SPECTRAL THEORY OF MONOTONE

HAMMERSTEIN OPERATORS

by

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1. Consider the linear integral equation,

$$(1) \qquad y(t) = \mu\int_\Omega K(t,s)p(s)y(s)ds,$$

where $K(s,t)$ is a real-valued symmetric positive definite kernel and $p(s)$ is a positive function. Let L denote the inverse of the integral operator $u \to \int_\Omega K(\cdot,s)u(s)ds$, and for a function y in the domain of L, $y \neq 0$, (all functions are assumed to be real valued) define the Rayleigh quotient $J(y)$ for (1) by,

$$J(y) = \int_\Omega y(t)[Ly](t)dt / \int_\Omega p(t)y^2(t)dt.$$

If $y_1 \neq 0$ and y_1 is in the domain of L and if $y_2 = \int_\Omega K(\cdot,s)p(s)y_1(s)ds$, then several applications of the Schwarz inequality show that,

$$J(y_2) \leq J(y_1),$$

with equality only if y_1 is an eigenfunction of (1). On the basis of this fact, when the integral operator in (1) is compact, one can develop the complete spectral theory of (1).

We wish to show here that the approach indicated above for the study of (1) has a simple and natural extension for the study of the non-linear integral equation,

$$y(t) = \mu \int_{\Omega} K(t,s) f(s,y(s))ds, \tag{2}$$

where $K(t,s)$ is as above and $f(t,y)$ is an odd function of y,

$$f(t,y) = -f(t,-y),$$

and satisfies,

$$yf(t,y) > 0, \qquad y \neq 0,$$

and

$$f(t,y_2) \geq f(t,y_1), \quad y_2 \geq y_1.$$

We cannot define a Rayleigh quotient for (2), but the problem of minimizing the Rayleigh quotient $J(y)$ for (1) can be generalized to either of the dual variational problems,

$$\int_{\Omega} y(t)[Ly](t)dt = \min., \qquad \int_{\Omega}\int_0^{y(t)} f(t,\eta)d\eta dt = \text{const.},$$

$$\int_{\Omega}\int_0^{y(t)} f(t,\eta)d\eta dt = \max., \qquad \int_{\Omega} y(t)[Ly](t)dt = \text{const.}$$

By studying the first of these two variational problems, we shall prove here a variant of a theorem of Sobolev, [17], concerning (2); Sobolev's work treats the second of the above variational problems. For a discussion of Sobolev's theorem and related results see [8].

The Lyusternik-Schnirelman theory upon which the theorem of Sobolev is based has undergone considerable development in recent years, see [14],[16], with fruitful applications to the theory of non-linear elliptic boundary value problems, see [1],[2]. The methods employed here differ substantially and in several respects from the standard methods of the Lyusternik-Schnirelman theory. We believe that for the problem at hand these methods are simpler and more natural, and therefore should be of interest. The central idea of using an iteration operator in the variational study of a non-linear problem was suggested by the work in the series of papers [11],[12], [13]. The notion of 'genus' which we use was introduced by Krasnosel'skii, and is treated in [8]; see also [4].

§2. Let X be a real infinite dimensional Banach space, X^* its dual space, and let the value of a linear functional $y \in X^*$ on an element $x \in X$ be denoted (y,x). Let $A : X \to X^*$ be a compact linear operator, take

$$Y = \overline{AX}, \tag{3}$$

and assume that A is symmetric,

$$(Ax_1, x_2) = (Ax_2, x_1). \qquad x_1, x_2 \in X, \tag{4}$$

and positive definite,

$$(Ax, x) > 0, \qquad x \in X \setminus \{0\}. \tag{5}$$

Note that if X is reflexive then (5) implies that $Y = X^*$.

Let $\Phi : Y \to X$ be a continuous nonlinear mapping and assume that Φ is the gradient (Fréchet derivative) of a real valued even functional $\gamma(y)$ on Y satisfying

$\gamma(0) = 0$. (The gradient of a functional on Y is actually an operator from Y to Y^*, however, X is canonically isometric to a subspace of Y^* so that Φ determines in an obvious way a mapping of Y into Y^*, it is this mapping which, properly speaking, is the gradient of γ; clearly this problem does not arise when X is reflexive.) The fact that γ is even implies that Φ is odd,

$$\Phi(y) = -\Phi(-y), \tag{6}$$

we assume also that Φ is positive definite,

$$(y, \Phi(y)) > 0, \qquad y \in Y \setminus \{0\}, \tag{7}$$

and monotone,

$$(y_2 - y_1, \Phi(y_2) - \Phi(y_1)) \geq 0, \quad y_1, y_2 \in Y. \tag{8}$$

The result which we shall prove is the following.

Theorem 1. _Under the above assumptions concerning_ A _and_ Φ, _the eigenvalue problem_,

$$y = \mu A \Phi(y), \tag{9}$$

has infinitely many eigenvectors satisfying,

$$\gamma(y) = c, \tag{10}$$

for every $c > 0$.

§3. We begin the proof of the theorem stated above by establishing several results concerning the linear operator A. If we let $\mathfrak{R}$ denote the range of A then, by (3), $\mathfrak{R}$ is dense in Y, and by (5), A has an inverse $L : \mathfrak{R} \to X$. From (4), (5) and the Schwarz inequality there follows

(9) $$(Ax_1, x_2)^2 \leq (Ax_1, x_1)(Ax_2, x_2).$$

and thus, since

$$\|Ax\|_{X^*} = \sup\{(Ax, x') : \|x'\|_X \leq 1\},$$

we have

$$\|Ax\|_{X^*}^2 \leq \|A\|(Ax, x),$$

and for $y \in \mathfrak{R}$, we have,

(10) $$\|y\|_{X^*}^2 \leq \|A\|(y, Ly).$$

If we complete $\mathfrak{R}$ with respect to the inner product $\langle y, y \rangle = (y, Ly)$ we obtain a Hilbert space $H \subseteq Y$ with

(11) $$\|i\eta\|_{X^*}^2 \leq \|A\| \; \|\eta\|_H^2,$$

where i denotes the inclusion mapping $H \subseteq X^*$. It is clear that for $\eta, \eta' \in H$ and $i\eta = Ax \in \mathfrak{R}$,

(12) $$\langle \eta', \eta \rangle = (i\eta', x).$$

Hence if we define $B : X \to H$ by $iBx = Ax$, then

$$\|Bx\|_H^2 = \langle Bx, Bx \rangle = (Ax, x) \leq \|A\| \|x\|_X^2,$$

so that B is a continuous linear mapping of X into H. Moreover, upon taking $\eta = Bx$ in (12) we get

$$\langle \eta', Bx \rangle = (i\eta', x),$$

from which it follows that

(13) $$B^* = i.$$

We next show that the compactness of A imples the compactness of both B and i.

Lemma 2. <u>The mapping</u> $B : X \to H$ <u>is compact</u>.

<u>Proof</u>. Let $\{x_n\}$ be a bounded sequence in X. There is no loss of generality in assuming that $\{Ax_n\}$ converges strongly in X^*, since A is compact, but then,

$$\|B(x_n - x_m)\|^2 = \langle B(x_n - x_m), B(x_n - x_m)\rangle,$$

$$= (A(x_n - x_m), (x_n - x_m)) \to 0,$$

as $n,m \to \infty$, and thus B is compact.

By standard results, the compactness of B implies the compactness of $i = B^*$, thus we have the following.

Lemma 3. <u>The imbedding</u> $i : H \subseteq X^*$ <u>is compact</u>.

§4. We turn our attention now to the non-linear operator Φ, and observe that, for $y_1, y_2 \in Y$ we have

$$\gamma(y_2) - \gamma(y_1) = \int_0^1 (y_2 - y_1, \Phi(y_1 + t(y_2 - y_i)))dt,$$

(see [18]), but by (8),

$$(y_2 - y_1, \Phi(y_1 + t(y_2 - y_1))) \geq (y_2 - y_1, \Phi(y_1)),$$

so that

(14) $$\gamma(y_2) - \gamma(y_1) \geq (y_2 - y_1, \Phi(y_1)).$$

Next we define $\varphi : H \to R, \Psi : H \to H$, by

(15) $$\varphi = \gamma \cdot i, \qquad \Psi = B \cdot \Phi \cdot i;$$

then the continuity of γ and Φ and the compactness of i imply the following result.

Lemma 4. _The mapping_ φ _is continuous and_ Ψ _is completely continuous._

From (7), (15), (14), and (13) we get

$$(16) \qquad \langle\eta,\Psi(\eta)\rangle > 0, \qquad \eta\in H\backslash\{0\},$$

and

$$(17) \qquad \langle\eta' - \eta,\Psi(\eta)\rangle \leq \varphi(\eta') - \varphi(\eta), \quad \eta',\eta\in H.$$

Lemma 5. _Let_ $c > 0$, _then for_ $\eta\in H\backslash\{0\}$, _there exists a unique_ $\alpha > 0$ _such that_ $\varphi(\alpha\eta) = c$. _Moreover,_ α _is a continuous function of_ η _on_ $H\backslash\{0\}$.

Proof. By hypothesis, $\gamma(0) = 0$, thus, from (15) it follows that $\varphi(0) = 0$. By taking $\eta' = \alpha\eta$ in (17) we get

$$(18) \qquad \varphi(\alpha\eta) \geq \varphi(\eta) + (\alpha-1) \langle \eta,\Psi(\eta)\rangle,$$

hence (16) implies that $\varphi(\alpha\eta) \to \infty$ as $\alpha \to \infty$, for $\eta \neq 0$. The existence of α then follows from the continuity of φ; uniqueness follows from (16) and (18). The continuity follows from the continuity of φ and the uniqueness of α.

Let $c > 0$ be fixed, put

$$(19) \qquad \Sigma = \Sigma_c = \{\eta\in H : \varphi(\eta) = c\},$$

and define $\sigma : H\backslash\{0\} \to \Sigma_c$ by

$$(20) \qquad \sigma(\eta) = \alpha\Psi(\eta),$$

where $\alpha > 0$ is chosen so that $\sigma(\eta)\in\Sigma$.

Lemma 6. _The mapping_ $\sigma : H\backslash\{0\} \to \Sigma$ is _odd and completely continuous. If_ $\eta\in\Sigma$, _then_

(21) $$\|\sigma(\eta)\|_H \leq \|\eta\|_H .$$

<u>with equality only if</u> η <u>is an eigenvector of</u>

(22) $$\eta = \mu\Psi(\eta).$$

<u>Proof</u>. The oddness and continuity of σ follow respectively from (6), and the definition of Ψ, and Lemmas 4 and 5. If $\eta' = \sigma(\eta)$, then since $\varphi(\eta') = \varphi(\eta)$, (17) implies

$$<\eta' - \eta, \Psi(\eta)> \leq 0,$$

but by (20), this is equivalent to

$$<\eta' - \eta, \eta'> \leq 0,$$

or

$$<\eta', \eta'> \leq <\eta, \eta'> \leq <\eta, \eta>^{\frac{1}{2}}<\eta', \eta'>^{\frac{1}{2}},$$

which implies (21). From the way in which the Schwarz inequality was used, it is clear that equality can hold in (21) only if η' and η are proportional, i.e. only if η is an eigenvector of (20).

§5. Let S denote the class of closed subsets of $H\backslash\{0\}$ which are symmetric through the origin. For a non-empty set $F \in S$, the <u>genus</u> of F, $\rho(F)$, is the supremum of the set of non-negative integers n such that every odd continuous map of F into R^{n-1} has a zero in F; here we understand $R^0 = \{0\}$. The genus of the empty set is zero.

Below, the letter 'F', with or without subscript, will always denote a set in the class S. The genus has the following properties.

1. If there exists an odd continuous map $h : F_1 \to F_2$, then $\rho(F_1) \leq \rho(F_2)$, in particular, if $F_1 \subseteq F_2$, then $\rho(F_1) \leq \rho(F_2)$.

2. $\rho(F_1 \cup F_2) \leq \rho(F_1) + \rho(F_2)$.

3. If F is compact then $\rho(F) < \infty$ and F has a neighborhood U such that $\overline{U} \in S$ and $\rho(\overline{U}) = \rho(F)$.

4. If $\{F_n\}$ is a decreasing sequence of compact sets and if $F = \bigcap_{n=1}^{\infty} F_n$ then $F \in S$ and

$$\rho(F) = \lim_{n\to\infty} \rho(F_n).$$

5. If there exists an odd homeomorphism of F onto the n-sphere then $\rho(F) = n + 1$.

For a proof of the above assertions see [4].

§6. In this section we complete the proof of Theorem 1. Let the number $c > 0$ be fixed, and let Σ be defined by (19). We shall call a set F <u>admissible</u> if F is compact, $F \in S$, and $F \subseteq \Sigma$. The class of admissible sets will be denoted by $\mathfrak{F}$, and we shall take, for $n = 1, 2, \ldots$,

$$\mathfrak{F}_n = \{F \in \mathfrak{F} : \rho(F) \geq n\}.$$

Lemma 7. *For any positive integer* n, *the class* $\mathfrak{F}_n$ *is non-empty*.

Proof. Let M be a subspace of H of dimension n, then it follows from Lemma 5 that the mapping $\eta \to \|\eta\|_H^{-1}\eta$ on $\Sigma \cap M$ is a homeomorphism onto the unit sphere in M. It thus follows from property 5 of the genus that $\rho(\Sigma \cap M) = n$, and thus $\Sigma \cap M \in \mathfrak{F}_n$.

We now define the numbers $\lambda_n = \lambda_n(c)$ by,

$$\lambda_n = \frac{1}{2c} \inf_{F\in\mathfrak{F}} \max_{\eta\in F} \|\eta\|_H^2 , \tag{23}$$

$$= \frac{1}{8c} \inf_{F\in\mathfrak{F}} [\text{diam } F]^2,$$

for $n \geq 1$, and for convenience we define $\lambda_0 = 0$. Since Σ is closed and does not contain zero, and because of the definition (23), it is clear that the sequence $\lambda_1, \lambda_2, \ldots,$ is a non-decreasing sequence of positive numbers. We say that λ_n has multiplicity m if,

$$\lambda_{k-1} < \lambda_k = \lambda_{k+1} = \ldots = \lambda_{k+m-1} < \lambda_{k+m},$$

for some $k \geq 1$, where $k \leq n \leq k+m-1$. We also make the definitions, for $\lambda \geq 0$,

$$\Sigma(\lambda) = \{\eta\in\Sigma : \|\eta\|_H^2 \leq 2c\lambda\},$$

$$\Sigma^1(\lambda) = \overline{\sigma(\Sigma(\lambda))},$$

and note that because $\Sigma(\lambda)$ is closed and symmetric, and because of Lemma 6,

$$\Sigma^1(\lambda) \subset \Sigma(\lambda), \tag{24}$$

and $\Sigma^1(\lambda)$ is compact and symmetric, i.e.,

(25) $\Sigma^1(\lambda) \in \mathfrak{F}$, $\quad (\text{diam } \Sigma^1(\lambda))^2 \leq 8\lambda c$.

On the other hand, by property 1 of the genus, since σ is odd, and because of (24),

$$\rho(\Sigma^1(\lambda)) = \rho(\Sigma(\lambda)).$$

Since $\Sigma^1(\lambda)$ is compact, it follows from property 3 that $\rho(\Sigma(\lambda)) < \infty$ for any $\lambda > 0$.

If we introduce the 'spectral function'

$$\tau(\lambda) = \rho(\Sigma^1(\lambda)),$$

then $\tau(\lambda)$ is a monotone integer valued function of λ for $\lambda \geq 0$,

$$\tau(\lambda) = 0, \qquad 0 \leq \lambda < \lambda_1,$$

and since,

$$\Sigma^1(\lambda) = \bigcap_{\mu>\lambda} \Sigma^1(\mu),$$

it follows from property 4 that $\tau(\lambda)$ is right continuous. Finally from (23) and (24) and the right continuity of τ it follows that,

(26) $\tau(\lambda) = n, \qquad \lambda_n \leq \lambda < \lambda_{n+1}$.

Thus the discontinuities of $\tau(\lambda)$ occur only at the numbers λ_n given by (23) and

(27) $\tau(\lambda_n) - \tau(\lambda_n - 0) = \text{multiplicity of } \lambda_n$.

It follows from (26) that no finite λ-interval can contain more than finitely many of the numbers λ_n, thus

(28) $$\lim_{n\to\infty} \lambda_n = \infty,$$

Now let E_n denote the set of solutions η in $\Sigma = \Sigma_c$ of (22) such that $\|\eta\|_H^2 = 2c\lambda_n$, (clearly Σ_n is admissible and $E_n \subseteq \Sigma^1(\lambda_n)$), and choose, by property 3, a neighborhood U of E_n such that $\overline{U} \in S$ and $\rho(\overline{U}) = \rho(E_n)$. It then follows from Lemma 6 that

$$[\mathrm{diam}\ (\sigma(\Sigma^1(\lambda_n)\backslash U))]^2 < 8c\lambda_n,$$

so that from the definition of τ and (27),

$$\rho(\sigma(\Sigma^1(\lambda_n)\backslash U)) \leq \tau(\lambda_n - 0)$$

$$\leq \tau(\lambda_n) - \text{mult. of } \lambda_n.$$

Using this last inequality, property 1 of the genus, and the definition of τ, we obtain,

$$\rho(\Sigma^1(\lambda_n)\backslash U) \leq \rho(\Sigma^1(\lambda_n)) - \text{mult. of } \lambda_n.$$

It then follows from property 2 of the genus that

$$\rho(E_n) = \rho(\overline{U}) \geq \text{mult. of } \lambda_n,$$

in particular, E_n is not empty. We have thus proved the following.

(*) <u>The eigenvalue problem</u> (22) <u>has infinitely many eigenvectors</u> η <u>satisfying</u>

(29) $$\varphi(\eta) = c.$$

The set $E_n(c)$ of eigenvectors η of (22) satisfying (29) and

$$\|\eta\|_H^2 = 2c\lambda n,$$

where λ_n is given by (23), is a set of genus $\geq$ multiplicity of λ_n.

It is clear from (15) and the definition of B that if η is an eigenvector of (22) satisfying (29) then $y = i\eta$ is an eigenvector of (9) and satisfies (10). Thus (*) implies Theorem 1. Moreover, the following is valid.

(**) The set of eigenvectors y of (9) which satisfy (10) and

$$(y, Ly) = 2c\lambda_n(c), \tag{30}$$

is a set of genus $\geq$ multiplicity of λ_n. The numbers $\lambda_n(c)$ can be determined as follows

$$\lambda_n(c) = \frac{1}{2c} \inf_{G \in \mathcal{G}_n} \sup_{y \in G} (y, Ly), \tag{31}$$

where $\mathcal{G}_n$ is the class of symmetric subsets G of $\mathcal{R}\setminus\{0\}$ which are closed in $\mathbf{Y}$, have genus $\geq n$ and satisfy

$$G \subseteq \{y \in Y : \gamma(y) = C\}.$$

Here the genus is to be understood to be relative to the $\mathbf{Y}$-topology.

Proof. First we observe that the H and Y topologies coincide on compact subsets of H. Thus the genus of E_n relative to the H topology and the genus of $i(E_n)$ relative to the Y topology are the same. Since the set of eigenvectors of (9) satisfying (10) and (30) is just $i(E_n)$, the first assertion above is proved. To prove the second assertion, let $G \in \mathcal{G}_n$ and let $\lambda = \frac{1}{2c}\sup_{y\in G}(y,Ly) < \infty$, then $G = i(F)$, where F is a closed symmetric subset of H and $F \subseteq \Sigma(\lambda)$. From (15) and (20), and the continuity of α in (20) it follows that $\sigma(F)$ is the image, under an odd completely continuous transformation, of G. Thus $\sigma(F)$ is compact and has genus $\geq n$. However, we clearly have $[\operatorname{diam}(\sigma(F))]^2 \leq 8c\lambda$ and thus it follows from (23) that

$$\lambda_n \leq \frac{1}{2c} \inf_{G\in\mathcal{G}_n} \sup_{y\in G} (y,Ly).$$

Making use of the observation at the beginning of the proof we conclude that $i(\sigma(\Sigma^{1}(\lambda_n))) \in \mathcal{G}_n$, and this, together with the above inequality, implies (31).

Remark. The existence of an infinity of eigenfunctions y of the problem (9) satisfying

$$(y,Ly) = \hat{c},$$

for an arbitrary $c > 0$, follows by applying Theorem 4.3 of [8] to the operator Ψ, provided that Ψ is uniformly differentiable on bounded sets; it is not required that Φ satisfy (8). The theorem of Sobolev, [17], quoted earlier, follows from this result. For a discussion of the multiplicity of solutions in this case, when the associated critical values are repeated, see [3].

§7. It is of interest to show that the above results do include the complete spectral theory of (9) in the linear case. Thus suppose that Φ is linear and observe that in this case,

$$\gamma(y) = \int_0^1 (y, \Phi(ty))dt$$

$$= \frac{1}{2} (y, \Phi(y)).$$

Thus, for $y, h \in Y$,

$$\gamma(y+h) = \frac{1}{2}(y, \Phi(y)) + \frac{1}{2}(h, \Phi(y)) + \frac{1}{2}(y, \Phi(h)) + \frac{1}{2}(h, \Phi(h)),$$

while on the other hand the definition of the gradient gives

$$\gamma(y + h) = \gamma(y) + (h, \Phi(y)) + o(\|h\|),$$

$$= \frac{1}{2}(y, \Phi(y)) + (h, \Phi(y)) + o(\|h\|),$$

as $\|h\| \to 0$. Comparison of these two formulas shows that Φ is symmetric,

$$(y_1, \Phi(y_2)) = (y_2, \Phi(y_1)), \quad y_1, y_2 \in Y.$$

Using this together with (7), (15) and Lemma 2 we conclude that Ψ is a compact self-adjoint positive definite operator and $\varphi(\eta) = \frac{1}{2} \langle \eta, \Psi\eta \rangle$; see the discussion of the operator χ_1 in section 8 below.

Let $0 < \mu_1 \leq \mu_2 \leq \cdots$ denote the characteristic values of Ψ and let $M_1 \subseteq M_2 \subseteq \cdots$ be the corresponding sequence of invariant subspaces for Ψ, i.e. M_n is spanned by those eigenvectors of Ψ corresponding to the first n characteristic values. Suppose that $\mu_n < \mu_{n+1}$ and for some $c > 0$ let $\Sigma_c(\lambda)$ be defined as in section 6.

If $\mu_n \leq \lambda < \mu_{n+1}$ then for $\eta \in \Sigma_c(\lambda)$,

$$\|\eta\|_H^2 \leq 2\lambda c = \lambda < \eta, \Psi(\eta) >$$

$$< \mu_{n+1} <\eta, \Psi(\eta)>.$$

Thus if P_n denotes the orthogonal projection of H onto M_n then $P_n\eta \neq 0$ for $\eta \in \Sigma_c(\lambda)$, since

$$\|\eta\|_H^2 \geq \mu_{n+1}<\eta, \Psi(\eta)>$$

for η in the kernel of P_n. It follows that the map $\eta \rightarrow (2c)^{\frac{1}{2}}<P_n\eta, \Psi(\eta)>^{-\frac{1}{2}} P_n\eta$ maps $\Sigma_c(\lambda)$ onto $\Sigma_c \cap M_n$. Since this mapping is odd and continuous on $\Sigma_c(\lambda)$ it follows that $n = \rho(\Sigma_c \cap M_n) \geq \rho(\Sigma_c(\lambda))$. On the other hand, for $\eta \in M_n$,

$$\|\eta\|_H^2 \leq \mu_n<\eta, \Psi(\eta)>$$

and thus $\Sigma_c \cap M \subseteq \Sigma_c(\mu_n) \subseteq \Sigma_c(\lambda)$,

from which we conclude that

$$\rho(\Sigma_c(\lambda)) = n, \qquad \mu_n \leq \lambda < \mu_{n+1} .$$

Combined with the results of section 6, this yields the following.

(***) <u>For the linear problem and for any</u> $c > 0$, <u>the number</u> $\lambda_n(c)$ <u>is just the</u> n^{th} <u>characteristic value, in increasing order, of the operator</u> Ψ.

We require also the following result concerning the linear case.

Lemma 8. <u>Let</u> F <u>be a compact symmetric subset of</u>

$H\setminus\{0\}$, *with* $\rho(F) \geq n$, *then*

$$\max_{\eta\in F} \frac{\|\eta\|_H^2}{\langle\eta,\Psi\eta\rangle} \geq \mu_n ,$$

where μ_n *is the* n^{th} *characteristic value, in increasing order, of the linear operator* Ψ.

Proof. Let F' be the image of F under $\eta \to (2c)^{\frac{1}{2}}\langle\eta,\Psi\eta\rangle^{-\frac{1}{2}}\eta$, where $c > 0$. Then $F' \subseteq \Sigma_c$ and $\rho(F') \geq n$, thus by $(*)$,

$$\max_{\eta\in F} \frac{\|\eta\|_H^2}{\langle\eta,\Psi\eta\rangle} = \frac{1}{2c} \max_{\eta\in F'} \|\eta\|_H^2$$

$$\geq \mu_n .$$

§8. In this section we shall derive a 'principle of linearization' for the problem (9); cf. [Ch. VI, §2,8]. What we shall prove is the following.

(†) *Let* Φ *have a Fréchet derivative* χ *at* $y = 0$, *and suppose that the continuous linear operator* $\chi : Y \to X$ *is symmetric*,

(32) $$(y_1,\chi y_2) = (y_2,\chi y_1), \qquad y_1,y_2\in Y,$$

and positive definite,

(33) $$(y,\chi y) > 0, \qquad y\in Y\setminus\{0\}.$$

Let $\kappa_1,\kappa_2,\kappa_3,\ldots,$ *be the characteristic values, arranged in increasing order, of the linear operator* $A\chi$, *then the numbers* $\lambda_n(c)$, *given by* (31) *satisfy*

(34) $$\lim_{c \to 0} \lambda_n(c) = \kappa_n .$$

Moreover, given $\epsilon > 0$, there exists $c_1(\epsilon) > 0$ such that for $0 < c < c_1$, the characteristic numbers μ corresponding to those eigenvectors η of (9) which satisfy (10) and (30) all lie in the interval $(\kappa_n - \epsilon, \kappa_n + \epsilon)$. Finally, if $\tilde{N}_n$ is the eigenspace of $A\chi$ corresponding to the characteristic number κ_n, and if $\tilde{E}_n(c)$ is the set of eigenvectors of (9) satisfying (10) and (30), then

$$\lim_{c \to 0} \max_{y \in \tilde{E}_n(c)} \frac{\operatorname{dist}(y, \tilde{N}_n)}{\|y\|} = 0.$$

Remark. The above assertion implies that, in the terminology of [8], each characteristic value of $A\chi$ is a bifurcation point for the problem (9), compare Theorem 2.2, [8,p.332].

Proof of (†). From the definition of the Fréchet derivative

$$\Phi(y) = \chi y + \omega(y), \qquad y \in Y,$$

where

$$\|\omega(y)\| = o(\|y\|), \qquad \text{as } y \to 0.$$

Thus, for $y \in Y$,

$$\gamma(y) = \int_0^1 (y, \Phi(ty))\,dt,$$

$$= \frac{1}{2}(y, \chi y) + o(\|y\|^2), \qquad \text{as } y \to 0,$$

and consequently, from (15),(12) and (13), for $\eta \in H$,

(35) $$\varphi(\eta) = \frac{1}{2}\langle\eta, \chi_1\eta\rangle + o(\|i\eta\|^2), \quad \text{as} \quad \eta \to 0,$$

where $\chi_1 = B\chi i$.

If we make the natural identification of X with a subspace of Y^*, then (32) implies that $\chi^* \supseteq \chi$. Since the range of i is contained in the domain of χ it follows, using (13), that the operator $\chi_1 = B\chi i$ is self-adjoint. Since

$$\langle\eta, \chi_1\eta\rangle = (i\eta, \chi i\eta),$$

it follows from (33) that χ_1 is positive definite, and from Lemma 2 it follows that χ_1 is compact. Clearly the characteristic values $\{\kappa_n\}$ of $A\chi$ are also characteristic values of χ_1, thus, χ_1 has an invariant subspace M of dimension n such that

(36) $$\|\eta\|_H^2 \leq \kappa_n \langle \eta, \chi_1\eta\rangle, \qquad \eta\in M.$$

Let $\epsilon > 0$ be given and, on the strength of (35), choose $c_1 > 0$ such that

(37) $$|\varphi(\eta) - \frac{1}{2}\langle\eta, \chi_1\eta\rangle| \leq \frac{1}{2}\kappa_n^{-1}\epsilon\|\eta\|_H^2$$

for

(38) $$\|\eta\|_H^2 \leq 2(1-\epsilon)^{-1}\kappa_n c_1 .$$

From (36) and (37) we then have

$$\varphi(\eta) \geq \frac{1}{2}\kappa_n^{-1}(1-\epsilon)\|\eta\|_H^2, \qquad \eta\in M, \|\eta\|_H^2 \leq 2(1-\epsilon)^{-1}\kappa_n c',$$

from which, it follows, by the use of (18), that for $0 < c < c'$,

$$\frac{1}{8}[\text{diam}](M \cap \Sigma(c))]^2 \leq \kappa_n(1-\epsilon)^{-1}c.$$

Hence, from (23) and since $\rho(M \cap \Sigma(c)) = n$,

$$\lambda_n(c) \leq \kappa_n(1-\epsilon)^{-1}.$$

Letting ϵ tend to zero, we obtain finally,

(39) $$\limsup_{c \to 0} \lambda_n(c) \leq \kappa_n .$$

Suppose now that

(40) $$\liminf_{c \to 0} \lambda_n(c) < \kappa < \kappa_n ,$$

and then choose $\epsilon > 0$ so that

(41) $$(1-\epsilon)^{-1}\kappa < \kappa_n .$$

With this choice of ϵ, let $c' > 0$ be chosen as above so that (37) holds when η satisfies (38). In view of the above supposition, there exists a number c and a set F such that $0 < c < c'$, F is admissible for the given choice of c, $\rho(F) \geq n$ and

(42) $$\frac{1}{2c} \max_{\eta \in F} \|\eta\|_H^2 < \kappa.$$

From (37), (42) and the fact that $F \subseteq \Sigma_c$, there follows, for $\eta \in F$,

$$\frac{1}{2}\langle\eta, X_1\eta\rangle \geq c(1-\epsilon).$$

Hence, by (42) and (41), (since $\kappa < \kappa_n$),

$$\max_{\eta\in F} \frac{\|\eta\|_H^2}{\langle\eta, \chi_1\eta\rangle} \leq \frac{1}{2c(1-\epsilon)} \max_{\eta\in F} \|\eta\|_H^2 ,$$

$$\leq \frac{1}{1-\epsilon} \kappa,$$

$$< \kappa_n .$$

In view of Lemma 8 this is impossible, we conclude therefore that (40) cannot hold, and thus from (39) we have (34).

Since for an eigenvector η of (22) the corresponding characteristic value μ is given by

$$\mu = \frac{\|\eta\|_H^2}{\langle\eta, \Psi(\eta)\rangle},$$

we have, for $\eta\in\Sigma_c$

$$\mu = \frac{\varphi(\eta)}{\frac{1}{2}\langle\eta, \Psi(\eta)\rangle} \cdot \frac{1}{2c}\|\eta\|_H^2 ,$$

thus if $\eta\in E_n(c)$,

(43) $$\mu = \frac{\varphi(\eta)\lambda_n(c)}{\frac{1}{2}\langle\eta, \Psi(\eta)\rangle} .$$

By definition

$$[\text{diam } E_n(c)]^2 = 8c\lambda_n(c),$$

so that from (35) and (34), for $\eta\in E_n(c)$, the characteristic value μ which is given by (43), satisfies

$$\mu = \mu(\eta) = \kappa_n(1 + o(c)), \quad \text{as} \quad c \to 0,$$

from which follows the second assertion of (†); notice that $\tilde{E}_n(c) = i(E_n(c))$.

The final assertion of (†) follows from the second assertion of (†) and a result from [8,pp.194-195].

§9. We conclude with several examples to which the above results are applicable. First, for the verification of condition (5), the following result will be useful.

Lemma 9. _The compact symmetric operator_ $A : X \to X^*$ _satisfies condition_ (5) _if and only if_ $Y = \overline{AX}$ _is total for_ X _and_

$$(44) \qquad (Ax,x) \geq 0, \qquad x \in X.$$

Proof. Condition (5) clearly implies (44), and if $x \in X$ and $(y,x) = 0$ for every $y \in Y$, then in particular $(Ax,x) = 0$, and hence $x = 0$, so that Y is indeed total for X. Conversely, suppose that (44) holds and that Y is total for X. Then if $x \in X$, $x \neq 0$, there exists $y \in Y$ such that $(y,x) \neq 0$. Since AX is dense in Y it follows that there exists $x' \in X$ such that $(Ax',x) \neq 0$. But from the symmetry of A, (44) and the Schwarz inequality there follows,

$$0 < (Ax',x)^2 \leq (Ax',x')(Ax,x),$$

and thus we conclude that (5) holds.

Now let Ω be a bounded region in Euclidean n-space, let $K(t,s)$ be a symmetric kernel defined for $(t,s) \in \Omega \times \Omega$, and let $f(t,y)$, defined for $(t,y) \in \Omega \times R$ satisfy the Carathéodory conditions, [8]. Assume moreover that $f(t,y)$ satisfies the conditions set down in section 1, namely

$$(45) \qquad f(t,y) = -f(t,-y), \qquad t\in\Omega, y\in R,$$

$$(46) \qquad yf(t,y) > 0, \qquad t\in\Omega, y\in R, y \neq 0,$$

and

$$(47) \qquad f(t,y_2) \geq f(t,y_1) \qquad t\in\Omega, y_1, y_2\in R, y_2 \geq y_1.$$

With X and Y yet to be specified, we take A to be the integral operator

$$Au = \int_\Omega K(\cdot,s)u(s)ds,$$

and Φ to be the Nemytsky operator

$$\Phi(y) = f(\cdot,y(\cdot)),$$

finally we take

$$(48) \qquad \gamma(y) = \int_\Omega \int_0^{y(t)} f(t,u)du\, dt.$$

We consider first the case where $K(t,s)$ is continuous on $\overline{\Omega} \times \overline{\Omega}$ and $f(t,y)$ is continuous on $\overline{\Omega} \times R$. In this case we take $X = L^1(\overline{\Omega})$, and Y is then determined and will be a subspace of $C(\overline{\Omega}) \subseteq L^\infty(\overline{\Omega}) = (L^1(\overline{\Omega}))^*$. The complete continuity of $A : L^1(\Omega) \to C(\overline{\Omega})$ and the continuity of $\Phi : C(\overline{\Omega}) \to L^1(\Omega)$ are easily verified; weaker conditions on $f(t,y)$ suffice for the latter continuity. The symmetry of A follows from the symmetry of $K(t,s)$. The operator Φ, which can be regarded as an operator from $C(\Omega)$ into $(C(\overline{\Omega}))^*$, since the latter contains a subspace naturally isomorphic to $L^1(\overline{\Omega})$, is the gradient of γ, given by

(48). The properties (6),(7) and (8) for Φ follow from (45),(46) and (47). If $K(t,s)$ is non-negative definite in the ordinary sense, i.e. if $A|L^2(\Omega)$ is non-negative definite, then A satisfies (44), thus the applicability of Theorem 1 hinges on the totality of Y for $L^1(\Omega)$. This depends on more special properties of the kernel $K(t,s)$. However, if Ω is a bounded interval and if $K(t,s)$ is the Green's function for a regular self-adjoint two-point boundary value problem on Ω, then Y will contain all continuous functions which vanish identically near the endpoints of Ω, and consequently will be total for $L^1(\overline{\Omega})$. Thus Theorem 1 implies the following.

Theorem 2. Let

$$\ell = \sum_{k=1}^{m} \frac{d^k}{dt^k} p_{m-k}(t)\frac{d^k}{dt^k}, \quad p_j \in C^{m-j}([a,b]), j=1,\dots,m, \tag{49}$$

be a formally self-adjoint regular differential operator of order 2m on [a,b], let

$$M_k(y) = \sum_{j=0}^{2m-1} (\alpha_{kj}\, y^{(j)}(a) + \beta_{kj}\, y^{(j)}(b)) = 0, \quad k=1,\dots,2m, \tag{50}$$

be self-adjoint boundary conditions for ℓ, and suppose that if $y \in C^{2m}([a,b])$, $y \not\equiv 0$ in $[a,b]$, and y satisfies (50) then

$$\int_a^b y(t)[\ell y](t)dt > 0.$$

Let $f(t,y)$ be continuous on $[a,b] \times R$ and satisfy (45), (46), and (47).

Then if $c > 0$, the problem

(51) $\ell(y) = \mu f(t,y)$, $M_k(y) = 0$, $k = 1,\ldots,2m$,

<u>has infinitely many eigenfunctions</u> y <u>satisfying</u>

$$\gamma(y) = \int_a^b \int_0^{y(t)} f(t,u)\,du\,dt = c.$$

The principle of linearization derived in section 8 applies to the problem (51), provided there exists a positive function $q(t)$ on $[a,b]$ such that

$$\lim_{\substack{y\to 0\\ y\neq 0}} y^{-1} f(t,y) = q(t), \qquad \text{uniformly with respect to } t,$$

Substantially weaker conditions actually suffice for the Fréchet differentiability at zero of the Nemytsky operator from C to L^1.

We next consider the case where the kernel $K(t,s)$ is singular. We are primarily interested in the particular case where $K(t,s)$ is the Green's function associated with an elliptic boundary value problem, thus, of the various conditions on K implying complete continuity of A we shall consider only the one which is satisfied by such a Green's function, namely,

$$\sup_{t\in\Omega} \int_\Omega |K(t,s)|^a ds < \infty, \qquad a > 1 \tag{52}$$

for some $a > 1$. Concerning $f(t,y)$ we then assume

$$|f(t,y)| \leq c|y|^b + d, \tag{53}$$

for some b such that

$$1 < b < 2a - 1, \tag{54}$$

and we take $\underline{X} = L^q(\Omega)$, where,

(55) $$q > 2a/(2a - 1).$$

If $\frac{1}{p} + \frac{1}{q} = 1$, we have by [7, Theorem 95.6, p. 658], that $A : L^q(\Omega) \to L^p(\Omega)$ is completely continuous, and by [18, Theorem 19.1, p. 154] that $\Phi : L^p(\Omega) \to L^q(\Omega)$ is continuous, Also, Φ is the gradient of the functional γ on $L^p(\Omega)$; see [18, §6.3, p.63]. Assume that $K(t,s)$ is positive definite in the ordinary sense, then since, (because of the reflexivity of $L^p(\Omega)$), no proper subspace of $L^p(\Omega)$ is total for $L^q(\Omega)$, the applicability of our main result hinges in this case on the density of the range of A in $L^p(\Omega)$. We note however that in any case, in the presence of conditions (52), (53), (54) and (55) and when $\frac{1}{p} + \frac{1}{q} = 1$, any L^p-eigenfunction of (2) actually belongs to $L^\infty(\Omega)$; see the proof of Theorem 3, [5].

In order for the Nemytsky operator from $L^p(\Omega)$ to $L^q(\Omega)$ $(p > 2, \frac{1}{p} + \frac{1}{q} = 1)$ to be Frechet differentiable at 0 it suffices that

$$f(t,y) = y(P(t) + q(t,y))$$

where $P \in L^r(\Omega)$, $r = p/(p-2)$, $q(t,y)$ satisfies the Carathéodory conditions, $q(t,0) \equiv 0$, and

(56) $$|q(t,y)| \leq m_1|y|^\delta + m_2,$$

where $m_1, m_2 > 0$, $\delta = (p-2)^{-1}$. If $P(t)$ is positive almost everywhere on Ω, then the principle of linearization is applicable in this case. In the presence of (52),(53),(54), the small L^p-solutions of (2), ($p > 2a$), can be shown to be shall in the L^∞ norm also, and thus the growth condition (56) on $g(t,y)$ can be dropped when these conditions hold.

We will not state a general theorem concerning the equation (2) with a singular kernel but rather we state the following result, which is a principal application of such a theorem.

Theorem 3. Let Ω be a bounded region of class C^{2m} in R^n, and let

$$\tau = \sum_{|\alpha|,|\beta|\leq m} (-1)^{|\alpha|} D^\alpha a_{\alpha\beta}(t) D^\beta,$$

where $D^\alpha = \partial^{|\alpha|}/\partial t_1^{a_1} \ldots \partial t_n^{\alpha_n}, |\alpha| = \alpha_1 + \ldots + \alpha_n$,

and for some $\mu : 0 < \mu < 1$,

$$a_{\alpha\beta} \in C^{\max(|\alpha|,|\beta|),\mu}, \qquad \text{all } \alpha,\beta.$$

Moreover assume that τ is formally self-adjoint and that there exists $k_o > 0$, such that

$$\sum_{|\alpha|,|\beta|\leq m} \int_\Omega a_{\alpha\beta}(t) D^\alpha y(t) D^\beta y(t)\,dt \geq k_o \sum_{|\alpha|\leq m} \int_\Omega |D^\alpha y|\,dt,$$

for all $y \in C_o^{2m}(\Omega)$.

Let $f(t,y)$ be uniformly Hölder continuous on $\Omega \times R$ and satisfy (45),(46),(47) and (53) with

$$(57) \qquad 1 \leq b, \quad b(n-2m) < n + 2m.$$

Then for any $c > 0$ *the eigenvalue problem*

$$(58) \qquad \tau y = \mu f(t,y), \qquad D^{\alpha}_{y}|_{\partial\Omega} = 0, \qquad |\alpha| \leq m - 1,$$

has infinitely many eigenfunctions $y \in C^{2m}(\Omega) \cap C^{m-1}(\overline{\Omega})$ *and satisfying*

$$\gamma(y) = \int_{\Omega}\int_{0}^{y(t)} f(t,u)\,du\,dt = c.$$

Proof. It follows from results of [6] that (58) is equivalent to an integral equation of the form (2), and where (52) is satisfied for $a(n-2m) < n$. The theorem then follows from the remarks above, preceding its statement, and with the use of the arguments employed in the proof of the main theorem in [6]. The sharpness of the condition (53),(56) is shown by an example in [15].

For the special case where $\tau = -\Delta$, the hypothesis can be weakened slightly, compare Theorems 4 and 5 in [5].

We remark finally that X and Y in Theorem 1 can also be taken to be Orlicz spaces other than L^p. Continuity conditions for the integral operator A and the Nemytsky operator Φ, when X and Y are Orlicz spaces, are given in [9]. In particular, by taking X and Y to be Orlicz spaces one can replace the growth condition (53),(56) in Theorem 3 by an exponential growth condition when $\tau = -\Delta$ and $n = 2$; see for example the hypothesis of the main theorem of [10].

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