple proof, see [16].) Thus, $r_{n-1} = r_i$ for $i = 0, 1, 2, \ldots, n$.

The assumption that $z_1(t)$ is positive and non-decreasing then allows us to apply Theorem 2 to equation (11) and obtain Theorem 4.

We make a few remarks relating this work to others.

Remark 1. Theorems 1 and 2 are immediately applicable to the following self-adjoint equation

$$(12) \qquad (p(t)x^{(m)})^{(m)} - q(t)x = 0 ,$$

which is a natural extension of equation (1). In this case, we have: if $q(t) \geq 0$ or $q \in \mathcal{L}_2[0, \infty)$, then equation (12) is not of limit circle type. Asymptotic results concerning (12) may be found in Hunt [8].

Remark 2. For other proofs of Weyl's limit point criterion concerning equation (1), we refer the reader to Coddington and Levinson [2], Dunford and Schwartz [3], Everitt [4] and Wong [15]. For another proof of Hartman's theorem, see Putnam [11] and Naimark [9]. Extensions of these results concerning equation (1) to second order systems may be found in Hartman [7], Chapter XI, section 9.

Remark 3. Results concerning property (W) and disconjugate solutions of (10) were discussed in Polya [10] and a summary of these results may be found in Hartman [7], Chapter IV, section 8. It was shown that if $Lz = 0$ is disconjugate on $[0, \infty)$ then the operator L has property W on $(0, \infty)$ with respect to a system of solutions $\{z_1, \ldots, z_n\}$ where $z_i^{(j)}(0) = \delta_{n-i,j}$, $i,j = 1,2,\ldots,n$. Here by disconjugacy of L, we mean that no solution of $Lz = 0$ can have more than n zeros on $[0, \infty)$ counting multiplicities.

Remark 4. There are $\mathcal{L}_p$ analogues of Theorems 2 and 4. The same argument given in the proof of Theorem 2 provides the following extension: If $p_{n-i} = p_i$, $i = 0,1,2,\ldots,n$, and $q(t) \in \mathcal{L}_p[0, \infty)$, $p > 1$; then not all solutions of (2) belong to $\mathcal{L}_q[0, \infty)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 5. Finally, we note that Wintner [13] contains results on non-existence of any $\mathcal{L}_2$ solutions of equation (1). Extensions of Wintner's theorem to nonlinear and n^{th} order equations may be found in Burlak [1], Wong [14], and Hallam [5]. These results complement those discussed in this note. A similar result to Theorem 2 may also be found in Zettl [17] under slightly different hypothesis.

We close our discussion with a few examples by demonstrating how Theorems 3 and 4 may be used to obtain results for equations not of the form (12).

Example 1. Consider the fourth order equation

(13) $$Ly = y^{iv} - y'' = b(t)y ,$$

where the operator L is self-adjoint and $Lz = 0$ is clearly disconjugate on $[0, \infty)$. Using Remark 3, the operator L has property (W) with respect to a system of solutions $\{z_1, \ldots, z_n\}$ on $(0, \infty)$. In particular $z_1(t) = -t + \frac{1}{2} e^{t} - \frac{1}{2} e^{-t}$ which will satisfy the hypothesis of both Theorems 3 and 4 on $[\epsilon, \infty)$, for $\epsilon > 0$. Thus if either $b(t) \geq 0$ or $b \in \mathcal{L}_2[0, \infty)$ then not all solutions of (13) can belong to $\mathcal{L}_2[\epsilon, \infty)$, hence equation (13) is not of limit circle type.

Example 2. Consider the third order equation

(14) $$Ly = y''' + y' = b(t)y$$

We note that third order equations cannot be self adjoint so Theorem 4 is not applicable. Since $Lz = 0$ is disconjugate on $[0, \infty)$, Theorem 3 is applicable. The solution z_1 of $Lz = 0$ satisfying $z_1^{(j-1)} = \delta_{nj}$, $j = 1, \ldots, n$ is easily determined to be $z_1(t) = -1 + \frac{1}{2}(e^{x} + e^{-x})$ which is positive and nondecreasing on $[\epsilon, \infty)$. Thus it follows from an application of Theorem 3 that if

$b(t) \geq 0$ for $t \geq 0$ then equation (14) is not of limit circle type.

Example 3. Consider the fourth order equation

$$Ly = y^{(iv)} - 2y''' - y'' + 2y' = b(t)y \ , \tag{15}$$

which is not self adjoint. Following Example 2, we find the solution in question is $z_1(t) = \frac{1}{6} e^{-x} - \frac{1}{2} + \frac{1}{3} e^{2x}$ which is positive and nondecreasing on $[\epsilon, \infty)$. Thus $b(t) \geq 0$, $t \geq 0$ implies that equation (15) is not of limit circle type.

Example 4. Consider the second order equation

$$Ly = y'' + p(t)y = b(t)y \tag{16}$$

where $p(t) \geq 0$ and $Lz = 0$ has non-oscillatory solution, say $z_1(t)$. By definition, $z_1(t) > 0$ on $[T, \infty)$ for some $T > 0$. Since $p(t) \geq 0$, we must also have $z_1(t) \geq 0$. Thus, in this case conditions on z_1 both in Theorems 3 and 4 are satisfied on $[T, \infty)$. Consequently, if $b(t) \geq 0$ or $b \in \mathcal{L}_2[0, \infty)$, then equation (16) is not of limit circle type. Various conditions for nonoscillation of the second order equation $z'' + p(t)z = 0$ may be found in [7], Chapter XI.

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