THE SPECTRAL SEQUENCE OF A FIBRATION IN GENERALIZED COHOMOLOGY EXISTS FOR AN ARBITRARY MAPPING

by

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As shown by[1] DOLD, ATIYAH-HIRZEBRUCH, and DYER [1969], any fibration $p:E \to B$ (B a simply-connected simplicial complex) and generalized cohomology theory h^* have a corresponding spectral sequence $E_2^{**}(p)$, $E_3^{**}(p), \ldots$ with

$$\begin{cases} E_2^{n,j}(p) = H^n(B;h^j(F)) & \text{(F the fiber)}, \\ d_r^{n,j} \text{ a map of the form } E_r^{n,j}(p) \to E_r^{n+r,j-r+1}(p) & . \end{cases}$$

If B has dimension $k < \infty$ then the sequence converges to the bigraded group associated with a length-$\leq$-k filtration of $h^*(E)$.[2]

We shall show in this paper that an analogous spectral sequence occurs under the following more general conditions: $f:X \to Y$ is an arbitrary continuous function from a given space X to any paracompact hausdorff space Y, and h^* is a generalized cohomology theory defined on the category of pairs of prespaces[3] of X and their maps. Our spectral sequence reduces to the one above when p is substituted for f. (Cf. also section 2 below.)

1. By a prespace of X we here mean any space M with an associated map $u_M:M \to X$ (e.g. take $M \subset X$ and u_M the inclusion map); by a prespace pair we mean a pair (M,N) of prespaces, with N a subspace of the space M and $u_N = u_M|_N$; by a map of one prespace pair (M',N') to another (M,N) we mean a continuous function φ from the space M' to M such that $\varphi N' \subset N$ and $u_M\varphi = u_{M'}$.

[1] Cf. also BECKER.

[2] A relative cohomology version of this spectral sequence is also given by DYER [1969]. Moreover, by reversing appropriate arrows the spectral sequence of p can be seen to exist equally well for generalized homology. However, the general spectral sequence of this paper (section 4) probably fails to exist for generalized homology, because passage to inverse limits is not an exact functor.

[3] "prespace" = a notion more general than subspace and convenient for this particular paper; defined in section 1.

Standard cohomology on the category of pairs of prespaces of X can be given by the formula $h^*(M,N) = H^*(M,N;\, u_M^*\mathcal{S})$, where $\mathcal{S}$ is some sheaf over X independent of (M,N) and $u_M^*\mathcal{S}$ is the pull-back of $\mathcal{S}$ to a sheaf over M.

Generalized cohomology (i.e., cohomology that does not satisfy the dimension axiom) is described in BECKER for prespace pairs that are polyhedral. Denoting by h'^* any one of BECKER's theories, we may define another theory h^* on all prespace pairs by setting $h^*(M,N) = h'^*(PM,PN)$. Here PM designates the singular polyhedron of M, and the associated map $u_{PM}: PM \longrightarrow X$ is $u_M \varkappa_M$, where $\varkappa_M$ is the associated weak homotopy equivalence $PM \longrightarrow M$. For (M,N) polyhedral, there is a natural isomorphism $h^*(M,N) \xrightarrow{\cong} h'^*(PM,PN) \xrightarrow[\cong]{\text{induced by } \varkappa_M} h'^*(M,N)$, so h^* may be regarded as an extension of h'^*.

Both examples of h^* satisfy the axiom below:

<u>Infinite Product Axiom</u>: Given a pair (M,N) of prespaces of X, and given subspaces $\{M^i \mid i \in I\}$ of M such that

$$\begin{cases} \bigcup \mathring{M}^i = M , \\ M^i \cap M^j = N \quad \text{whenever } i,j \in I \text{ are unequal,} \end{cases}$$

the homomorphism

$$h^*(M,N) \longrightarrow \prod_{i \in I} h^*(M^i,N)$$

induced by inclusion is an isomorphism.

(To verify this for the above generalized cohomology theory, note that $h'^*(\bigcup PM^i, PN) \longrightarrow \prod_i h'^*(PM^i,PN)$ is an isomorphism and that $\bigcup PM^i \xrightarrow{\subset} PM$ is a homotopy equivalence. The verification for the sheaf cohomology theory follows from the usual definitions, as given, e.g., in BREDON.)

From this axiom for the case of I a two-member set it follows that an excision map $(M-V, N-V) \xrightarrow{\subset} (M,N)$ induces an h^* isomorphism for arbitrary (i.e., not necessarily open) sets V of M which satisfy the usual condition $\overline{V} \subset \mathring{N}$. For example, if $U \cup W = M$ for U and W open, then $h^*(U \cup W, W) \longrightarrow h^*(U, U \cap W)$ is an isomorphism. (Take $V = W-U$.)

2. In constructing the spectral sequence of this paper, we shall first prove the statement below by the lemmas which succeed it (some of whose proofs are standard and are therefore omitted). This proposition is simply a generalized cohomology analogue of the spectral sequence of GODEMENT [1957, p.212].[1]

Proposition 2.1. Let (X,A) be a space pair, $\mathfrak{U} = \{U^i \mid i \in I\}$ an open covering of X, and h* a generalized cohomology theory on the pairs of prespaces of X. Suppose that h* satisfies the infinite product axiom. Then there is a corresponding spectral sequence $E_1^{**}(\mathfrak{U})$, $E_2^{**}(\mathfrak{U})$, ... with

$$E_1^{n,j}(\mathfrak{U}) = \prod_{\sigma \in I^{n+1}} h^j(U^\sigma, A \cap U^\sigma)$$

(where for $\sigma = (i_0, \ldots, i_n) \in I^{n+1}$) U^σ designates $U^{i_0} \cap \ldots \cap U^{i_n}$),

$d_1^{n,j}: E_1^{n,j}(\mathfrak{U}) \longrightarrow E_1^{n+1,j}(\mathfrak{U})$ the standard coboundary operator (which sends a member $\xi = \{\xi^\sigma \mid \sigma \in I^{n+1}\} \in E_1^{n,j}(\mathfrak{U})$ into the member $\eta = \{\eta^\sigma \mid \sigma \in I^{n+2}\}$ that has the formula

$$\eta^\sigma = \sum_{q=0}^{n+1} (-1)^q \, \xi^{i_0, \ldots, i_{q-1}, i_{q+1}, \ldots, i_{n+1}} \Big|_{U^\sigma, A \cap U^\sigma}$$

for $\sigma = (i_0, \ldots, i_{n+1}) \in I^{n+2}$).

If the order k of $\mathfrak{U}$ is finite, then the sequence converges to the bigraded group associated with a length-$\leq$-k filtration of h*(X,A).

For reference we state:

Lemma 2.2. Let $\ldots K_n \subset K_{n+1} \subset \ldots$ be subspaces of a space K such that $\cup K_n = K$ and $\cap K_n$ is a given subspace L of K. There is

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[1] Compare with LERAY [1950, p.]. MCCORD (unpublished) has also worked on this question and is thanked for the information that DYER [1957], and originally P. CONNER (unpublished), conceived of the space G of this section. Unaware of these possible sources for G, the author defined G and proved Proposition 2.1 in a doctoral thesis at New York University (1969). Cf. CAIN [1968].

Also proved in the thesis is that when $h^*(M,N) = H^*(M,N; u_M^* \mathcal{S})$ for some sheaf $\mathcal{S}$ over X, the spectral sequence of Proposition 2.1 is the same as that of GODEMENT [1957, p.212], while the one of Proposition 4 is the same as that of LERAY [1950, p.] and of BREDON [1967, p.140].

then a spectral sequence E_1^{**}, $E_2^{**}, \ldots$ with

$$E_r^{n,j} = \frac{\mathrm{Im}[h^{n+j}(K_{n+r-1},K_{n-1}) \longrightarrow h^{n+j}(K_{n+r-1},K_{n-r})]}{\mathrm{Im}[h^{n+j}(K_{n+r-1},K_n) \longrightarrow h^{n+j}(K_{n+r-1},K_{n-r})]} .$$

The homomorphism $h^{n+j}(K_{n+r-1},K_{n-1}) \xrightarrow{\partial} h^{n+1+j}(K_{n+2r-1},K_{n+r-1})$ induces the required map $d_r^{n,j}: E_r^{n,j} \longrightarrow E_r^{n+r,j-r+1}$ in the spectral sequence, while the homomorphism $h^{n+j}(K_{n+r},K_{n-1}) \longrightarrow h^{n+j}(K_{n+r-1},K_{n-1})$ (induced by inclusion) induces the required isomorphism $E_{r+1}^{n,j} \xrightarrow{\cong} \mathcal{H}^{n,j}E_r^{**}$. If the filtration of (K,L) has length $k < \infty$, then the spectral sequence converges to the bigraded group associated with a length-$\leq$-k filtration $\ldots F^n h^*(K,L) \supset F^{n+1}h^*(K,L) \supset \ldots$ of $h^*(K,L)$, where $F^n h^*(K,L) = \mathrm{Im}[h^*(K,K_{n-1}) \longrightarrow h^*(K,L)]$.

(For the proof, see J.T.SCHWARTZ.)

In what follows let N denote the nerve of the given open covering $\mathfrak{U}$, where in this context N is by definition a triangulated space. For each finite set $s \subset I$ for which the associated set $\bigcap_{i \in s} U^i = U^s$ is non-empty, let [s] denote the corresponding simplex of N, and let (s) denote the interior $[s]-\partial[s]$ of [s]. (In particular, (s)=[s] if s is a singleton.)

The <u>graph</u> of $\mathfrak{U}$ is the subspace G of X×N with the formula

$$G = \cup U^s \times (s),$$

where, in taking the union, s ranges through the collection of subsets of I for which $U^s \neq \emptyset$. To regard G as a prespace of X, take $u_G: G \longrightarrow X$ to be the restriction of the coordinate projector $X \times N \longrightarrow X$. Then, for subspaces $B \subset X$ and $K \subset N$ write G^B to denote $(B \times N) \cap G$, G_K to denote $(X \times K) \cap G$, and G_K^B to denote $G^B \cap G_K$.

<u>Definition 2.3.</u> let $G_n = G^A \cup G_{N_n}$ $(n \in Z)$; let

$$E^{n,j}(\mathfrak{U}) = \frac{\mathrm{Im}[h^{n+j}(G_{n+r-1},G_{n-1}) \longrightarrow h^{n+j}(G_{n+r-1},G_{n-r})]}{\mathrm{Im}[h^{n+j}(G_{n+r-1},G_n) \longrightarrow h^{n+j}(G_{n+r-1},G_{n-r})]}$$

for $n,j,r \in Z$ and $r \geq 2$; $d_r^{n,j}$ and the isomorphism $E_{r+1}^{n,j}(\mathfrak{U}) \cong \mathcal{H}^{n,j}E_r^{**}(\mathfrak{U})$ are to be as in Lemma 2.2.

(Note that when the order k of $\mathfrak{U}$ is finite, the filtration of (G,G^A) has length $\leq k$, so this spectral sequence converges to the bigraded group associated with a length-$\leq$-k filtration of $h^*(G,G^A)$. See Lemma 2.9, though.)

<u>Lemma 2.4.</u> For each n-simplex [s] of N the pair $U^s \cap (X,A) \times ([s],\partial[s])$ is contained in (G_n,G_{n-1}), and there is an induced isomorphism[1]

$$h^*(G_n,G_{n-1}) \xrightarrow{\cong} \prod_{\dim[s]=n} h^*\big(U^s \cap (X,A) \times ([s],\partial[s])\big) \quad .$$

We first construct a deformation that will be useful in the proof.

<u>PROOF:</u> For $n-1<a<n$ let $[s]_a$ be the closed set

$$\Big\{x \in [s] \,\Big|\, \min_{i \in s} x^i \leq \frac{a-(n-1)}{n+1}\Big\} \subset [s] ,$$

let $[s]^a = \Big\{x \in [s] \,\Big|\, \min_{i \in s} x^i \geq \frac{a-(n-1)}{n+1}\Big\} \subset [s]$,

{for $i \in I$ and $x \in N$, x^i designates the i^{th} barycentric coordinate of x}

and for $n-1<b<a<n$ let $[s]^b_a = [s]^b \cap [s]_a$. One can deform $[s]_a$ into $\partial[s]$ by a deformation that leaves each point of $\partial[s]$ invariant; this deformation can be extended to a deformation of [s] (into itself); as [s] ranges through the n-simplexes of N we piece together these deformations to get a deformation of N_n into itself; extending further a simplex at a time, we obtain finally the required deformation D^a_t: $N \longrightarrow N$ $(0 \leq t \leq 1)$ that leaves invariant each point of N_{n-1} and each simplex of N and that deforms the set $N_a = N_{n-1} \cup \big(\cup\{[s]_a \,|\, \dim[s]=n\}\big)$ invariantly into N_{n-1}. Next, we note that certain inclusion maps induce h^*-isomorphisms by the excision and homotopy axioms:

One open set of the space G_n is $V = G_n - G_{N_a}$, where we now assume $n<a<n+1$; since $N_n - N_a = \emptyset$ we have $V = G^A - G_{N_a}$, implying that $\overline{V} \subset G^A - G_{N_{a-\epsilon}} \subset$ interior of G_{n-1} (for small positive ϵ). Therefore, the homomorphism

$$\begin{array}{c} h^*(G_n,G_{n-1}) \longrightarrow h^*(G_n - V,\ G_{n-1} - V) \\ \qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad h^*(G^A_{N_a} \cup G_{N_n}\ ,\ G^A_{N_a} \cup G_{N_{n-1}}) \end{array}$$

induced by inclusion is an isomorphism. But $1_X \times D^a_t$ $(0 \leq t \leq 1)$ is a family of maps of the prespace $X \times N$ of X into itself, and it deforms $G^A_{N_a}$ into $G^A_{N_n}$, keeping invariant the points of G_{N_n}. Thus, we so

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[1] Notation: $L \cap (M,N)$ designates $(L \cap M, L \cap N)$, and $(K,L) \times (M,N)$ designates $(K \times M,\ K \times N \cup L \times M)$.

far have an isomorphism

$$\begin{array}{c} h^*(G_n, G_{n-1}) \\ \text{(excision axiom)} \Big\downarrow \cong \\ h^*(G^A_{N_a} \cup G_{N_n},\ G^A_{N_a} \cup G_{N_{n-1}}) \\ \text{(homotopy axiom)} \Big\downarrow \cong \\ h^*(G_{N_n},\ G^A_{N_n} \cup G_{N_{n-1}}) \end{array}$$

which is induced by inclusion.

For $n-1<a<n$ the family of maps $1_X \times D^a_t$ $(0 \le t \le 1)$ of the prespace $X \times N$ (of X) into itself deforms G_{N_a} invariantly into $G_{N_{n-1}}$ and keeps invariant G_{N_n}, $G^A_{N_n}$, and the points of $G_{N_{n-1}}$. This implies that the homomorphism

$$\begin{array}{c} h^*(G_{N_n},\ G^A_{N_n} \cup G_{N_a}) \\ \text{(homotopy axiom)} \Big\downarrow \cong \\ h^*(G_{N_n},\ G^A_{N_n} \cup G_{N_{n-1}}) \end{array}$$

that is induced by inclusion is a third isomorphism. For a fourth, note that one open set of the space G_{N_n} is $V = G_{N_n} - G_{K^b}$, where $n-1<b<a$ and $K^b = \cup\{[s]^b \mid \dim[s]=n\}$. Thus, we get an isomorphism

$$\begin{array}{c} h^*(G_{N_n},\ G^A_{N_n} \cup G_{N_a}) \\ \text{(excision axiom)} \Big\downarrow \cong \\ h^*\left(G_{N_n} - V,\ (G^A_{N_n} \cup G_{N_a}) - V\right) \\ \Big\| \\ h^*(G_{N_n \cap K^b},\ G^A_{N_n \cap K^b} \cup G_{N_a \cap K^b}) \end{array}$$

induced by inclusion, because $B = G_{N_n} - G_{K^b} = G_{N_a} - G_{K^b}$ implies that $\overline{V} \subset G_{N_a} - G_{K^{b+\epsilon}} \subset$ interior of $G^A_{N_n} \cup G_{N_a}$ (for small positive ϵ).

The isomorphisms and maps that are of interest here form a commutative diagram:

$$
\begin{array}{ccc}
h^*(G_n, G_{n-1}) & \searrow & \\
\downarrow \cong & & \\
h^*(G_{N_n}, G^A_{N_n} \cup G_{N_{n-1}}) & \longrightarrow & \prod_{\dim[s]=n} h^*(U^s \cap (X,A) \times ([s], \partial[s])) \\
\text{(homotopy axiom)} \ \uparrow \cong & & \cong \uparrow \ \text{(homotopy axiom)} \\
h^*(G_{N_n}, G^A_{N_n} \cup G_{N_a}) & \longrightarrow & \prod_{\dim[s]=n} h^*(U^s \cap (X,A) \times ([s], [s]_a)) \\
\text{(excision axiom)} \ \downarrow \cong & & \cong \downarrow \ \text{(excision axiom)} \\
h^*(G_{N_n \cap K^b}, G_{N_n \cap K^b} \cup G_{N_a \cap K^b}) & \xrightarrow{\cong} & \prod_{\dim[s]=n} h^*(U^s \cap (X,A) \times ([s]^b, [s]^b_a))
\end{array}
$$

(The maps in the right column are analogues of those in the left cloumn; the map at the bottom is the isomorphism

$$
\begin{array}{c}
h^*\left(G_{N_n \cap K^b}, G^A_{N_n \cap K^b} \cup G_{N_a \cap K^b}\right) \\
\cong \downarrow \ \text{(infinite product axiom)} \\
\prod_{\dim[s]=n} h^*\left(G_{[s]^b} \cup G^A_{N_n \cap K^b} \cup G_{N_a \cap K^b},\ G^A_{N_n \cap K^b} \cup G_{N_a \cap K^b}\right) \\
\cong \downarrow \ \text{(excision axiom)} \\
\prod_{\dim[s]=n} h^*\left(G_{[s]^b},\ G^A_{[s]^b} \cup G_{[s]^b_a}\right) \\
\| \\
\prod_{\dim[s]=n} h^*\left(U^s \cap (X,A) \times ([s]^b, [s]^b_a)\right) \quad ;
\end{array}
$$

all maps here are induced by inclusion.) By the commutativity and the indicated isomorphisms, it follows that the upper diagonal map is an isomorphism. Q.E.D.

<u>Lemma 2.5.</u> If (M,N) is any pair of prespaces of X, then a generalized cohomology theory h'^{*} on the category of CW complex-subcomplex pairs is defined by setting $h'^{*}(K,L) = h^{*}((M,N)\times(K,L))$, where the required map $u_{M\times K}: M\times K \longrightarrow X$ is taken to be

$$M\times K \xrightarrow{\text{coordinate projector}} M \xrightarrow{u_M} X .$$

(The proof is elementary and is omitted.)

<u>Lemma 2.6.</u> The group $h^{n+j}(G_n,G_{n-1})$ is isomorphic to the subgroup of alternating non-degenerate members of $E_1^{n,j}(\mathcal{U})$ (Cf. Prop. 2.1).

<u>PROOF</u>: Take the isomorphism to be the 1:1 map

$$\begin{array}{c} h^{n+j}(G_n,G_{n-1}) \\ \Big\downarrow \cong \ \text{(Lemma 2.4)} \\ \prod_{\dim[s]=n} h^{n+j}\big(U^s\cap(X,A)\times([s],\partial[s])\big) \\ \Big\downarrow \ \prod_{\dim[s]=n}\Big(\prod_{\sigma\in s^{n+1}}\varphi^{s,\sigma}\Big) \\ \prod_{\dim[s]=n}\Big(\prod_{\sigma\in s^{n+1}} h^j\big(U^\sigma\cap(X,A)\big)\Big) \\ \| \\ E_1^{n,j}(\mathcal{U}) \end{array} ,$$

where for each n-simplex $[s]$ of N and member $\sigma = (i_0,.,.,i_n)$ of s^{n+1} we define a homomorphism $\varphi^{s,\sigma}: h^{n+j}(U^s\cap(X,A)\times([s],\partial[s])) \longrightarrow h^j(U^\sigma\cap(X,A))$ as follows: if $i_0,\ldots,i_n$ are not distinct then $\varphi^{s,\sigma} = 0$; otherwise, we set $\sigma_q = (i_q,i_{q+1},\ldots,i_n)$ for $q = 0,1,\ldots,n$ and we define $\varphi^{s,\sigma}$ to be the isomorphism below, where $h'^{*}(\cdot)$ stands for $h^{*}(U^s\cap(X,A)\times(\cdot))$:

$$h^{n+j}(U^s \cap (X,A) \times ([s], \partial[s]))$$

$$\Vert$$

$$h'^{n+j}([\sigma_o], \partial[\sigma_o])$$

$$\vdots$$

$$h'^{n-q+j}([\sigma_q], \partial[\sigma_q])$$

$$\uparrow \partial \cong \quad \begin{pmatrix}\text{by the exact sequence for the triple } ([\sigma_q], \partial[\sigma_q], \partial[\sigma_q] - (\sigma_{q+1})), \\ \text{since } ([\sigma_q], \partial[\sigma_q] - (\sigma_{q+1})) \text{ is homotopy-equivalent to} \\ ([i_n], [i_n])\end{pmatrix}$$

$$h'^{n-(q+1)+j}(\partial[\sigma_q], \partial[\sigma_q] - (\sigma_{q+1}))$$

$$\downarrow \cong \quad (\text{excision for polyhedral pairs})$$

$$h'^{n-(q+1)+j}([\sigma_{q+1}], \partial[\sigma_{q+1}])$$

$$\vdots$$

$$h'^{j}([\sigma_n], \partial[\sigma_n])$$

$$\Vert$$

$$h^{j}(U^\sigma \cap (X,A) \times [i_n])$$

$$\Vert\wr$$

$$h^{j}(U^\sigma \cap (X,A)) \quad .$$

By the following argument the map $\varphi^{s,\sigma}$ changes sign when the order of σ undergoes an odd permutation: if $\nu : [s] \longrightarrow [s]$ is a simplicial homeomorphism that carries σ into a new ordering σ', then $\varphi^{s,\sigma'} = \varphi^{s,\sigma} \circ h'^{n+j}(\nu) = \varphi^{s,\sigma} \circ (\text{degree}(\nu) \cdot \text{identity}) = \pm\varphi^{s,\sigma}$, depending upon whether ν is even or odd.

The statement of the Lemma now follows. Q.E.D.

<u>Lemma 2.7.</u> The diagram

$$\begin{array}{ccc} h^{n+j}(G_n,G_{n-1}) & \xrightarrow{\ \partial\ } & h^{n+1+j}(G_{n+1},G_n) \\ \Big\downarrow \text{(Lemma 2.6)} & & \Big\downarrow \text{(Lemma 2.6)} \\ E_1^{n,j}(u) & \xrightarrow{\ d_1^{n,j}\ } & E_1^{n+1,j}(u) \end{array}$$

commutes.

<u>PROOF</u>: Let $\xi \in h^{n+j}(G_n,G_{n-1})$, and let [s] be an (n+1)-simplex of N with σ a total ordering of s. Let $h'^{*}(\cdot) = h^{*}(U^s \cap (X,A) \times (\cdot))$. Let ξ' denote the image of ξ under the homomorphism $h^{n+j}(G_n,G_{n-1}) \longrightarrow h^{n+j}(U^s \cap (X,A) \times ([s]_n,[s]_{n-1}) = h'^{n+j}([s]_n,[s]_{n-1})$ (induced by inclusion); let $\eta = \partial\xi$ for ∂ as above, and let η' denote the image of η under an analogous map $h^{n+1+j}(G_{n+1},G_n) \longrightarrow h'^{n+1+j}([s],[s]_n)$. Then $\eta' = \partial'\xi'$, where ∂' is the map $h'^{n+j}([s]_n,[s]_{n-1}) \xrightarrow{\partial'} h'^{n+1+j}([s],[s]_n)$.

The condition

$$\sum_{q=0}^{n+1} (-1)^q \left(\varphi^{s_q,\sigma_q}\left(\xi\Big|_{U^{s_q}\cap (X,A) \times ([s_q],\partial[s_q])}\right)\right)\Big|_{U^s\cap(X,A)}$$

$$= \varphi^{s,\sigma}\left(\eta\Big|_{U^s\cap (X,A)\times([s],\partial[s])}\right),$$

$$\text{i.e., } \varphi^{s,\sigma}(\eta'),$$

is what we want to prove, where $s_q = s - \{i_q\}$ and $\sigma_q = (i_0,\dots,i_{q-1}, i_{q+1},\dots,i_{n+1})$. Evidently the q^{th} term ($q=0,1,\dots,n+1$) on the left-hand side is $(-1)^q \varphi^{s_q,\sigma_q}\left(\xi'\Big|_{[s_q],\partial[s_q]}\right)$, and there is a corresponding isomorphism

$$h'^{n+j}([s_q],\partial[s_q]) \xleftarrow[\cong]{\alpha^q} h'^{n+j}(\partial[s],\partial[s]-(s^q)) \xrightarrow[\cong]{\beta^q\ (\partial)} h'^{n+1+j}([s],\partial[s])$$

with the property that $\varphi^{s^q,\sigma^q}\alpha^q(\beta^q)^{-1} = \varphi^{s,(i_q,i_0,i_1,\dots,i_{q-1},i_{q+1},\dots,i_{n+1})} = (-1)^q\varphi^{s,\sigma}$. In the commutative diagram

$$\begin{array}{ccc}
h'^{n+j}([s]_n,[s]_{n-1}) & \xrightarrow{\partial'} & h'^{n+1+j}([s],[s]_n) \\
\text{(direct sum property for polyhedra)}\ \Big\downarrow \theta \ \cong & \nwarrow & \Big\uparrow \sum_q \beta^q \\
\sum_q \oplus h'^{n+j}([s^q],\partial[s^q]) & \xleftarrow[\cong]{\Sigma \oplus \alpha^q} & \sum_q \oplus h'^{n+j}(\partial[s],\partial[s]-(s^q))
\end{array}$$

one deduces that $\partial' = (\sum_q \beta^q(\alpha^q)^{-1}\theta)$. This completes the proof, for we have

$$\varphi^{s,\sigma}(\eta') = \varphi^{s,\sigma}\partial'\xi' = \varphi^{s,\sigma}\sum_q \beta^q(\alpha^q)^{-1}\theta\,\xi'$$

$$= \sum_q (-1)^q \varphi^{s^q,\sigma^q}\xi'\Big|_{[s^q],\partial[s^q]}$$

as required. Q.E.D.

<u>Lemma 2.8.</u> The isomorphism of Lemma 2.6 induces an isomorphism $E_2^{n,j}(\mathfrak{U}) \xrightarrow{\cong} \mathfrak{X}^{n,j}E_1^{n,j}(\mathfrak{U})$.

(This is a standard result on the equivalence of alternating non-degenerate cochains to all cochains in determining standard cohomology groups. The proof is in GODEMENT.)

<u>Lemma 2.9.</u> If the order k of $\mathfrak{U}$ is finite, then the prespace map $u_G: G \longrightarrow X$ induces an isomorphism $H^*(X,A) \xrightarrow{\cong} h^*(G,G^A)$.

<u>PROOF</u>: If the statement were true for $A = \emptyset$, then it would be true for $A \neq \emptyset$ by the diagrammatic 5-lemma for the diagram

$$\begin{array}{ccccccccc}
h^{q-1}(X) & \longrightarrow & h^{q-1}(A) & \longrightarrow & h^q(X,A) & \longrightarrow & h^q(X) & \longrightarrow & h^q(A) \\
\downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong \\
h^{q-1}(G) & \longrightarrow & h^{q-1}(G^A) & \longrightarrow & h^q(G,G^A) & \longrightarrow & h^q(G) & \longrightarrow & h^q(G^A)
\end{array} .$$

Therefore, assume $A = \emptyset$. Set $X^n = \cup\{U^s \mid \dim[s]=n\}$ for $n \in Z$.

It suffices to prove that for $0 \le n \le k$ the map $\theta_n^*: h^*(X^n,X^{n+1}) \longrightarrow h^*(G^{X^n},G^{X^{n+1}})$ induced by u_G is an isomorphism, for an inductive use of the diagrammatic 5-lemma would then lead from the validity of the isomorphism $h^*(X^k) \xrightarrow{\cong} h^*(G^{X^k})$ to the validity of the required isomorphism $h^*(X^0) \xrightarrow{\cong} h^*(G^{X^0})$ in k steps.

In examining θ_n^* we have the commutative diagram

$$\begin{array}{ccccc}
h^*(X^n, X^{n+1}) & \xrightarrow[\text{product axiom})]{\text{(direct}\ \cong} & \prod_s h^*(U^s \cup X^{n+1}, X^{n+1}) & \xrightarrow[\text{sion axiom})]{\text{(exci-}\ \cong} & \prod_s h^*(U^s, U^s \cap X^{n+1}) \\
\downarrow & & \downarrow & & \downarrow \prod_s \theta_s^* \\
h^*(G^{X^n}, G^{X^{n+1}}) & \xrightarrow{('')\ \cong} & \prod_s h^*(G^{U^s \cup X^{n+1}}, G^{X^{n+1}}) & \xrightarrow{('')\ \cong} & \prod_s h^*(G^{U^s}, G^{U^s \cap X^{n+1}})
\end{array}$$

(all direct products taken over the collection of n-simplexes [s] of N), which implies that it suffices to show that $\theta_s^*: h^*(U^s, U^s \cap X^{n+1}) \xrightarrow{\cong} h^*(G^{U^s}, G^{U^s \cap X^{n+1}})$ for all n-simplexes [s] of N.

In examining θ_s^*, note first that

$$G^{U^s} = (U^s \times N) \cap G = (U^s \times N) \cap (\bigcup\{U^{s''} \times (s'') \mid s'' \subset I\})$$

$$= \bigcup\{U^{s\ s''} \times (s'') \mid s'' \subset I\} = \bigcup\{U^{s'} \times (s'') \mid s' \subset I,\ s' \supset s\ s''\}$$

$$= \bigcup\{U^{s'} \times [s'] \mid s \subset s' \subset I\}, \qquad (*)$$

and secondly, that

$$G^{U^s \cap X^{n+1}} = \bigcup\{G^{U^s \cap U^{s''}} \mid \dim[s''] = n+1\} = \bigcup\{G^{U^{s'''}} \mid s \subsetneqq s''' \subset I\}$$

$$= \bigcup\{\bigcup\{U^{s'} \times [s'] \mid s''' \subset s' \subset I\} \mid s \subsetneqq s'''\}$$

$$= \bigcup\{U^{s'} \times [s'] \mid s \subsetneqq s' \subset I\}. \qquad (**)$$

Recall also that $\overline{\text{Star}}[s]$ is by definition the set $\bigcup\{[s'] \mid s \subset s'\}$ and possesses a (non-unique) simplicial retraction $r: \overline{\text{Star}}[s] \longrightarrow [s]$, which in turn provides a deformation $D_t: \overline{\text{Star}}[s] \longrightarrow \overline{\text{Star}}[s]$ $(0 \le t \le 1)$ by the formula $D_t(x)^i = (1-t)x^i + tr(x)^i$, where x^i $(i \in I)$ denotes the i^{th} barycentric coordinate of a point $x \in \overline{\text{Star}}[s]$. This deformation keeps invariant each simplex of $\overline{\text{Star}}[s]$ and point of [s] and deforms $\overline{\text{Star}}[s]$ into [s].

Then $1_{U^s} \times D_t$ $(0 \le t \le 1)$ is a family of maps of the prespace $U^s \times \overline{\text{Star}}[s]$ of X that deforms each term of (*) into the corresponding term of $\bigcup\{U^{s'} \times [s] \mid s \subset s'\} = U^s \times [s]$, and that deforms each term of (**) into the corresponding term of $\bigcup\{U^{s'}\times[s] \mid s \subsetneq s'\} = U^{s}\cap X^{n+1}\times[s]$. Thus, we have θ^*_s an isomorphism by the commutative diagram

$$\begin{array}{ccc} h^*(U^s, U^s\cap X^{n+1}) & \xrightarrow{\ \theta^*_s\ } & h^*(G^{U^s}, G^{U^s\cap X^{n+1}}) \\ & \searrow^{\text{(induced by } u_G)}_{\text{(homotopy axiom) } \cong} & \Big\downarrow {\scriptstyle \text{(homotopy axiom) } \cong \ \text{(induced by inclusion)}} \\ & & h^*(U^s\times[s], U^s\cap X^{n+1}\times[s]) \end{array} .$$

The statement of the lemma now follows. Q.E.D.

These lemmas suffice to prove Proposition 2.1, for the required groups $E_r^{n,j}(\mathfrak{U})$ and maps $d_r^{n,j}: E_r^{n,j}(\mathfrak{U}) \longrightarrow E_r^{n+r,j-r+1}(\mathfrak{U})$ ($n,j,r\in Z$; $r\geq 1$) are given in Definition 2.3 and in the statement of the Proposition, while the required isomorphisms $E_{r+1}^{n,j}(\mathfrak{U}) = H^{nj}E_r^{**}(\mathfrak{U})$ and $E_\infty^{n,j}(\mathfrak{U}) = F^n h^{n+j}(X,A)/F^{n+1}h^{n+j}(X,A)$, as well as the filtration $\ldots F^n h^*(X,A)\supset F^{n+1}h^*(X,A)\supset\ldots$ can be deduced immediately from Lemmas 2.2, 2.8, and 2.9.

It can easily be shown that the spectral sequence $E_2^{**}(p), \ldots$ of the introduction is the same as the spectral sequence $E_2^{**}(\mathfrak{U}), \ldots$, where $\mathfrak{U} = \{p^{-1}\text{star}(v) \mid v\in B_o\}$. The proof: the map $u_G: G \longrightarrow E$ carries the filtration of G into the filtration $\ldots p^{-1}B_n\subset p^{-1}B_{n+1}\subset\ldots$ of E by which $E_2^{**}(p), \ldots$ is defined as in Lemma 2.2, and the induced map $E_2^{**}(p) \longrightarrow E_2^{**}(\mathfrak{U})$ is an isomorphism.

3. Suppose that $\mathfrak{U}' = \{U'^j \mid j\in I'\}$ is a refinement of the open covering $\mathfrak{U}$ of section 2, and let $\pi: I' \longrightarrow I$ be any refinement projector. Then there is a corresponding homomorphism $\pi_1: E_1^{n,j}(\mathfrak{U}) \longrightarrow E_1^{n,j}(\mathfrak{U}')$ given by the standard formula: π_1 sends a member $\xi = \{\xi^\sigma \mid \sigma\in I^{n+1}\}$ of $E_1^{n,j}(\mathfrak{U})$ into the member $\xi' = \{\xi'^\sigma \mid \sigma\in I'^{n+1}\}$ of $E_1^{n,j}(\mathfrak{U}')$ with

$$\xi'^\sigma = \xi^{\pi(\sigma)}\Big|_{U'\sigma, A\cap U'\sigma} \qquad (\sigma\in I'^{n+1}) .$$

Furthermore, the following proposition is true:

<u>Proposition 3.1.</u> π_1 **is part of a unique spectral sequence** homomorphism $\{\pi_r : E_r^{**}(\mathfrak{U}) \longrightarrow E_r^{**}(\mathfrak{U}') \mid r \geq 1\}$. If both $\mathfrak{U}$ and $\mathfrak{U}'$ have finite order, then the filtrations $\{F^n h^*(X,A) \mid n \in Z\}$ and $\{F'^n h^*(X,A) \mid n \in Z\}$ that are induced by them satisfy the condition

$$F^n h^q(X,A) \subset F'^n h^q(X,A) \quad ,$$

and the diagram

$$\begin{array}{ccc}
E_\infty^{n,j}(\mathfrak{U}) & \xrightarrow[\text{(induced by } \pi_1)]{\pi_\infty} & E_\infty^{n,j}(\mathfrak{U}') \\
\Big\downarrow \cong & & \Big\downarrow \cong \\
\dfrac{F^n h^{n+j}(X,A)}{F^{n+1} h^{n+j}(X,A)} & \xrightarrow[\text{identity)}]{\text{(induced by the}} & \dfrac{F'^n h^{n+j}(X,A)}{F'^{n+1} h^{n+j}(X,A)}
\end{array}$$

is commutative.

<u>PROOF</u>: Clearly, if π_1 is part of such a spectral sequence homomorphism, then the latter is unique, because its portion π_{r+1} is fully determined by the portion π_r for $r = 1,2,\ldots$.

π determines a simplicial map $\nu : N' \longrightarrow N$, where N' denotes the nerve of $\mathfrak{U}'$. Moreover, if $[s]$ is any simplex of N', then $1_X \times \nu$ maps $U'^s \times (s)$ into $U^{\pi(s)} \times (\pi s)$, so the restriction of $1_X \times \nu$ to the graph G' of $\mathfrak{U}'$ defines a map $\gamma : G' \longrightarrow G$. Since $\gamma(G'^B_K) \subset G^B_{\nu(K)}$ holds for subspaces B of X and K of N', it follows in particular that $\gamma(G'_n) \subset G_n$ for all $n \in Z$. We conclude thus far that there is a homomorphism from the spectral sequence of Lemma 2.2 for $\ldots G_n \subset G_{n+1} \ldots$ to the spectral sequence of Lemma 2.2 for $\ldots G'_n \subset G'_{n+1} \ldots$, this homomorphism being induced by γ.

Add to this the fact that the isomorphism of Lemma 2.6 has the following commutative diagram:

$$\begin{array}{ccc}
h^{n+j}(G_n,G_{n-1}) & \xrightarrow{\text{induced by } \gamma} & h^{n+j}(G'_n,G'_{n-1}) \\
\downarrow \cong & & \downarrow \cong \\
\prod_{\dim[s]=n} h^{n+j}(U^s \cap (X,A) \times ([s],\partial[s])) & \xrightarrow{\text{"}} & \prod_{\dim[s']=n} h^{n+j}(U'^{s'} \cap (X,A) \times ([s'],\partial[s'])) \\
\prod_{\dim[s]=n}\left(\prod_{\sigma\in s^{n+1}} \varphi^{s,\sigma}\right) \downarrow & & \downarrow \prod_{\dim[s']=n}\left(\prod_{\sigma'\in s'^{n+1}} \varphi^{s',\sigma'}\right) \\
\prod_{\dim[s]=n}\left(\prod_{\sigma\in s^{n+1}} h^j(U^\sigma, A\cap U^\sigma)\right) & \xrightarrow{\text{induced by } \pi_1} & \prod_{\dim[s']=n}\left(\prod_{\sigma'\in s'^{n+1}} h^j(U'^{\sigma'}, A\cap U'^{\sigma'})\right) \\
\| & & \| \\
E_1^{n,j}(\mathfrak{U}) & & E_1^{n,j}(\mathfrak{U}')
\end{array}$$

(Simply examine Lemma 2.6 and its proof.) There is thus a homomorphism from $E_1^{**}(\mathfrak{U}), E_2^{**}(\mathfrak{U}), \ldots$ to $E_1^{**}(\mathfrak{U}'), E_2^{**}(\mathfrak{U}'), \ldots$ that is in effect induced by π.

Suppose that $\mathfrak{U}$ and $\mathfrak{U}'$ both have finite order. The commutative diagram

$$\begin{array}{ccccc}
h^*(G,G_{n-1}) & \longrightarrow & h^*(G,G^A) & \xleftarrow{\cong} & h^*(X,A) \\
\downarrow & & \downarrow & & \downarrow = \\
h^*(G',G'_{n-1}) & \longrightarrow & h^*(G',G'^A) & \xleftarrow{\cong} & h^*(X,A)
\end{array}$$

(left vertical maps induced by γ, left horizontal maps by inclusion, right horizontal maps by u_G and $u_{G'}$) implies that the rest of Proposition 3.1 is true. Q.E.D.

<u>Proposition 3.2.</u> The (convergent) spectral sequence of Proposition 2.1 depends functorially upon $\mathfrak{U}$, if by a morphism of open coverings we mean a refinement projector. If instead we mean merely a refinement, then the (convergent) spectral sequence $E_2^{**}(\mathfrak{U}), E_3^{**}(\mathfrak{U}), \ldots$ depends functorially upon $\mathfrak{U}$.

PROOF: $E_1^{**}(\mathfrak{U})$ alone depends functorially upon $\mathfrak{U}$ in the first sense.

Proposition 3.1 insures that a refinement projector induces not simply a map of $E_1^{**}(\mathfrak{U})$ but of the entire spectral sequence, and since this map is unique, the entire spectral sequence cannot be anything but functorial in $\mathfrak{U}$.

If $\mathfrak{U}$ and $\mathfrak{U}'$ are two open coverings, and $\mathfrak{U}'$ refines $\mathfrak{U}$, then it is known that for each $j \in Z$ the chain map $\pi_1 : E_1^{*,j}(\mathfrak{U}) \to E_1^{*,j}(\mathfrak{U}')$, which is induced by a refinement projector π, is, to within chain homotopy, independent of the choice of π. (Cf. GODEMENT [1957].) It follows that π_2 is altogether independent of the choice of π, and the same must therefore be true for all π_r, $r \geq 2$, by the fact that π_r determines π_{r+1}. The second conclusion of Proposition 3.2 now follows. Q.E.D.

4. We now conclude by proving the proposition below:

Proposition 4. Let (X,A) be a space pair, $f: X \to Y$ a map into a paracompact hausdorff space Y, and h^* a generalized cohomology theory on the pairs of prespaces of X. Suppose h^* satisfies the infinite product axiom. Then there is a corresponding spectral sequence $E_2^{**}(f), E_3^{**}(f), \ldots$ with $E_2^{n,j}(f) = H^n(Y; L^j(f))$, where $L^j(f)$ is the induced sheaf of the presheaf

$$P^j(f) = \{h^j(f^{-1}V, A \cap f^{-1}V) \mid V \text{ open} \subset X\}.$$

If the covering dimension k of Y is finite then the sequence converges to the bigraded group associated with a length-$\leq$-k filtration of $h^*(X,A)$.

PROOF: For each open covering $\mathfrak{V} = \{V^i \mid i \in I\}$ of Y let $f^{-1}\mathfrak{V} = \{f^{-1}V^i \mid i \in I\}$. By Proposition 3.2 the spectral sequence $E_2^{**}(f^{-1}\mathfrak{V}), E_3^{**}(f^{-1}\mathfrak{V}), \ldots$ depends functorially upon $\mathfrak{V}$, if by a morphism of open coverings we mean merely a refinement. Define the required $E_r^{**}(f)$ to be $\varinjlim_{\mathfrak{V}} E_r^{**}(f^{-1}V)$ (as usual, we allow $\mathfrak{V}$ to vary only through a <u>set</u> of open coverings that is cofinal in the class of all open coverings of Y).

It follows that $E_2^{n,j}(f) = \check{H}^n(Y;P^j(f))$, the ČECH cohomology group of Y with coefficients in the presheaf $P^j(f)$. This group is isomorphic to the required $H^n(Y;L^j(f))$ (Cf. GODEMENT [1957, p.].)

Define the required $d_r^{n,j}: E_r^{n,j}(f) \longrightarrow E_r^{n+r,j-r+1}(f)$ to be the $\varinjlim_{\mathcal{U}}$ of $d_r^{n,j}: E_r^{n,j}(f^{-1}\mathcal{U}) \longrightarrow E_r^{n+r,j-r+1}(f^{-1}\mathcal{U})$. It follows that the property $d_r^{n,j}\, d_r^{n-r,j+r-1} = 0$ is preserved, since passage to direct limits is an additive functor. This functor is in fact exact, so there is an isomorphism

$$\mathcal{H}^{n,j}E_r^{**}(f) \cong \varinjlim_{\mathcal{U}} \mathcal{H}^{n,j}E_r^{**}(f^{-1}\mathcal{U})$$

(Cf. CARTAN-EILENBERG [195 , p.]). This isomorphism, together with the $\varinjlim_{\mathcal{U}}$ of the isomorphism $E_{r+1}^{n,j}(f^{-1}\mathcal{U}) \cong \mathcal{H}^{n,j}E_r^{**}(f^{-1}\mathcal{U})$, provides the required isomorphism $E_{r+1}^{n,j}(f) \cong \mathcal{H}^{n,j}E_r^{**}(f)$, to complete the first conclusion of the proposition.

In proving the second conclusion of the proposition, note that the operation $\varinjlim_{\mathcal{U}}$ depends only on $\mathcal{U}$ of order $\leq k$. Considering only such $\mathcal{U}$, then, we have that the filtration of $h^*(X,A)$ that is induced by $f^{-1}\mathcal{U}$, to be denoted here $\{F_{\mathcal{U}}^n h^*(X,A) \mid n \in Z\}$, has length $\leq k$. As noted earlier, the condition $F_{\mathcal{U}}^n h^q(X,A) \subset F_{\mathcal{U}'}^n h^q(X,A)$ holds if $\mathcal{U}'$ is a second open covering that refines $\mathcal{U}$, so we define the required filtration $\{F_f^n h^*(X,A) \mid n \in Z\}$ by the formula $F_f^n h^q(X,A) = \bigcup_{\mathcal{U}} F_{\mathcal{U}}^n h^q(X,A)$ (where the union is taken over a set of order-$\leq$-k coverings $\mathcal{U}$ that is cofinal in the class of all order-$\leq$-k coverings).

Since $\varinjlim_{\mathcal{U}}$ is an exact functor it follows easily that there is an isomorphism

$$\varinjlim_{\mathcal{U}}\left(F_{\mathcal{U}}^n h^q(X,A)/F_{\mathcal{U}}^{n+1}h^q(X,A)\right) = F_f^n h^q(X,A)/F_f^{n+1}h^q(X,A).$$

Combining this with the $\varinjlim_{\mathcal{U}}$ of the isomorphism

$$E_k^{n,j}(f^{-1}\mathcal{U}) \cong E_\infty^{n,j}(f^{-1}\mathcal{U}) \cong \frac{F_{\mathcal{U}}^n h^{n+j}(X,A)}{F_{\mathcal{U}}^{n+1}h^{n+j}(X,A)},$$

we obtain the required isomorphism

$$E_\infty^{n,j}(f) \cong E_k^{n,j}(f) \cong \frac{F_f^n h^{n+j}(X,A)}{F_f^{n+1}h^{n+j}(X\in A)},$$

to complete the proof of the second conclusion of the proposition. Q.E.D.

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