THE ORDINALS OF THE SYSTEMS OF SECOND ORDER ARITHMETIC WITH THE PROVABLY Δ_2^1-COMPREHENSION AXIOM AND WITH THE Δ_2^1-COMPREHENSION AXIOM RESPECTIVELY

by

Gaisi Takeuti and Mariko Yasugi

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Introduction.

We shall present our work in three parts: namely, the ordinal of second order arithmetic with the provably Δ_2^1-comprehension axiom (Chapter I), the ordinal of second order arithmetic with the Δ_2^1-comprehension axiom (Chapter II), and some applications of the reduction method which is adopted in Chapter I (Chapter III). We may remark here that all the systems which we are concerned with have the full induction.

Technically, Chapter I is a further development of the consistency proofs of some systems of second order arithmetic in [5]. We shall first introduce a new notion of blocks and by the help of it carry out the reduction of the proofs of second order arithmetic with the provably Δ_2^1-comprehension axiom (let us call this system $P\Delta_2^1$), and then prove that the reduction process halts, by transfinite induction along the well orderings of $O(\omega^n+1,\omega^n)$ for $n < \omega$, where $O(I,A)$ represents the system of ordinal diagrams (abbreviated to o.d.s. as usual) with the basic sets I and A.

*University of Illinois

The well-ordering of $O(\omega^n, \omega^n)$ is formally provable in a system with the Π^1_1-inductive definitions (cf. [5]) along the canonical ordering of ω^n. On the other hand it can be shown by a routine calculation that the Π^1_1-inductive definitions can be defined in $P\Delta^1_2$. Thus, we can conclude that the ordinal of $P\Delta^1_2$ is the limit of the order types of $O(\omega^n, \omega^n)$ for all n.

Chapter II starts with a revised version of the consistency proof of second order arithmetic with the Π^1_1-comprehension axiom and the Π^1_1-inductive definitions, which was first presented by Takeuti in the last chapter of [5]. As a consequence, the consistency of the system with the Π^1_1-inductive definitions along $w_n = \left.\omega^{\omega^{\cdot^{\cdot^{\omega}}}}\right\} n$ (let us call this system ID_n) can be proved by the system of o.d.s $O(w_n, w_n)$.

Now combining Friedman's result (cf. [1]) and a simple computation, we can claim that the Δ^1_2-comprehension axiom and the Π^1_1-inductive definitions along w_n (see above) are interdeducible in second order arithmetic with the Π^1_1-comprehension axiom. According to the remark on the o.d.s as quoted above implies that the ordinal of second order arithmetic with the Δ^1_2-comprehension axiom is the limit of the order types of $O(w_n, w_n)$ for all n.

For Chapter III, we shall briefly remark that one significance of our reduction method which was adopted in Chapter I exists in that, just as any previous consistency proofs, it supplies us with useful informations about the structure of the formal proofs of the concerning system. In fact by going through almost the same arguments as the case of the Π^1_1-comprehension

axiom, we can easily extend the results in [6] and [7] to the case of the provably Δ_2^1-comprehension axiom, by exploitation of our consistency proof.

Throughout this article, the acquaintance with the content of [5] is presupposed. Also we shall not repeat the references which are quoted in [5] but shall list only the more recent works in the related area.

Finally, we would like to take this opportunity to change some technical terms which were defined in [5]. Here is the glossary (the old terms are cited in the parentheses):

sequent (sequence)

abstract (variety)

auxiliary formula (subformula)

$\forall$ in the succedent ($\forall$ right) etc.

initial sequent (beginning sequence)

second order variable (f-variable)

and first order variable (t-variable).

CHAPTER I

The Ordinal of Second Order Arithmetic with the Provably Δ^1_2-Comprehension Axiom.

§1. A formulation of second order arithmetic with the provably Δ^1_2-comprehension axiom. We shall define a formal system of second order arithmetic with the provably Δ^1_2-comprehension axiom, say $P\Delta^1_2$, in the following manner. Let P be a (formal) proof of $P\Delta^1_2$ which is defined by the successive uses of the provably Δ^1_2-comprehension axiom at most n times, so to speak. Then we say that P is of dimension n (or, for short, dim n), and define D_n as the system of all proofs of $P\Delta^1_2$ which are of dim n. $P\Delta^1_2$ is then obtained as $\bigcup_n D_n$. Notice that D_o is actually the system with the Π^1_1-comprehension axiom (or SINN in [5]). Since all the proofs we deal with in this chapter are those of $P\Delta^1_2$, it is justifiable to simply talk about a proof of dim n, instead of somewhat lengthy expression like 'a proof of dim n with respect to the provably Δ^1_2-comprehension axiom'.

Definition 1.1. The system D_n which is the collection of the proofs of dim n, and the subsidiary proofs of dim i, $0 \leq i \leq n-1$, are defined by induction on n. D_o is the system SINN in Chapter 2 of [5], except that here we assume the quantifier $\exists$ on an f-variable as well as the related rules of inference. No subsidiary proof is involved in D_o. We assume that there is no substitution.

Suppose $D_o, \ldots, D_n$ have been defined in a manner that $D_o \subseteq D_1 \subseteq \ldots \subseteq D_n$ and suppose that for any given D_n-proof its subsidiary proofs of dim i, $0 \leq i \leq n-1$, are defined. Then we shall define D_{n+1}, the system of provably-Δ^1_2-proofs of dim n+1.

D_{n+1}-proofs are defined similarly to D_o-proofs except the non-semi-isolated comprehension axiom (i.e. $\forall$ left and $\exists$ right on an f-variable). The end sequent of such a proof is also called a main sequent.

Let P_o be a proof of D_{n+1} which has been already defined and which has the end sequent $A(V), \Gamma \rightarrow \Delta$ $(\Gamma \rightarrow \Delta, A(V))$, where V is a Π^1_2 or Σ^1_2-abstract, say $\{x\} \forall \varphi \exists \psi G(\varphi,\psi,x)$ or $\{x\} \exists \varphi \forall \psi F(\varphi,\psi,x)$ respectively. Suppose Q is a D_i-proof where $0 \leq i \leq n-1$ and whose end sequent is $\forall x(\forall \varphi \exists \psi G(\varphi,\psi,x) \equiv \exists \varphi \forall \psi H(\varphi,\psi,x))$, where H is arithmetical, or $\forall x(\exists \varphi \forall \psi F(\varphi,\psi,x) \equiv \forall \varphi \exists \psi H'(\varphi,\psi,x))$, where H' is arithmetical, respectively. Then a figure which is defined as

$$\left[Q, \quad \frac{P_o}{\forall \chi A(\chi), \Gamma \rightarrow \Delta} \right] \quad \left(\left[Q, \quad \frac{P_o}{\Gamma \rightarrow \Delta, \exists \chi A(\chi)} \right] \right)$$

is a proof of D_n. This is also called a proof of dim n. Note. More precisely the Q consists of two proofs Q_1 and Q_2, where the end sequent of Q_1 is $\forall x(\forall \varphi \exists \psi G(\varphi,\psi,x) \supset \exists \varphi \forall \psi H(\varphi,\psi,x))$ for the former case and $\forall x(\exists \varphi \forall \psi F(\varphi,\psi,x) \supset \forall \varphi \exists \psi H'(\varphi,\psi,x))$ for the latter case, while the end sequent of Q_2 is $\forall x(\exists \varphi \forall \psi H(\varphi,\psi,x) \supset \forall \varphi \exists \psi G(\varphi,\psi,x))$ for the former case and $\forall\ \chi A(\chi), \Gamma \rightarrow \Delta$ $(\Gamma \rightarrow \Delta, \exists\ \chi A(\chi))$ is called the end sequent of P. It is also called a main sequent of P. The main part of P consists of exactly the main sequents of P.

If i is the smallest number such that the Q above belongs to D_i (this can be decided effectively from Q), then Q is called a subsidiary proof (of P) of dim i. A subsidiary proof of Q of dim j is also called a subsidiary proof of P of dim j. Notice that for a Q as above, as a proof of D_i, we can talk about the main part of Q and the subsidiary proofs of Q, etc.

In the above definition, we may assume that G and H, and F and H', respectively contain exactly the same second order free variables (i.e. they actually occur). Also, we may assume that the eigen variables in subsidiary proofs do not occur in the main parts. Thus, we shall assume those restrictions on the variables throughout.

A proof of $P\Delta^1_2 = \bigcup_n D_n$ is called a $P\Delta^1_2$-proof.

Note. It is adequate to restrict the comprehension abstracts to semi-isolated ones and strictly Π^1_2- and Σ^1_2-ones.

§2. Transformations of the $P\Delta^1_2$-proofs. Given a $P\Delta^1_2$-proof, say P, we shall transform it gradually in four steps, say P^1, P^2, P^3, P^4 into a more convenient form for the reduction in a manner that $P\Delta^1_2$ is consistent if and only if the system which consists of P^4's as above is consistent. In most of those definitions we shall only outline how to carry out the transformations and list some consequences of the transformations.

This will be sufficient and convenient since the meaning of the transformations is intuitively clear but the precise definitions are lengthy and complicated.

Definition 1.2. Let P be a $P\Delta^1_2$-proof. The first transformation of P is defined as follows; we shall denote the resulting figure by P^1 or $T_1(P)$.

1) Change the proof so that all logical symbols are introduced by inferences.

2) The main part of P, say M(P), is transformed in a manner that if $A_1,\ldots,A_m \to B_1,\ldots,B_n$ is in M(P), then it is changed to $A'_1,\ldots,A'_m \to B'_1,\ldots,B'_n$, where A' is A or is obtained from A by changing some $\exists\varphi$ into $\neg\forall\varphi\neg$. Namely, suppose there is a $\exists$ on an f-variable in the antecedent such that the last descendent of its principal formula is not a comprehension-abstract. Let P be of the following form

$$P_0 \left\{ \begin{array}{c} \vdots \\ A(\alpha),\Gamma \to \Delta \end{array} \right. \quad I\ \frac{\qquad\qquad}{\begin{array}{c}\exists\varphi A(\varphi),\Gamma \to \Delta \\ \vdots\end{array}}\ ,$$

where I is the inference described as above, and suppose the main part of P_0 has been already transformed so that the end sequent of P_0 has turned to $A'(\alpha),\Gamma' \to \Delta'$. Then change I to:

$$\begin{array}{c} \vdots \\ A'(\alpha), \Gamma' \rightarrow \Delta' \\ \hline \Gamma' \rightarrow \Delta', \neg A'(\alpha) \\ \hline \Gamma' \rightarrow \Delta', \forall \varphi \neg A'(\varphi) \\ \hline \neg \forall \varphi \neg A'(\varphi), \Gamma' \rightarrow \Delta' \end{array} .$$

Similarly for $\exists$ on an f-variable in the succedent. Notice that if the auxiliary formula of I in P is A(V), then the corresponding formula in the transformed sequent is A'(V) (V unchanged). Thus, for example,

$$\begin{array}{c} \left[Q, \quad P'_o \left\{ \begin{array}{c} \vdots \\ \Gamma' \rightarrow \Delta', A'(V') \\ \hline \neg A'(V'), \Gamma' \rightarrow \Delta' \\ \hline \forall \varphi \neg A'(\varphi), \Gamma' \rightarrow \Delta' \end{array} \right. \right] \\ \hline \Gamma' \rightarrow \Delta', \neg \forall \varphi \neg A'(\varphi) \end{array} .$$

3) Let Q be a subsidiary proof of P. Then the main part of Q is changed as in 2) except that if the last descendent of $\exists \varphi A(\varphi)$ is in the end sequent of Q, then do not change I.

With 1) - 3), the transformation of P, $T_1(P)$ is completed.

<u>Corollary</u>. For any D_n-proof P, P^1, or $T_1(P)$, satisfies the following properties.

(1) The comprehension abstracts are semi-isolated, Π^1_2 or Σ^1_2. The relation between the comprehension abstract V and the end sequent of the related subsidiary proof satisfy the relation as for P. Therefore, P^1 is a special case of D_n-proofs.

(2) The second order $\exists$ is introduced by an inference.

(3) If a $\exists\varphi A(\varphi)$ occurs in the main part of P^1, then either $A(\varphi)$ is of the form $\forall\psi B(\varphi,\psi)$, where $B(\varphi,\psi)$ is arithmetical, and there is a descendent of $\exists\varphi A(\varphi)$ which has the identical form as $\exists\varphi A(\varphi)$ (up to some terms) and is used as a comprehension-abstract, or $A(\varphi)$ is arithmetical and is of the form $A(V,\varphi)$ and there is an introduction of second order $\forall$, thus resulting in $\forall\psi\exists\varphi A(\varphi,\psi)$ and there is a descendent of it which is used as a comprehension abstract.

(4) If a $\exists\varphi A(\varphi)$ occurs in the main part of a subsidiary proof Q' of P', then a similar situation as in (3) holds except that the last descendent of $\exists\varphi A(\varphi)$ may occur in the end sequent of Q'. The end sequent of Q' is the same as that of Q.

Proof. First prove a number of T_1-invariant properties of P and Q according to the definition of T_1. Then (1), (3) and (4) follow as the corollaries. (2) is obvious.

Proposition 1.1. Let $(P\Delta_2^1)^1$ be the system of $T_1(P)$'s for all P of $P\Delta_2^1$. Then $P\Delta_2^1$ is consistent iff $(P\Delta_2^1)^1$ is consistent.

Definition 1.3. Second transformation, i.e. the transformation of P^1, or $T_1(P)$, say T_2. We shall denote $T_2(P^1)$, or $T_2(T_1(P))$ by P^2.

1) First change the main part of P^1 as follows. Let I_1 be a second order $\forall$ (in the antecedent or the succedent) such that its principal formula is of the form $\forall\psi F(V,\psi)$, where

$F(\alpha,\psi)$ is arithmetical and there is a descendent of the form $\exists\varphi\forall\psi F(\varphi,\psi)$. Take, as an example, $\forall$ in the succedent:

$$I_1 \quad \frac{\Gamma \to \Delta, F(V,\alpha)}{\Gamma \to \Delta, \forall\psi F(V,\psi)} .$$

Change I_i as follows.

$$\frac{\dfrac{\Gamma' \to \Delta', F(V,\alpha)}{\neg F(V,\alpha), \Gamma' \to \Delta'}}{\exists\psi \neg F(V,\psi), \Gamma' \to \Delta'} .$$

Let I_2 be a second order $\exists$ (in the antecendent or the succedent) such that the principal formula of I_2 is of the form $\exists\varphi\forall\psi F(\varphi,\psi)$, where $F(\varphi,\psi)$ is arithmetical. Consider the case where I_2 is a $\exists$ in the succedent:

$$I_2 \quad \left[\ Q^1, \quad \frac{\Gamma \to \Delta, \forall\psi F(V,\psi)}{\Gamma \to \Delta, \exists\varphi\forall\psi F(\varphi,\psi)}\ \right] .$$

Notice that the upper sequent of I_2 has been changed to $\exists\psi \neg F(V,\psi), \Gamma' \to \Delta'$. Then define a proof as:

$$\frac{\left[\ Q^1, \quad \dfrac{\exists\psi \neg F(V,\psi), \Gamma' \to \Delta'}{\forall\varphi\exists\psi \neg F(\varphi,\psi), \Gamma' \to \Delta'}\ \right]}{\Gamma' \to \Delta' \quad \neg\forall\varphi\exists\psi \neg F(\varphi,\psi)} .$$

(Of course we must prove that I_2 has the above form. Actually we should define $T_2(R)$ for every subproof R of P^1, and show that except the cases I_1 and I_2 $T_2(R)$ is a 'copy' of R.)

2) Let Q^1 be a subsidiary proof of P^1. Then the transformation on the main part of Q^1 is defined as in 1) except that the ancestors of the formulas in the end sequent of Q^1 are untouched. P^2, or $T_2(P^1)$, is defined as the figure which is obtained from P^1 by 1) and 2) above. $T_2(Q^1)$, or Q^2, is called a subsidiary proof of P^2.

Corollary. For any $(P\Delta^1_2)^1$-proof P^1, $T_2(P^1)$, or P^2, satisfies the following properties.

(1) The comprehension abstracts are either semi-isolated, of the strictly Π^1_2-form or of the form $\neg A$, where A is of the strictly Π^1_2-form.

(2) A second order $\exists$ is introduced by an inference.

(3) If a $\exists \varphi A(V,\varphi)$ occurs in the main part of P^2, then $A(\alpha,\varphi)$ is arithmetical and there is a descendent of $\exists \varphi A(V,\varphi)$ of the form $\forall \psi \exists \varphi A(\psi,\varphi)$ or $\neg\forall \psi \exists \varphi A(\psi,\varphi)$ which is used as a comprehension abstract.

(4) If a $\exists \varphi A(\varphi)$ occurs in the main part of a subsidiary proof Q^2 of P^2, then either the same situation as in (3) holds or the last descendent of $\exists \varphi A(\varphi)$ occurs in the end sequent of Q^2. The end sequent of Q^2 is the same as that of Q^1.

Proof. First prove a number of T_2-invariant properties of P^1 and Q^1 according to the definition of T_2. Then (1), (3) and (4) follow as corollaries.

Proposition 1.2. Let $(P\Delta^1_2)^2$ be the system of all P^2's for all P^1 in $(P\Delta^1_2)^1$. Then $(P\Delta^1_2)^1$ is consistent iff $(P\Delta^1_2)^2$ is consistent.

Definition 1.4. The third transformation, T_3. Transform each subproof R in the main part of a proof P^2 of $(P\Delta_2^1)^2$ into $\tilde{R}$ as follows by induction on the number of inferences in R. If S is a sequent in R and $A(V)$ is a formula with indicated occurrences of V, then $\tilde{S}$ and $A(V)$ in $\tilde{R}$ are defined in a manner that $\tilde{S}$ consists of the formulas $\tilde{B}$ corresponding to the formulas of S, say B, and $\widetilde{A(V)}$ is $\tilde{A}(\tilde{V})$, where $\tilde{V}$ is V if V is semi-isolated and a Σ_2^1-abstract if V is Π_2^1. Moreover B and $\tilde{B}$ have exactly the same free variables, and the end sequent of $\tilde{P}^2$ is the same as that of P^2. First we define $\sim$ operation to all subsidiary proofs, say X, of dim i in P^2 by induction on i. $i = 0$. X is a SINN proof. Therefore define $\tilde{X}$ as X itself.

Assume that $\sim$ has been defined for $i \leq m$. Suppose X is a subsidiary proof of dim $m + 1$ in P^2. Let R be a subproof of the main part of X. Define $\tilde{R}$ by induction on the number of inferences in $\tilde{R}$ in the same manner as was described for P^2. Recall that for any subsidiary proof in X, $\sim$ has been already defined, since its dimension is less than the dimension of X.

1) The initial sequents remain unchanged.

Suppose R is of the form

$$I\ \frac{R_1}{S} \quad \text{or} \quad I\ \frac{R_1 R_2}{S}\ .$$

2) I is not second order $\forall$ or $\exists$. Then $\tilde{R}$ is

$$I\ \frac{\tilde{R}_1}{\tilde{S}} \quad \text{or} \quad I\ \frac{\tilde{R}_1 \tilde{R}_2}{\tilde{S}}$$

respectively.

3) I is second order $\forall$ in the antecedent and R is of the form:

$$\left[Q^2, \qquad \frac{R_1 \left\{ \begin{array}{c} \vdots \\ A(V), \Gamma \rightarrow \Delta \end{array} \right.}{\forall \varphi A(\varphi), \Gamma \rightarrow \Delta} \right]$$

$\tilde{R}_1$ is already defined and its end sequent is $\tilde{A}(\tilde{V}), \tilde{\Gamma} \rightarrow \tilde{\Delta}$. First define R' as

$$\frac{\tilde{R}_1}{\forall \varphi \tilde{A}(\varphi), \tilde{\Gamma} \rightarrow \tilde{\Delta}} .$$

If $\forall \varphi A(\varphi)$ is of the form $\forall \varphi \exists \psi B(\varphi, \psi, s)$ and there is a Q^* in X whose end sequent is of the form $\forall x(\exists \varphi \forall \psi C(\varphi, \psi, x) \equiv \forall \varphi \exists \psi B(\varphi, \psi, x))$ or $\forall x(\exists \varphi \forall \psi B_1(\varphi, \psi, x) \equiv \forall \varphi \exists \psi D(\varphi, \psi, x))$, where B is $\neg B_1$, then define $\tilde{R}$ as follows. Recall that Q^* consists of two proofs, Q_1^* and Q_2^*, and $\tilde{Q}_1^*$ and $\tilde{Q}_2^*$ have been defined. For the first case $\tilde{R}$ is defined as

$$\frac{\widetilde{Q_2(s)} \left\{ \begin{array}{c} \vdots \\ \exists \varphi \forall \psi C(\varphi, \psi, s) \rightarrow \forall \varphi \exists \psi B(\varphi, \psi, s) \end{array} \right. \qquad R' \left\{ \begin{array}{c} \vdots \\ \forall \varphi \exists \psi B(\varphi, \psi, s), \Gamma' \rightarrow \Delta' \end{array} \right.}{\exists \varphi \forall \psi C(\varphi, \psi, s), \tilde{\Gamma} \rightarrow \tilde{\Delta}} ,$$

where $Q_2(s)$ is determined from Q_2 in an obvious manner. Recall that B and C have the same second order free variables. For the second case, $\tilde{R}$ is defined as follows. First change the main part of $\widetilde{Q_1}(s)$, i.e.

$$\exists \varphi \forall B_1(\varphi, \psi, s) \overset{\vdots}{\rightarrow} \forall \varphi \exists \psi D(\varphi, \psi, s)$$

in order to obtain

$$\exists\, \varphi\, \forall\, \psi \neg D(\varphi,\psi,s) \overset{\vdots}{\rightarrow} \forall\, \varphi\, \exists\, \psi \neg B_1(\varphi,\psi,s)$$

by copying $\tilde{Q}_1(s)$. Call the resulting figure $\tilde{\tilde{Q}}_1(s)$. $\tilde{R}$ is now defined as

$$\frac{\tilde{\tilde{Q}}_1(s)\left\{\exists\varphi\forall\psi\neg D(\varphi,\psi,s) \overset{\vdots}{\rightarrow} \forall\varphi\exists\psi\neg B_1(\varphi,\psi,s) \qquad R'\right\{ \forall\varphi\exists\psi B(\varphi,\psi,s),\Gamma' \overset{\vdots}{\rightarrow} \Delta'}{\exists\varphi\forall\psi\; D(\varphi,\psi,s),\Gamma' \rightarrow \Delta'} .$$

If the above condition is not satisfied, then define $\tilde{R}$ to be R'.

4) I is second order $\forall$ in the succedent:

$$I \quad \frac{R_1 \left\{ \Gamma \overset{\vdots}{\rightarrow} \Delta,\; A(\alpha)\right.}{\Gamma \rightarrow \Delta,\; \forall\; \varphi A(\varphi)} .$$

First define R' as

$$\frac{\tilde{R}_1}{\tilde{\Gamma} \rightarrow \tilde{\Delta},\; \forall\; \varphi\tilde{A}(\varphi)} .$$

Suppose $\forall\; \varphi A(\varphi)$ is of the form $\forall\; \varphi\; \exists\; \psi B(\varphi,\psi)$ and there is a Q^* in X as in 3). For the first case (see 3)) $\tilde{R}$ is defined as:

$$\frac{R'\left\{\Gamma' \overset{\vdots}{\rightarrow} \Delta',\forall\; \varphi\; \exists\; \psi B(\varphi,\psi,s) \qquad \tilde{Q}_1(s)\right\{ \forall\; \varphi\; \exists\; \psi B(\varphi,\psi,s) \overset{\vdots}{\rightarrow} \exists\; \varphi\; \forall\; \psi C(\varphi,\psi,s)}{\tilde{\Gamma} \rightarrow \tilde{\Delta},\exists\; \varphi\; \forall\; \psi C(\varphi,\psi,s)} .$$

For the second case $\tilde{R}$ is defined as:

$$\dfrac{R' \left\{ \begin{array}{c} \vdots \\ \Gamma' \to \Delta', \forall\varphi\exists\psi B(\varphi,\psi,s) \end{array} \right. \qquad \tilde{\tilde{Q}}_2(s) \left\{ \begin{array}{c} \vdots \\ \forall\varphi\exists\psi B(\varphi,\psi,s) \to \exists\varphi\forall\psi \neg D(\varphi,\psi,s) \end{array} \right.}{\tilde{\Gamma} \to \tilde{\Delta}, \exists\varphi\forall\psi\neg D(\varphi,\psi,s)}$$

where $\tilde{\tilde{Q}}_2(s)$ is obtained from $\tilde{Q}_2(s)$ as in 3). If this is not the case, then define $\tilde{R}$ as R'.

5) I is second order $\exists$ (in the antecedent or the succedent).

Recall that any Σ^1_2-formula has a descendent in the end sequent of X in the same form. Let R be of the form

$$\dfrac{R_1 \left\{ \begin{array}{c} \vdots \\ \Gamma \to \Delta, A(V) \end{array} \right.}{\Gamma \to \Delta, \exists\varphi A(\varphi)} ,$$

for example. Then define $\tilde{R}$ as

$$\dfrac{\tilde{R}_1 \left\{ \begin{array}{c} \vdots \\ \tilde{\Gamma} \to \tilde{\Delta}, \tilde{A}(\tilde{V}) \end{array} \right.}{\tilde{\Gamma} \to \tilde{\Delta}, \exists\varphi\tilde{A}(\varphi)} .$$

$\tilde{P}_2$, i.e. the transformation of the main part of P_2 is defined just as the transformation of X above, except that here the 'if' clause necessarily holds. Define P^3, or $T_3(P^2)$, to be $\tilde{P}^2$ and call the system of $T_3(P^2)$ for all $(P\Delta^1_2)^2$-proofs P^2 $(P\Delta^1_2)^3$.

Corollary. The proofs of $(P\Delta^1_2)^3$ satisfy the following properties.

(1) P^3 is a proof of second order arithmetic in the ordinary formulation.

(2) The second order existential quantifiers are introduced by inferences.

(3) A formula of the form $\forall \varphi \exists \psi G(\varphi,\psi)$ is cut out in the same form, while a formula of the form $\exists \varphi \forall \psi F(\varphi,\psi)$ has a descendent which is a comprehension abstract.

(4) The comprehension abstracts are semi-isolated, or of the form $\{x\} \exists \varphi \forall \psi B(\varphi,\psi,x)$ or $\{x\}\neg\exists \varphi \forall \psi B(\varphi,\psi,x)$, which is determined by P^2 (hence by P).

Proof. Notice that the main part of a P (a $P\Delta^1_2$-proof) or of one of its subsidiary proofs is a proof of second order arithmetic with formally Σ^1_2- and Π^1_2-comprehension abstracts. This implies that the main part of a P^2 or Q^2, where Q is a subsidiary proof of P, is a proof of second order arithmetic with formally Π^1_2-comprehension abstracts. Also, the second order existential quantifiers are introduced by inferences in P^2.

(1) and (2): From the above remark: by induction onthe number of inferences in R.

(3) and (4): In the first sequent in which a formula of the form $\forall \varphi \exists \psi G(\varphi,\psi)$ is introduced, it is cut out by the definition of $\tilde{R}$, and is replaced with a formula of the form $\exists \varphi \forall \psi F(\varphi,\psi)$. If a formula B has a part $\forall \varphi \exists \psi G$ in P, then its corresponding formula in P^3 has a part $\exists \varphi \forall \psi F$ correspondingly. The Σ^1_2-form does not occur in any other way. All this is proved by induction on the number of inferences in R.

Therefore, comprehension abstracts in P^3 are semi-isolated or Σ^1_2 or $\neg\Sigma^1_2$, since those in P^2 are semi-isolated or Π^1_2 or $\neg\Pi^1_2$. Also, as every Π^1_2-formula in P^2 except those in the end sequents of subsidiary proofs has a descendent which is used as a comprehension abstract, every Σ^1_2-formula in P^3 has a descendent which is used as a comprehension abstract.

Proposition 1.3. Call the system of $T_3(P^2)$'s for all P^2 of $(P\Delta^1_2)^2$, $(P\Delta^1_2)^3$. Then $(P\Delta^1_2)^2$ is consistent if and only if $(P\Delta^1_2)^3$ is.

Definition 1.5. The fourth transformation T_4, i.e. the transformation of the $(P\Delta^1_2)^3$-proofs, say P^3, changes a formula of the form $\forall\varphi\,\exists\psi B(\varphi,\psi)$ to $\forall\varphi\neg\forall\psi\neg B(\varphi,\psi)$ throughout P^3. How to change P^3 to fit this condition should be selfevident. Recall that second order $\exists$ are introduced by inferences only. The system of T_4 (P^3)'s for all P^3 is called $(P\Delta^1_2)^4$.

Proposition 1.4. $(P\Delta^1_2)^3$ is consistent if and only if $(P\Delta^1_2)^4$ is.

Theorem 1.1. In order to prove the consistency of $P\Delta^1_2$ it suffices to prove the consistency of $(P\Delta^1_2)^4$.

§3. The reducible proofs with degree.

In order to prove the consistency of $(P\Delta^1_2)^4$, we shall first abstract the characteristic properties which the proofs of $(P\Delta^1_2)^4$ possess. We end up with the notion of reducible proofs with degree.

Definition 1.6. Let $\mathfrak{F}$ be a set of abstracts such that no abstract in $\mathfrak{F}$ contains any first order free-variable and $\mathfrak{F}$ contains all atomic abstracts. An abstract or a formula A is called an $\mathfrak{F}$-abstract or an $\mathfrak{F}$-formula if one of the following is the case.

1) A is an abstract of $\mathfrak{F}$ or, if A is a formula of the form $A(s_1,s_2,\ldots,s_n)$, where $\{x_1,\ldots,x_n\}A(x_1,\ldots,x_n)$ is an abstract of $\mathfrak{F}$ and $s_1,\ldots,s_n$ are terms respectively.

2) A is semi-isolated.

3) A is obtained by several applications of substitution, starting from formulas and abstracts of 1) and 2).

Note. For the sake of simplicity we deal with abstracts of one argument only.

Definition 1.7. For each i such that $1 \leq i \leq n$ let $\mathfrak{F}_i$ be a finite set of Σ^1_2-abstracts. Let $\mathfrak{F} = \mathfrak{F}_1 \cup \ldots \cup \mathfrak{F}_n$. Then an $\mathfrak{F}$-formula or an $\mathfrak{F}$-abstract is called $\mathfrak{F}$-reducible (of dim n).

Definition 1.8. A formula or an abstract A is called $\mathfrak{F}$-admissible if it satisfies one of the following.

1) A is $\mathfrak{F}$-reducible.

2) A contains no second order $\exists$.

3) Let $A(\alpha)$ be $\mathfrak{F}$-admissible and V be $\mathfrak{F}$-reducible. Then A(V) is admissible.

Note. 1) We omit '$\mathfrak{F}$-' once we shall have fixed $\mathfrak{F}$ and say simply 'reducible (of dim n)' or 'admissible'.

2) Notice that in 3) above, V must be reducible. This means that a non-reducible abstract cannot be substituted into a reducible formula.

3) We did not require that $\mathfrak{F}_i$ and $\mathfrak{F}_j$ be mutually exclusive when i and j are distinct. In order to distinguish $\mathfrak{F}_i$'s for distinct i's, we assume that an abstract of $\mathfrak{F}_i$ is indexed by i; (i,V), for example, although we do not write the indices explicitly.

Assumption. In the following we shall consider only the admissible (relative to some $\mathfrak{F}$) formulas and abstracts. Thus we shall not mention it at each time. A proof of second order arithmetic with ($\mathfrak{F}$-)admissible formulas only is called an ($\mathfrak{F}$-) admissible proof.

Now we can relate those definitions to the system $(P\Delta_2^1)^4$ in §2. Let P be a proof of D_n. We shall define $\mathfrak{F}_1,\ldots,\mathfrak{F}_n$ as follows. For every i such that $1 \leq i \leq n$, let Q be a subsidiary proof of dim i whose end sequent is, say, $\forall x(\exists \varphi \forall \psi F(\varphi,\psi,x) \equiv \forall \varphi \exists \psi G(\varphi,\psi,x))$. Then $\{xy_1\ldots y_n\} \exists \varphi \forall \psi \tilde{F}(\varphi,\psi,x)$ and $\{xz_1\ldots z_n\} \exists \varphi \forall \psi \neg \tilde{G}(\varphi,\psi,x)$ belong to $\mathfrak{F}_i$, where $\tilde{F}(\varphi,\psi,x)$ is obtained from $F(\varphi,\psi,x)$ by replacing all first order free variables by (distinct) bound variables $y_1,\ldots,y_m$; similarly with $\tilde{G}(\varphi,\psi,x)$. Only those kinds of abstracts belong to $\mathfrak{F}_i$. Define $\mathfrak{F} = \mathfrak{F}_1 \cup \ldots \cup \mathfrak{F}_n$.

Proposition 1.5. Given a proof P of D_n and let $\mathfrak{F}$ be the related set of abstracts defined as above. Then P^4 consists of $\mathfrak{F}$-admissible formulas only and the comprehension abstracts are $\mathfrak{F}$-reducible.

Next we shall define some functions from quasi- or semi-formulas to numbers (natural or ordinal).

Definition 1.9. The function δ_i from quasi-formulas to natural numbers is defined as follows for each i such that $1 \leq i \leq n$.

1) $\delta_i(A) = 0$ if A does not involve any abstract from $\mathfrak{F}_i$.

2) $\delta_i(A \wedge B) = \max(\delta_i(A), \delta_i(B))$; $\delta_i(\neg A) = \delta_i(A)$; $\delta_i(\forall x\, A(x)) = \delta_i(A(x))$; $\delta_i(\forall \varphi A(\varphi)) = \delta_i(A(\varphi))$.

3) $\delta_i(\exists \varphi A(\varphi)) = 0$ if $\exists \varphi\, A(\varphi)$ is of the form $\exists \varphi \forall \psi F(\varphi, \psi, V_1, \ldots, V_k, s)$, where $\{x\} \exists \varphi \forall \psi F(\varphi, \psi, \beta_1, \ldots, \beta_k, x)$ for some $\beta_1, \ldots, \beta_k$, x belongs to $\mathfrak{F}_j$ for j such that $i + 1 \leq j \leq n$.

$= \delta_i(A(\varphi)) + 1$ if the above abstract belongs to $\mathfrak{F}_i$,

$= \delta_i(A(\varphi))$ if the above abstract belongs to $\mathfrak{F}_j$ for a j such that $1 \leq j < i$.

Note. We shall call such an abstract as in 3) the type of $\exists \varphi A(\varphi)$.

4) $\delta_i(\{x\}A(x)) = \delta_i(A(x))$.

Note. The above definition is complete, since we deal with admissible formulas only and we assume that each abstract of an $\mathfrak{F}_j$ is indexed by j, so that in defining $\delta_i(\exists \varphi A(\varphi))$ we can uniquely determine the j such that the abstract which is mentioned there belongs to $\mathfrak{F}_j$.

Definition 1.10. The function λ from quasi-formulas to natural numbers is defined as follows. $\lambda(A) = 0$ if A is atomic; $\lambda(A \wedge B) = \max(\lambda(A), \lambda(B)) + 1$; $\lambda(\quad A) = \lambda(A) + 1$; $\lambda(\forall\, x\, A(x)) = \lambda(A(x)) + 1$; $\lambda(\forall\, \varphi A(\varphi)) = \lambda(A(\varphi)) + 1$; $\lambda(\exists\, \varphi A(\varphi)) = 0$; $\lambda(\{x\}A(x)) = \lambda(A(x))$.

Note. We may regard λ as a δ of dim 0, or δ_0.

Definition 1.11. 1) The function δ is defined as

$$\delta(A) = \omega^n \cdot \delta_n(A) + \omega^{n-1} \cdot \delta_{n-1}(A) + \ldots + \omega \cdot \delta_1(A).$$

2) The function 1 from quasi-formulas to ω^{n+1} is defined as $1(A) = \delta(A) + \lambda(A)$.

Definition 1.12. Let i be any number such that $1 \leq i \leq n$. The function dp_i from (second order free variables; quasi-formulas) to natural numbers, is defined as follows.

1) Let $\underline{\beta}$ be an indicated occurrence of β in a quasi-formula A. Then $dp_i(\underline{\beta};A)$ is defined as follows, where we assume that $\underline{\beta}$ actually occurs in A; otherwise $dp_i(\underline{\beta};A) = 0$.

$dp_i(\underline{\beta};A) = 0$ if no abstract of $\mathfrak{F}_i$ is involved in A.

$dp_i(\underline{\beta}; \neg A) = dp_i(\underline{\beta};A)$; $dp_i(\underline{\beta};\forall\, x\, A(x)) = dp_i(\underline{\beta};A(x))$;

$dp_i(\underline{\beta};\forall\, \varphi A(\varphi)) = dp_i(\underline{\beta};A(\varphi))$; $dp_i(\underline{\beta};B \wedge C) = \max(dp_i(\underline{\beta};B), dp_i(\underline{\beta};C))$;

$$dp_i(\underline{\beta};\exists\, \varphi A(\varphi)) = \begin{cases} 0 & \text{if } \exists\, \varphi A(\varphi) \text{ is of the form } \exists\, \varphi\, \forall\, \psi F(\varphi,\psi) \\ & \text{and the type of } \exists\, \varphi A(\varphi) \text{ belongs to } \mathfrak{F}_j \\ & \text{for some } j \text{ such that } i + 1 \leq j \leq n; \\ dp_i(\underline{\beta};A(\varphi)) + 1 & \text{if the type of } \exists\, \varphi A(\varphi) \\ & \text{belongs to } \mathfrak{F}_i; \end{cases}$$

$= dp_i(\underline{\beta};A(\varphi))$ if the type of $\exists \varphi A(\varphi)$ belongs to $\mathfrak{F}_j$ for some j such that $1 \leq j < i$.

2) $dp_i(\beta;A)$ is now defined as

$$dp_i(\beta;A) = \max_{\underline{\beta} \text{ for all occurrences of } \beta \text{ in } A} dp_i(\underline{\beta};A).$$

3) $dp_i(\beta;\{x\}A(x)) = dp_i(\beta;A(x))$, of β in A.

4) $dp(\beta;A) = \omega^n \cdot dp_n(\beta;A) + \omega^{n-1} \cdot dp_{n-1}(\beta;A) + \ldots + \omega \cdot dp_1(\beta;A)$.

Definition 1.13. Fix an admissible proof P. The grade of an occurrence of a formula is defined as follows.

1) The γ-degree (relative to a class of reducible formulas) is defined similarly to the γ-degree relative to the class of semi-isolated formulas. Namely $\gamma(A) = 0$ if A is reducible. Suppose now that the concerning quasi-formulas are non-reducible. Then $\gamma(A \wedge B) = \max(\gamma(A), \gamma(B)) + 1$; $\gamma(\neg A) = \gamma(A) + 1$; $\gamma(\forall xA(x)) = \gamma(A(x)) + 1$; $\gamma(\forall \varphi A(\varphi)) = \gamma(A(\varphi)) + 1$. $\gamma(\{x\}A(x)) = \gamma(A(x))$. Notice that this completes the definition since we are concerned with the admissible formulas only.

2) $v(P;A)$, or $v(A)$, is the number of eigen variables of a second order $\forall$ in the succedent, a second order $\exists$ in the antecedent which occur under the occurrence of A in P.

3) The grade of A, $g(A;P)$, or $g(A)$ for short, is defined to be $\omega^{n+2} \cdot \gamma(A) + \omega^{n+1} \cdot v(A) + \ell(A)$.

Definition 1.14. Some terminologies. The following are all considered in an admissible proof.

1) Let $\forall \varphi A(\varphi)$ be the principal formula of a second order $\forall$ in the succedent, say I. I is called semi-isolated, non-semi-isolated, reducible or non-reducible, according as $\forall \varphi A(\varphi)$ is semi-isolated, non-semi-isolated, reducible or non-reducible respectively.

2) Let

$$K \quad \frac{\Gamma \rightarrow \Theta, \forall \psi F(U,\psi)}{\Gamma \rightarrow \Theta, \exists \varphi \forall \psi F(\varphi,\psi)}$$

be an inference (in a proof) 'second order $\exists$ in the succedent'. K is then called a key inference and U is called a key abstract. We may also call the $\forall \psi F(U,\psi)$ and the $\exists \varphi \forall \psi F(\varphi,\psi)$ a key auxiliary formula and a key principal formula respectively. Notice that we are talking about the occurrences of $K, U, \forall \psi F(U,\psi)$ and $\exists \varphi \forall \psi F(\varphi,\psi)$ in an admissible proof.

3) If the type of the principal formula of an inference I belongs to $\mathfrak{F}_i$, then I is said to be of dim i.

Proposition 1.6. $\delta_i(A(U)) = \max(\delta_i(A(\gamma)), \delta_i(U) + dp_i(\gamma;A))$ if γ actually occurs in A and $dp_k(\gamma;A) = 0$ for all k such that $i + 1 \leq k \leq n$. (In the following we shall often omit the upper bound n for such k.)

Proof. By induction on the construction of $A(\gamma)$.

If $A(\gamma)$ is γ (or $\gamma(t)$), then $\delta_i(A(U)) = \delta_i(U)$ and $\delta_i(A(\gamma)) = 0 = dp_i(\gamma;A)$.

If $A(\gamma)$ is $B(\gamma) \wedge C(\gamma)$, then

$$\delta_i(A(U)) = \max(\delta_i(B(U)),\delta_i(C(U))) . \qquad (*)$$

Since $dp_k(\gamma;A) = 0$ for $k \geq i+1$, $dp_k(\gamma;B) = dp_k(\gamma;C) = 0$ for all such k.

Case 1) γ occurs both in B and in C.

$$(*) = \max(\max(\delta_i(B(\gamma)),\delta_i(U) + dp_i(\gamma;B(\gamma))),$$
$$\max(\delta_i(C(\gamma)),\delta_i(U) + dp_i(\gamma;C(\gamma))))$$

by induction hypothesis,

$$= \max(\delta_i(B(\gamma)),\delta_i(C(\gamma)),\delta_i(U) + dp_i(\gamma;B(\gamma)),\delta_i(U) + dp_i(\gamma;C(\gamma)))$$

$$= \max(\max(\delta_i(B(\gamma)),\delta_i(C(\gamma))),$$
$$\delta_i(U) + \max(dp_i(\gamma;B(\gamma)),dp_i(\gamma;C(\gamma)))) \qquad (**)$$

Case 2) B contains γ but C does not.

$$(*) = \max(\max(\delta_i(B(\gamma)),\delta_i(U) + dp_i(\gamma;B)),\delta_i(C))$$

by induction hypothesis,

$= (**)$ (without γ in C), since $dp_i(\gamma;A) = dp_i(\gamma;B)$.

Case 3) C contains γ but B does not. Similarly.

In any case $(**) = \max(\delta_i(A),\delta_i(U) + dp_i(\gamma;A))$.

For other logical symbols, the proposition follows immediately from the induction hypothesis.

If $A(\gamma)$ is of the form $\exists\varphi\forall\psi A(\varphi,\psi,\gamma)$, then the type of $A(\gamma)$ does not belong to an $\mathfrak{F}_k$ for $k \geq i+1$, since otherwise $dp_\ell(\gamma;A) > 0$ for some $j \geq k \geq i+1$, contradicting the assumption. Suppose $A(\gamma)$ is of the form $\exists\varphi\forall\psi A(\varphi,\psi,\gamma)$ and its type belongs to $\mathfrak{F}_i$. Then

$$\begin{aligned}\delta_i(A(U)) &= \delta_i(\forall\ \psi A(\varphi,\psi,U)) + 1\\ &= \max(\delta_i(\forall\ \psi A(\varphi,\psi,\gamma) + 1),\\ &\qquad \delta_i(U) + dp_i(\gamma;\forall\ \psi A(\varphi,\psi,\gamma)) + 1)\end{aligned}$$

by induction hypothesis,

$$= \max(\delta_i(A)\ , \delta_i(U) + dp_i(\gamma;A)).$$

Suppose the type of A as above belongs to a $\mathfrak{F}_k$, where $1 \leq k < i$. Then a similar argument as for the case $k = i$ goes through without '+1'.

For any other case, the proposition is proved easily.

The following proposition is proved similarly to Proposition 1.6.

Proposition 1.7. $dp_i(\beta;A(U)) = \max(dp_i(\beta;A(\gamma)),$
$dp_i(\beta;U) + dp_i(\gamma;A(\gamma)))$

if γ actually occurs in $A(\gamma)$, β actually occurs in U and $dp_k(\gamma;A(\gamma)) = 0$ for $i + 1 \leq k \leq n$.

Let us now fix n and an $\mathfrak{F} = \mathfrak{F}_1 \cup \ldots \cup \mathfrak{F}_n$, a finite set of Σ_2^1-abstracts (cf. Definition 1.7).

Definition 1.15. A proof of second order arithmetic in the tree form formulation with $G^1 L C$ as its logical basis (see [5] for the precise definition) (including substitution as one of the rules of inference) is called ($\mathfrak{F}$-) reducible if it satisfies the following.

1) The proof consists of ($\mathfrak{J}$-) admissible formulas only.

2) The comprehension abstracts are reducible.

A reducible proof is said to be of dim n if $F = F_1 \cup \ldots \cup F_n$ as above.

Corollary. In a reducible proof, every second order $\exists$ in the succedent is a key inference.

Definition 1.16. Consider a (reducible) proof P. A formula in P, say A, is called a direct descendent of B if A is a descendent of B and no logical inference applies to any descendent of B above A.

We shall now define the proofs with degree.

Definition 1.17. The notion of the proofs with degree consists of two conditions which are imposed on the reducible proofs; the condition on blocks and the condition on degree. In the following i,j, etc. denote any number $\leq$ n.

1) The axioms on blocks. (A block is mostly denoted by B.) A set of (occurrences of) formulas in a reducible proof (of dim n) is called an i-block if it satisfies the following conditions B1-B5.

B1. An i-block B has certain closure properties. Namely:

B1.1. If a formula belongs to B, then all its ancestors belong to B.

B1.2. Let D_1 and D_2 be a pair of formulas which satisfies the following: D_1 and D_2 are the left and the right D respectively in a logical beginning sequent $D \rightarrow D$, the $A(s)$ and the $A(t)$ respectively in an equality axiom $s = t, A(s) \rightarrow A(t)$, or the left and the right auxiliary formula respectively of an induction. Then D_1 belongs to B if and only if D_2 belongs to B.

B1.3. If a (the) auxiliary formula of a logical inference belongs to B, then its principal formula belongs to B.

B2. An i-block B excludes some formulas. Namely:

B2.1. If $i \leq j$, then the auxiliary formula of a second order $\exists$ in the antecedent of dim j (cf. Definition 1.14) does not belong to B.

B2.2. Let j be any number such that $i + 1 \leq j \leq n$. The auxiliary formula of a key inference of dim j (cf. Definition 1.14) does not belong to B.

B2.3. Let G be the auxiliary formula of a key inference of dim i and C be a descendent of G such that C is the auxiliary formula of a strong inference (i.e. a logical inference, an induction, a cut or a substitution). Then C does not belong to B.

B3. Suppose a second order eigen variable α which is not a substitution variable occurs both inside and outside an i-block B. If the key principal formula, say F, of a key inference of dim i, belongs to B then α occurs in F.

B4. In an i-block B, some inequality relations hold for δ and dp. Let A be any semi-formula in B which is not a key principal formula of dim i or a descendent of a key principal formula of dim i, and which is not in the end piece of P. It should be noted that A may be a (proper) subformula of a key principal formula of dim i. Suppose F is a key principal formula of dim i which belongs to B. Then the following three conditions hold.

B4.1. $\delta_i(A) < \delta_i(F)$ and $\delta_j(A) \leq \delta_j(F)$ for all j such that $i + 1 \leq j \leq n$.

B4.2. Suppose α is a substitution variable which occurs in F. Then $dp_i(\alpha;A) < dp_i(\alpha;F)$ and $dp_j(\alpha;A) \leq dp_j(\alpha;F)$ for all j such that $i + 1 \leq j \leq n$.

B4.3. Suppose that α is the eigen variable of a second order $\forall$ or $\exists$ or α is a substitution variable which does not occur in F. Then $dp_j(\alpha;A) = 0$ for all j such that $i \leq j \leq n$.

B5. For any i-block B, there is a subset of B, denoted by E or E_B, which satisfies certain closure properties. Such an E is called the entrance of B. Namely, E is the entrance of B if the following five conditions hold.

B5.1. The auxiliary formula of a key inference of dim i which is in B belongs to E.

B5.2. If a formula belongs to E, then all its ancestors belong to E and all its descendents which belong to B belong to E.

B5.3. Let us denote the complement of E relative to B by B - E. If a formula belongs to B - E, then all its descendents belong to B.

B5.4. Let D_1 and D_2 be the left and the right cut formula respectively of a cut. Then D_1 belongs to B - E iff D_2 belongs to B - E.

B5.5 If a non-reducible formula belongs to E, then it occurs in the antecendent of a sequent and is of the form $\forall \varphi \neg \forall \psi \neg G(\varphi,\psi)$, where $\neg \forall \psi \neg G(\varphi,\psi)$ is reducible.

Note. E may be empty.

2) Now, a reducible proof P is called a proof with blocks if it satisfies the following C1 - C3.

C1. For every key inference K of dim i in P, there is an i-block B such that the principal formula of K belongs to B.

C2. The blocks of the same dimension do not intersect one another.

C3. If B_1 is an i-block, B_2 is a j-block, where $i < j$, and B_1 and B_2 are not disjoint, then B_1 is included by B_2.

3) Let P be a proof with blocks (of dim n). If there is a function d from semi-formulas and substitutions of P to $\omega^{n+1} + 1$ which satisfies the following conditions D1 - D6, then d is called a degree function of P.

D1. $d(A) = 0$ if A is explicit in P.

Assume that A is implicit in P.

D2. $d(A) = \omega^{n+1}$ if A is not reducible.

D3. Let Af(J,A;P) express the fact that a substitution J affects A in P and let dp(J;A) be dp(α;A) where α is the eigen variable of J. Then

$$d(A) = \max_{Af(J,A;P)} (d(J;P) + dp(J;A), \delta(A)) + \lambda(A)$$

if A is reducible.

D4. Let J be a substitution in P. Then d(A) < d(J) if A is an implicit formula which occurs in the upper sequent of J.

D5. $0 < d(J) < \omega^{n+1}$ for any substitution J in P.

D6. Let B be an i-block of P and let F be a key principal formula of dim i which belongs to B. If the eigen variable of a substitution J occurs in B but not in F, then d(J) < d(F).

Note. We may denote d(J) and d(A) for a J and A in P by d(J;P) and d(A;P) respectively, as d depends on P, although d is not necessarily uniquely determined for a P.

4) A proof with degree (of dim n) is a proof with blocks (of dim n) for which a degree function can be defined and in which all substitutions are under any logical inference and induction.

Proposition 1.8. For every proof of D_n, say P, its fourth transformation P^4 is a proof with degree of dim n.

Proof. By Proposition 1.5 there exists an $\mathfrak{F}$ $(= \mathfrak{F}_1 \cup \ldots \cup \mathfrak{F}_n)$ for a P^4 as above such that P^4 consists of $\mathfrak{F}$-admissible formulas only and the comprehension abstracts are $\mathfrak{F}$-reducible. Also, a P^4 as above does not involve substitutions. Therefore we may define a degree function for P^4 according to D1 - D3. In particular, D3 is simplified as $d(A) = \delta(A) + \lambda(A) = \ell(A)$. Thus, we only have to show that P^4 is a proof with blocks.

Let i be any number such that $1 \leq i \leq n$. Let Q be a subsidiary proof in P of dim i. In P^4 let us call a

$T_4(\tilde{Q}_1(s))$ or $T_4(\tilde{\tilde{Q}}_2(s))$ (of dim i) a key subproof (of dim i) (cf. Definitions 1.4 and 1.5). If a key subproof of dim i occurs in P^4, then the set of all formulas in it is defined to be an i-block of P^4. Only such a subproof of P^4 determines an i-block. The formulas in the end sequent of a key subproof Q as well as all their ancestors form the entrance of Q. We must prove that all conditions on blocks are satisfied.

Suppose there is a key inference K of dim i in P^4. Then it can happen only in a key subproof of dim i. (See the process of the transformations of P to P^4, i.e. Definitions 1.2 - 1.5.) Therefore there is an i-block which the principal formula of K belongs to. Since no pair of key subproofs of the same dimension intersect each other, the blocks of the same dimension are mutually disjoint. Suppose Q is an i-block and Q' is a j-block, where $i < j$, and Q and Q' intersect. Then from the definition Q is included by Q'. This proves C1 - C3.

We now proceed to the proof of B1 - B5. As before, i,j, etc. denote any numbers $\leq n$.

B1.1. Any key subproof is closed with respect to ancestry.

B1.2. Any key subproof is closed with respect to those formulas as D_1 and D_2 in the condition B1.2.

B1.3. The first inference under a key subproof (i.e. the inference whose upper sequent (or one of the upper sequents) is the endsequent of the concerning key subproof) is a cut by definition (cf. Definition 1.4). Therefore if the subformula of a logical inference belongs to a key subproof, then its principal formula also belongs to the same key subproof.

B2.1. Suppose $j \geq i$ P^4 is so defined that a second order $\exists$ in the antecedent of dim j does not occur in any key subproof of dim i. In fact it occurs only in a $T_4(\tilde{\tilde{Q}}_1(s))$ or $T_4(\tilde{Q}_2(s))$ (which we denote by Q^*) of dim j. Then such a subproof does not intersect with any key subproof of dim j. Therefore if $i = j$, then B2.1 holds. If $i < j$, then a key subproof of dim i, say Q^4, may be contained in a Q^* (as the sets of occurrences of formulas). However, we may assume that the auxiliary formula of a second order $\exists$ in the antecedent in Q^* is introduced outside Q^4, since in P Q is a subsidiary proof of a Q' of dim j such that $Q^* = T_4 \ldots T_1\ Q'$, and the first inference under Q^4 is a cut (cf. the proof of B1.3 above).

B2.2. Suppose $j > i$. Suppose that Q and Q' are key subproofs of dim i and of dim j respectively and that Q and Q' intersect. (This implies, as was already proved, that Q is included by Q'.) Suppose Q is $T_4 \ldots T_1(Q_o)$ and Q' is $T_4 \ldots T_1(Q'_o)$, where Q_o and Q'_o are subsidiary proofs of dim i and dim j respectively in P. Then Q_o is a subsidiary proof of Q'_o, and the subformula of a key inference of dim j in Q'_o cannot belong to Q_o, and this property is preserved under four transformations.

B2.3. Let G be the auxiliary formula of a key inference of dim i. Then a descendent of G occurs in the end sequent of the key subproof (of dim i) which G belongs to. To the end sequent of it a cut, where the cut formula is not a descendent of G, applies. Therefore no strong inference applies to a descendent of G within the key subproof.

B3. Let Q^4 be a key subproof of dim i and an eigen variable α occurs both inside and outside Q^4. Then we may assume that in P α occurs both inside and outside Q. Recalling that Q is a subsidiary proof of P, we may assume that α occurs in the end sequent of Q. Therefore, as the end sequent of Q and the end sequent of Q^4 are essentially the same, α occurs in the end sequent of Q^4, or α occurs in the key principal formula of any key inference of dim i in Q^4.

B4. Let Q^4 be a key subproof of dim i in P^4. Let A be any semi-formula in Q^4 which is not a or a descendent of a key principal formula of dim i and is not in the end piece of P^4. Then by definition A does not involve any formula of dim j if $j \geq i$, since the only formulas whose types belong to $\mathfrak{F}_i$ are the key principal formulas and their direct descendents and no $\mathfrak{F}_\ell$, where $\ell \geq i + 1$, are involved. (Therefore the last condition on A is actually irrelevant.) Suppose F is a key principal formula of dim i in Q^4. By definition, F belongs to $\mathfrak{F}_i$.

B4.1. $\delta_i(A) = 0 < 1 \leq \delta_i(F)$ and $\delta_j(A) = 0 = \delta_j(F)$ for all j such that $i + 1 \leq j \leq n$ from the above remark.

B4.2. There is no substitution.

B4.3. Let α be an eigen variable. Then from the above remark $dp_j(\alpha;A) = 0$ if $i \leq j \leq n$.

B5. We have defined the entrance of a block (of dim i).

B5.1. Suppose a key auxiliary formula (of dim i) belongs to a key subproof (of dim i), say Q^4. Then its descendent occurs in the succedent of the end sequent of Q^4; that is, it is an ancestor of a formula in the end sequent of Q^4. Therefore by definition it belongs to the entrance of Q^4.

B5.2. If a formula belongs to an entrance, then by definition all its ancestors belong to the same entrance. Suppose A belongs to the entrance of a Q^4 and A' is a descendent of A which belongs to Q^4. Then A and A' are both the ancestors of a formula in the end sequent of Q^4 (or A' itslef occurs in the end sequent). Therefore by definition A' belongs to the same entrance.

B5.3. Consider a block, i.e. a key subproof, of dim i, say B, and its entrance E. Suppose a formula A belongs to B - E. Then A belongs to B but it is not an ancestor of a formula which occurs in the end sequent of B. From this it follows immediately that all descendents of A belong to B.

B5.4. Let D_1 and D_2 be the cut formulas of a cut. Suppose D_1 belongs to a B - E. This means that D_1 is in B and is not a or an ancestor of a formula in the end sequent of B. Therefore D_2 must satisfy the same condition, or D_2 belongs to B - E. The converse is proved with the same reasoning.

B5.5. The only non-reducible formulas which belong to the entrance of a B are either the formula C in the antecedent of the end sequent of B or its ancestors. But C is of the form $\forall \varphi \neg \forall \psi \neg G(\varphi,\psi)$, where $G(\varphi,\psi)$ is arithmetical. Therefore the only ancestors of C which are non-reducible have the same form as C and occur in the antecedent of the sequents.

As a corollary of Proposition 1.8 we have the following.

Theorem 1.2. If the system of the proofs with degree is consistent, then so is $P\Delta^1_2$.

Proof. By Proposition 1.8 and Theorem 1.1 in §2.

§4. Some Corollaries of the Definition in §3.

Corollary. Consider a proof with degree (of dim n), say P, and let i be any number such that $1 \leq i \leq n$.

1) We may restrict B1.1 to the immediate ancestor(s).

2) If a formula belongs to a B - E, then all its descendents belong to B - E.

3) The principal formula of a second order $\exists$ of dim i (and its descendents) does (do) not belong to any block of dim $\leq$ i.

4) The auxiliary formula of a key inference of dim i belongs to the block which its principal formula belongs to. Furthermore those formulas belong to the same entrances.

5) Let B be an i-block, F be a key principal formula of dim i which occurs in B and G be a key auxiliary formula of dim i which occurs in B. Then B,F and G satisfy the condition in B.4.

6) Let G and F be the auxiliary formula and the principal formula of a key inference (of dim i). Then $\ell(G) < \ell(F)$.

7) Let G and F be as in 5). Then $d(G) < d(F)$.

Proof. 6) Suppose F belongs to an i-block B. Then by 4) in the corollary G also belongs to B. B,F and G satisfy the condition in B4 by 5) above. Therefore by B4.1, $\delta_i(G) < \delta_i(F)$ and $\delta_j(G) \leq \delta_j(F)$ if $i + 1 \leq j \leq n$. Therefore $\delta(G) < \delta(F)$, and hence $\ell(G) = \delta(G) + \lambda(G) < \delta(F) + \lambda(F) = \ell(F)$ by definition of δ and λ.

In order to prove 7), we shall first prove the following.

Lemma. Let B be an i-block, F be a key principal formula of dim i which belongs to B and A be any semi-formula in B. Suppose B,F and A satisfy the following conditions.

1^o) Let α be a substitution variable of a substitution J in P. If α does not occur in F, then $dp_j(\alpha;A) = 0$ for all $j \geq 1$.

2^o) Let α be as above. If α occurs in B but not in F, then $d(J) < d(F)$.

3^o) Let α be as in 1^o). If α occurs in F, then $dp_i(\alpha;A) < dp_i(\alpha;F)$ and $dp_j(\alpha;A) \leq dp_j(\alpha;F)$, for all $j \geq i + 1$.

4^o) $\delta_i(A) < \delta_i(F)$ and $\delta_j(A) \leq \delta_j(F)$ for all $j \geq i + 1$.

Then $d(A) < d(F)$.

Proof. From the definition of d we have

$$(1) \quad d(F) = \max_{Af(J_1,F;P)} (d(J_1) + dp(J_1;F), \delta(F))$$

and

$$(2) \quad d(A) = \max_{Af(J_2,A;P)} (d(J_2) + dp(J_2;A), \delta(A)) + \lambda(A).$$

(1) is of the form

(3) $\omega^n \cdot m_n + \ldots + \omega \cdot m_1$, where $m_i > 0$,

by 3^o) and 4^o). Also from 4^o)

(4) $$\delta(A) + \lambda(A) < \delta(F).$$

Case 1) $d(A) = \delta(A) + \lambda(A)$.

Then from (4) $d(A) < \delta(F) \leq d(F)$.

Case 2) $d(A) = \max_{Af(J_2;A;P)} (d(J_2) + dp(J_2;A)) + \lambda(A)$

$$= d(J_o) + dp(J_o;A) + \lambda(A), \text{ say.}$$

This means that the eigen variable α of J_o occurs in A, and hence in B.

Subcase 2.1) α occurs in F. Then $dp(\alpha;A) < dp(\alpha;F)$ by 3^o), and $d(J_o) + dp(J_o;F)$ is counted in $d(F)$ (cf. (1)). Hence

(5) $d(J_o) + dp(J_o;A) < d(J_o) + dp(J_o;F) \leq d(F)$.

From (3) and (5), $d(A) = d(J_o) + dp(J_o;A) + \lambda(A) < d(F)$, since $\lambda(A) < \omega$.

Subcase 2.2) α does not occur in F. Then by 1^o) $dp_j(J_o;A) = 0$ for all $j \geq i$. Also $d(J_o) < d(F)$ by 2^o) since α occurs in B.

$$d(A) = d(J_o) + \omega^{i-1} dp_{i-1}(\alpha;A) + \ldots + \omega \cdot dp_1(\alpha;A) + \lambda(A) < d(F)$$

by (3). This completes the proof of the lemma.

Proof of 7). B,G and F satisfy the condition in Lemma. Hence, if 1^o) - 4^o) in Lemma are satisfied, then $d(G) < d(F)$ follows immediately from Lemma. Notice that F and G satisfy the conditions on F and A in B4. (G is not a key principal formula of dim i or a descendent of such, and G is not in the end piece of P.) Therefore: 1^o) follows from B4.3, 2^o) is exactly D6, 3^o) is B4.2, and 4^o) is B4.1.

§5. Theorems.

Theorem 1.3. The system of the proofs with degree of dim n (for every $\mathfrak{J}$ and n) is consistent. Furthermore, the consistency of such a system is proved by using the system of ordinal diagrams $O(\omega^{n+1} + 1, \omega^{2(n+1)} + 1)$.

Note. Although the theorem is stated relative to $\mathfrak{J}$, the proof of it which is carried out in the following sections is uniform in $\mathfrak{J}$.

The proof of Theorem 1.3 will be carried out in the following two sections.

One direction of our main theorem of this chapter now follows from Theorems 1.2 and 1.3.

Theorem 1.4. Let V_n be the order type of $O(\omega^n, \omega^n)$ (the system of ordinal diagram with both basic sets ω^n and ordered by $<_o$).

Let $V = \lim_{n<\omega} V_n$. Then the consistency of $P\Delta^1_2$ is proved by transfinite induction up to V.

§6. Reduction of the proofs with degree (of dim n) of the sequent $\rightarrow$, where we assume that an $\mathfrak{I} = \mathfrak{I}_1 \cup \ldots \cup \mathfrak{I}_n$ is fixed. At each step it is easy to see that the reduction from P to P' preserves the $\mathfrak{I}$-reducibility, hence P' is of dim n. This is due to the fact that the $\mathfrak{I}$-reducibility is preserved under the replacements of first order free variables by some terms, of second order free variables by other second order free variables, and the substitutions of abstracts. Therefore we shall not mention it at each time. We shall first show that the reducts are the proofs with degree. In the next section we shall assign the ordinal diagrams to the proofs and prove that with the reductions the ordinal diagrams decrease.

We may remark here that by changing some eigen variables in an appropriate manner, we can always avoid the clash of free variables. We assume that we do this alteration whenever it is necessary.

In this section and the next section we follow more or less the consistency proof in Chapter 2 of [5] and quote the corresponding numberings there with * whenever it is possible to do so. We assume that we are given a proof with degree (of dim n), say P, of the sequent $\rightarrow$, and carry out reduction to P, obtaining another proof of $\rightarrow$, say P'. After the definition of P', we shall define the blocks of P' and their entrances in a manner that for each i such that $1 \leq i \leq n$ an i-block in P , say B, and its entrance, say E, induces an i-block in P', say B', and its entrance, say E'. We then show that P' is a proof with degree, assuming that P is. We shall observe the above notational conventions throughout. A condition on a proof with degree, for example B3, for a proof P will be denoted by [B3,P].

6.0. Preliminary operations. See 8.1* and 8.2*. Notice that replacing a first order free variable by 0 preserves the property that a formula is an $\mathfrak{F}_1$-formula.

6.1. The end piece of P contains an induction. (See 8.3* for the detail.) Let J be an under-most induction in the end piece of P:

$$\begin{array}{ll} & Q(a) \\ & \vdots \\ S_1 & A(a),\Gamma \rightarrow \Delta, A(a') \\ \hline S_2 & A(0),\Gamma \rightarrow \Delta, A(t) \end{array} \; J \qquad ,$$

where t does not contain any free variable. The reduct P' is defined as follows according to two cases.

1^o t equals 0 (8.3.1*).

$$\begin{array}{ll} S_1' & A(0) \rightarrow A(0) \\ \hline & * \\ \hline S_1'' & A(0),\Gamma \rightarrow \Delta, A(0) \\ \hline S_2' & A(0),\Gamma \rightarrow \Delta, A(t) \end{array}$$

where * indicates that there may be several uses of weak inferences between S_1' and S_1''.

Since this is an easy case, we shall explain the typical, routine parts of the argument in some detail so that, for the less obvious cases, we can avoid such and yield to the crucial points only.

Let B be an i-block of P and E be its entrance. Then the induced i-block B' (entrance E') of P' is defined as follows: the explicitly indicated A(O)'s in the antecedents of $S_1', \ldots, S_1''$ belong to B' (E') if and only if the induction formula A(a) in S_1 belongs to B (E); the explicitly indicated A(O)'s in the succedents of $S_1', \ldots, S_1''$ belong to B' (E') if and only if the induction formula A(a') in S_1 belongs to B (E); the A(O) in the antecedent of S_2' belongs to B' (E') if and only if the A(O) in S_2 belongs to B (E); the A(t) in S_2' belongs to B' (E') if and only if the A(t) in S_2 belongs to B (E); a formula in Γ or Δ in one of $S_1', \ldots, S_1''$ belongs to B' (E') if and only if the corresponding formula in Γ or Δ in S_1 belongs to B (E) respectively; a formula in Γ or Δ in S_2' belongs to B' (E') if and only if the corresponding formula in Γ or Δ respectively in S_2 belongs to B (E). Any other formula in P' belongs to B' (E') if and only if the corresponding formula in P belongs to B (E). The blocks of P' are only those which are defined as above. We should note that for every (occurrence of every) formula in P', say A', there is a corresponding formula in P, say A, such that A' belongs to B' (E') if and only if A belongs to the corresponding B (E) for every block B' of P', and that A and A' are identical up to some terms.

We shall first show that P' is a proof with blocks.

We assume that $1 \leq i \leq n$ throughout.

C1. Suppose that there is a key inference of dim i, say K', in P'. Then there is the corresponding key inference K of dim i in P and it follows from [C,P] that the principal formula of K belongs to an i-block B. Therefore from the above definition, the principal formula of K belongs to B', the i-block which is induced from B.

C2. From the above note, it is easy to see that the blocks of the same dimension of P' do not intersect, since [C2,P] holds.

C3 also follows from the above note and [C3,P].

B1.1. Suppose that the A(O) in S_2' belongs to an i-block B'. Then by definition the A(O) in S_2 belongs to the corresponding B. Hence by [B1.1,P] the induction formula A(a) (in S_1) belongs to B. Therefore, by definition, the A(O)'s in the antecedent of $S_1', \ldots, S_1''$ belong to B'.

If A(O) in the antecedent of one of $S_1', \ldots, S_1''$ belongs to a B', then by definition the A(a) in S_1 belongs to B; hence by definition any ancestors of A(O) as above belong to B'. For a A(O) or A(t) in the succedent of a sequent, a similar argument as above goes through.

For a formula in Γ or Δ, B1.1 is obvious from the definition and [B1.1,P].

Since a formula in S_2' belongs to a B' if and only if the corresponding formula in S_2 belongs to B, B1.1 for other formulas in P' follows immediately from [B1.1,P] and the definition.

B1.2. If the A(O) in the antecedent of S_1' belongs to a B', then by definition, the A(a) in S_1 belongs to the corresponding B and hence, by [B1.2,P] A(a') belongs to B. Therefore, by definition, the A(O) in the succedent of S_1' belongs to B'. The converse is proved similarly. For any other pair of the D_1 and D_2, B1.2 follows from [B1.2,P].

B1.3. A (the) auxiliary formula A' of a logical inference, say I', in P' occurs in a thread which S_2' does not belong to. Therefore the corresponding formula A in P is a (the) auxiliary formula of the corresponding inference I in P. If A' is in a B', then, by definition, A is in B, and, by [B1.3,P], the principal formula of I belongs to B; hence, by definition, the principal formula of I' belongs to B'.

B2.1. Suppose $i \leq j \leq n$. The auxiliary formula of a second order $\exists$ in the antecedent of dim j in P', say A', occurs in a thread which S_2 does not belong to. Hence there is the corresponding formula, A, in P, which is the auxiliary formula of a second order $\exists$ in the antecedent of dim j. Thus by [B2.1,P], A does not belong to any i-block, which implies that A' does not belong to any i-block.

B2.2. Let j be any number such that $i + 1 \leq j \leq n$ and A' be the auxiliary formula of a key inference of dim j. Then the corresponding formula A in P is the auxiliary formula of a key inference of dim j, and hence, by [B2.2,P], does not belong to any i-block. This implies that A' does not belong to any i-block.

B2.3. If there is such a formula C in P', then it occurs in a thread which does not contain S_2'. (There is no induction under S_2.) So, this follows from [B2.3,P].

B3. We may assume that A(a) (hence A(O), A(t) and A(a')) does not contain any second order eigen variable except substitution variables. If α occurs elsewhere in P' inside of (outside) a B', then α occurs in P inside of (outside) B in the corresponding formula. Also if F' is a key principal formula of dim i in P' and belongs to a block B', then the corresponding F in P is a key principal formula of dim i which belongs to B. Therefore if a situation in B3 arises in P' for B', then the same situation in P arises for B and, by [B3,P], α occurs in F, which implies that α occurs in F'.

B4. Suppose B',A' and F' satisfy the conditions on B, A and F in B4 for P'. Then there are the corresponding formulas A and F and the corresponding block B in P, satisfying the same condition. Since neither F' nor A' is in the end piece of P', this situation happens in the untouched part of the proof. Recall also that A' is A itself and F' is F. Next, for any substitution in P', say J', there is the corresponding substitution in P, say J, and, if J' affects a formula in P', then J affects the corresponding formula in P. Based on all this, B4 follows from [B4,P].

B5. For every i-block B' of P', we have defined the entrance of B', say E', which is induced from the entrance of B, say E.

B5.1. The subformula of a key inference of dim i in P' occurs in a thread which does not contain S_2'. Therefore the situations for P and for P' are exactly the same.

B5.2. First part: Suppose, for example, that the A(O) in the antecedent of S_2' belongs to E'. Then by definition A(O) in S_2 belongs to the corresponding E and hence A(a) belongs to E by [B5.2,P]. So, by definition the A(O)'s in the antecedents of $S_1', \ldots, S_1''$ belong to E'. For other A(O)'s and A(t) and the formulas in Γ and Δ, similar arguments go through. For any other formulas, this follows from [B5.2,P].

Second part: Suppose, for example, the A(O) in the antecedent of S_1' belongs to an E' and the A(O) in the antecedent of S_2' belongs to the related B'. Then in P the A(a) in S_1 belongs to E and the A(O) in S_2 belongs to B. Therefore by [B5.2,P] A(O) belongs to E; hence in P' the A(O) in S_2' belongs to E'. For other A(O)'s and A(t) similar arguments go through. For any other formulas, this follows from [B5.2,P].

B5.3. Suppose, for example, the A(O) in the left hand side of one of $S_1', \ldots, S_1''$ belongs to a B' - E'. Then the A(a) in S_1 belongs to B - E. Hence by [B5.3,P] the A(O) in S_2 and its descendents belong to B. Hence in P' the A(O) in S_2' and its descendents belong to B'. For any other A(O)'s and A(t) similar arguments go through. For any other formula this follows from [B5.3,P].

B5.4. Suppose D_1' and D_2' satisfy the condition on D_1 and D_2 in B5.4. They are not above S_2'. Suppose D_1' belongs to a B' - E'. Then its corresponding formula in P, D_1, belongs to the corresponding B - E by definition and hence the other cut formula, D_2, which corresponds to D_2', belongs to B - E, and hence D_2' belongs to B' - E'.

B5.5. This follows from [B5.5,P], in virtue of the note after the definition of blocks, and entrances.

Next we must show that P' is a proof with degree. Define the degrees of substitutions as d(J';P') = d(J;P), where J' is a substitution in P' and J is the corresponding substitution in P. Notice that, by definition, for every substitution J' in P' there exists a corresponding substitution J in P. d(A) is defined as D1 - D3.

Recall that if A corresponds to A', then A and A' are identical up to some terms; and hence Af(J',A';P') if and only if Af(J,A;P), where J corresponds to J'. Therefore by definition d(A';P') = d(A;P) for any formula A' in P'.

D4. d(A';P') = d(A;P) (See above.)
$<$ d(J;P) (By [D4,P].)
= d(J';P') (By definition.)

D5. By definition and [D5,P].

D6. For any B',F' and J' of P' which satisfy the condition in D6, there are corresponding B,F and J of P. Therefore

d(J';P') = d(J;P) $<$ d(F;P) by [D6,P]
= d(F';P) (See above.).

2^o. t is equal to an n which is not O (8.3.2*). Define P' as follows.

$$
\begin{array}{llll}
 & Q(0) & & Q(0') \\
 & \vdots & & \vdots \\
S_1^1 & A(0),\Gamma \to \Delta, A(0') & S_1^2 & A(0'),\Gamma \to \Delta, A(0'') \\
\hline
\end{array}
$$

$$
\begin{array}{llll}
S_3 & \underline{A(0),\Gamma,\Gamma \to \Delta,\Delta,A(0'')} & & Q(0'') \\
\hline
 & & & \vdots \\
S_4 & A(0),\Gamma \to \Delta, A(0'') & S_1^3 & A(0''),\Gamma \to \Delta, A(0''') \\
\hline
\end{array}
$$

$$
\begin{array}{ll}
S_5 & A(0),\Gamma,\Gamma \to \Delta,\Delta,A(0''') \\
\hline \hline
S_6 & A(0),\Gamma \to \Delta, A(0''') \\
 & \vdots \\
S_{2n} & A(0),\Gamma \to \Delta, A(n) \\
\hline
S_2' & A(0),\Gamma \to \Delta, A(t) \\
 & \vdots \\
 & \to
\end{array}
$$

The i-blocks and the entrances of P' are induced from those of P as follows. The $A(0)$ in the antecedent of one of the $S_3,\ldots,S_4,\ldots,S_5,\ldots,S_{2n}$ belongs to an i-block B' (E') if and only if the $A(a)$ in S_1 belongs to B (E); the $A(0)$ in S_2' belongs to B' (E') if and only if the $A(0)$ in S_2 belongs to $B(E)$; a formula in a $Q(k)$, say A', belongs to B' if and only if its corresponding formula A in $Q(a)$ belongs to B. (Notice that one A in $Q(a)$ corresponds to an A' in any of $Q(0),\ldots,Q(n-1)$.); the $A(k)$ in the succedent of one of $S_3,\ldots,S_{2n}$ belongs to B' if and only if the $A(a')$ in S_1 belongs to B. A formula of $\Gamma(\Delta)$ in one of $S_3,\ldots,S_{2n}$ belongs to B' (E') if and only if its corresponding formula of $\Gamma(\Delta)$ in S_1 belongs to B (E). A formula in S_2' belongs to B' if and onlyif its corresponding formula in S_2 belongs to B.

Any other formula belongs to a B' if and only if its corresponding formula belongs to B.

Let A(k) be the cut formula in S_{2k} or S_1^{2k-1} (or S_1^1 or S_1^2 if k is 0"), k = 2,...,2n - 1. A(k) does not belong to any entrance. If an ancestor of A(k), say C', belongs to a block B', then C' belongs to its entrance if and only if A(k) does not belong to B'. A(n) in S_{2n} and its ancestors belong to E' if and only if A(a') belongs to E. Any other formula belongs to an E' if and only if its corresponding formula in P belongs to E. It should be noted that for every formula, say A', in P', and every i-block B', there is a corresponding formula A in P and A' belongs to B' if and only if A belongs to B.

C1. If there is a key inference of dim i in P', say K', then its principal formula A' is in one of the Q(k)'s or in a thread which does not contain S_2'. Therefore its corresponding formula A (as well as the corresponding inference K) is in Q(a) or in a thread which does not contain S_2. So, by definition A' belongs to B' if and only if A belongs to B. Therefore, by [C1,P] for every key principal formula A' in P' there is a block B' such that the corresponding A belongs to B, or A' belongs to B'. C2 and C3 follow from [C2,P] and [C3,P]. (See the note after the definition of the blocks and the entrances.)

B1.1. If the A(0) in the antecedent of one of $S_1^1, S_3, \ldots, S_{2n}$ belongs to a B', then, by definition, A(a) in S_1 belongs to B, and hence by [B1.1,P] all ancestors of the A(a) belong to B.

This implies that all ancestors of the concerning $A(0)$ belong to B'. If the $A(0)$ in S_2' belongs to B', then the $A(0)$ in S_2 belongs to B, and hence all its ancestors belong to B by [B1.1,P], which implies that all ancestors of the $A(0)$ in S_2' belong to B'.

Suppose, for example, the $A(0'')$ in S_3 belongs to B'. Then by definition $A(a')$ belongs to B. Hence all its ancestors belong to B, and so the $A(0'')$ in S_1^2 and all its ancestors belong to B'. If, as another example, one of the formulas in S_2', say C', belongs to B', then the corresponding formula in S_2 belongs to B, and hence the corresponding ancestors belong to B. This implies that all the ancestors of C' belong to B'.

B1.2. In a $Q(k)$ a formula belongs to B' if and only if the corresponding formula in $Q(a)$ belongs to B, and so the formulas in the beginning sequents and the equality axioms, and the induction formulas in a $Q(k)$ correspond to the formulas in the same kind of sequents in $Q(a)$. Hence for those formulas B1.2 follows from [B1.2,P]. For other sequents B1.2 follows from [B1.2,P] trivially.

B1.3. The auxiliary formula and the principal formula of a logical inference occur in a $Q(k)$ or in a thread which does not contain S_2'. Hence this follows from [B1.3,P].

B2.1. Let j be a number such that $i \leq j \leq n$. Then the auxiliary formula of a second order $\exists$ (or dim j) occurs within a $Q(k)$ or in a thread which does not contain S_2'. Therefore this follows from [B2.1,P].

B2.2. This is shown by a similar argument as in B2.1.

B2.3. Suppose there is a C' which satisfies the condition on C in B2.3 in a Q(k) or in a thread which does not contain S'_2. Then this follows from [B2.3,P]. Let C' be a cut formula A(k) which is a descendent of a key auxiliary formula of dim i. Then in P the corresponding C (A(a) or A(a')) is a descendent of a key auxiliary formula of dim i and is the auxiliary formula of a strong inference (i.e., an induction). Therefore C does not belong to any i-block, which implies that C' does not belong to any i-block.

B3. Suppose that in P' α' is a second order eigen variable of an inference I' and occurs in a formula C'. Then in P there is a corresponding inference I whose eigen variable α occurs in C, the formula which corresponds to C'. In virtue of this fact and the note after the definition of the blocks and the entrances, B3 follows from [B3,P].

B4. Let A' be a semi-formula in an i-block B' of P' which is not a key principal formula of dim i or its descendent and which is not in the end piece. Then the corresponding formula A in P belongs to B and satisfies the same condition as A' does. Let F' be a key principal formula of dim i and belongs to B'. Then the corresponding formula F in P is a key principal formula of dim i and belongs to B. Recall that A and A' and F and F' respectively are identical up to some terms. Therefore B4 follows directly from [B4,P].

B5. We have defined the entrance E' for each i-block B'.

B5.1. The auxiliary formula of a key inference of dim i occurs either within one of the Q(k)'s or in a thread which does not contain S'_2. Therefore it belongs to E' if and only if its corresponding formula in P belongs to E, except the case where it is one of the cut formulas A(k)'s and their ancestors. Hence, except the latter case, this follows from [B5.1,P]. As for the cut formulas A(k)'s and their ancestors, recall that to a cut formula A(k) corresponds A(a) or A(a') of P. Suppose an A(k) as above is the descendent of a key auxiliary formula of dim i, say G'. Then, in P, A(a) or A(a') is the descendent of a key auxiliary formula of dim i, say G, and A(a) or A(a') respectively is the auxiliary formula of a strong inference. Therefore [B2.3,P] implies that A(a) or A(a') respectively does not belong to B. Then, according to our definitions A(k) does not belong to B'. Therefore, by definition of E', an ancestor of A(k) belongs to E' if and only if its corresponding formula belongs to E. But by [B5.1,P] G (see above) belongs to E. So G' belongs to E'.

B5.2. First part: Suppose the A(0) in the antecedent of one of $S'_1, S_3, \ldots, S_4, \ldots, S_{2n}$ belongs to E'. Then, according to the definition, A(a) in S_1 belongs to E. Hence all its ancestors belong to E by [B5.2,P]. Therefore all the ancestors of the A(0) belong to E'. Similarly for the A(0) in S'_2. The cut formulas A(k)'s do not belong to any E' by definition. For A(n) in S_{2n} a similar argument goes through. For any other formula this follows from [B5.2,P], according to the definition.

Second part: Suppose the A(O) in the antecedent of one of $S_1', \ldots, S_{2n}$ belongs to E' and one of its descendents C' belongs to B'. Then A(a) in S_1 belongs to E and C (corresponding to C') belongs to B. So, by [B5.2,P] C belongs to E, and hence C' belongs to E'. If an A(k) belongs to B' then by definition no ancestor of A(k) nor A(k) itself belongs to E'. Suppose A(k) does not belong to B', and let C' be an ancestor of A(k). Then the C' belongs to E' if and only if A(k) does not belong to B'. On the basis of the above facts, it is easy to see that the second part holds for any formula bundle which contains a cut formula A(k). For any other formulas, this follows from [B5.2,P].

B5.3. Suppose an ancestor C' of a cut formula A(k) belongs to B' - E'. Then, by definition of E', A(k) belongs to B', and hence by definition all ancestors of A(k) belong to B' - E' by definition. Therefore all descendents of C' belong to B'.

For any other formula this follows from definition and [B5.3,P].

B5.4 Suppose a left cut formula A(k) (also called D_1) belongs to B' - E'. This means that A(a') belongs to B. Then by [B1.2,P] A(a) belongs to B. So, the right cut formula D_2 (corresponding to A(k)) belongs to B'. This together with definition implies that D_2 belongs to B' - E'. For other cut formulas, this follows from [B5.4,P].

B5.5. There is no formula in E' such that its corresponding formula in P does not belong to E. By [B5.5,P] the corresponding formula of a non-reducible formula in P in the succedent of a sequent does not belong to E; hence in P' a non-reducible formula in the succedent of a sequent does not belong to E'. If a non-reducible formula C' belongs to E', then its corresponding formula C belongs to E and is of the form $\forall\varphi\neg\forall\psi\neg G(\varphi,\psi)$, where $\neg\forall\psi\neg G(\varphi,\psi)$ is reducible. Then C' is of the form $\forall\varphi\neg\forall\psi\neg\tilde{G}(\varphi,\psi)$ where $\neg\forall\psi\neg\tilde{G}(\varphi,\psi)$ is reducible.

Now consider the conditions D1 - D6. It is easily seen that in P' all substitutions are under any logical inference or induction. For each substitution in P', say J', define d(J';P') as d(J';P') = d(J;P), where J is the corresponding substitution in P, and define d(A';P') as D1 - D3 for all semi-formulas A' in P'. Since substitutions do not occur in Q(a), it is easy to see that d(A';P') = d(A;P) for any A' in P', where A is its corresponding formula in P. D4- D6 are proved just as in 1°.

6.2. The end piece of P does not contain any induction but does contain an equality axiom. (See 8.4* for the detail.)

Let P be of the form

$$S \qquad s = t, A(s) \rightarrow A(t)$$

$$\vdots$$

$$\rightarrow$$

where s and t are equal to the numerals m and n respectively.

1°. $m = n \rightarrow$ is true (8.4.1*). The reduct P' is defined as follows.

$$\begin{array}{ll} S' & \dfrac{m = n \rightarrow}{\dfrac{\overline{\qquad * \qquad}}{}} \\ S'' & m = n, A(m) \rightarrow A(n) \\ & \overline{\overline{\qquad\qquad\qquad}} \\ S''' & s = t, A(s) \rightarrow A(t) \\ & \qquad\qquad \vdots \\ & \qquad\qquad \rightarrow \end{array}$$

The blocks and the entrances are defined as follows.

$m = n$ or $s = t$ in one of $S', \ldots, S'', \ldots, S'''$ belongs to a B' (E') if and only if $s = t$ in S belongs to B (E). $A(m)$ and $A(s)$ belong to B' (E') if and only if $A(s)$ in S belongs to B (E). $A(n)$ and $A(t)$ belongs to B' (E') if and only if $A(t)$ in S belongs to B (E). Any other formula in P' belongs to B' (E') if and only if its corresponding formula belongs to B (E).

In order to prove that P' is a proof with degree, just regard the set of occurrences of $m = n$ and $s = t$ as one unit corresponding to the $s = t$ in S and similary the set of occurrences of $A(m)$ and $A(s)$ ($A(n)$ and $A(t)$) as one unit corresponding to $A(s)$ ($A(t)$). Then everything follows from the condition on P.

2°. $\rightarrow m = n$ is true (8.4.2*).

Define P' as

$$\dfrac{\dfrac{\dfrac{A(m) \rightarrow A(n)}{\overline{\qquad * \qquad}}}{A(s) \rightarrow A(t)}}{\begin{array}{c} s = t, A(s) \rightarrow A(t) \\ \vdots \\ \rightarrow \end{array}}$$

The blocks and the entrances are defined similarly to 1^{o}.

6.3. The end piece of P contains neither induction nor equality axioms, but does contain a logical beginning sequent (8.5*).

$$\dfrac{S_1 \quad \Gamma \to \Delta, \tilde{D} \qquad\qquad \begin{array}{c} S_2 \quad D \to D \\ \vdots \\ S_3 \quad \tilde{D}, \Pi \to \Lambda_1, \tilde{D}, \Lambda_2 \end{array}}{S_4 \quad \Gamma, \Pi \to \Delta, \Lambda_1, \tilde{D}, \Lambda_2}$$

$$(S) \qquad \begin{array}{c} \vdots \\ \to \end{array}$$

where (S) stands for any sequent under S_1 and S_3 (hence S may be S_4). Define P' as

$$\begin{array}{ll} & \vdots \\ S'_1 & \Gamma \to \Delta, \tilde{D} \\ \hline\hline S'_4 & \Gamma, \Pi \to \Delta, \Lambda_1, \tilde{D}, \Lambda_2 \\ & \vdots \\ (S') & \to \end{array}$$

The D in S'_1 belongs to a B' if and only if the $\tilde{D}$ in S_1 belongs to B. The descendent of the $\tilde{D}$ in a S', say C', belongs to B' if and only if both the $\tilde{D}$ in S_1 and the corresponding C in S belongs to B. A formula of Γ or Δ in S'_1 belongs to B' if and only if its corresponding formula in S_1 belongs to B. A formula of $\Gamma, \Pi, \Delta, \Lambda_1$ or Λ_2 in a sequent between S'_1 and S'_4, including S'_4, belongs to B' if and only if the corresponding formula in S_4 belongs to B.

Any other formula belongs to B' if and only if the corresponding formula in P belongs to B.

The definition of the entrances of P' is kind of complicated. Let B' be an i-block of P'. We shall define the entrance of B', say E'.

Case 1) The lowermost descendent of the $\tilde{D}$ in S_1', say C_1, which is naturally a left cut formula, does not belong to any block (of P'). Then any ancestor of C_1 belongs to E' if and only if it belongs to B'. Let C_2 be the right cut formula for C_1. Then all ancestors of C_2 as well as C_2 itself which belong to B' belong to E'.

Case 2) The C_1 as in Case 1) belongs to B'. (This implies that all ancestors of C_1 belong to B', and hence all descendents of the $\tilde{D}$ in S_1' belong to B'.)

Subcase 2.1) The C_2 as above belongs to B'. Then neither C_1 nor C_2 nor their ancestors belong to E'.

Subcase 2.2) C_2 does not belong to B'. All ancestors of C_1 as well as C_1 belong to E' and all ancestors of C_2 as well as C_2 which are in B' belong to E'.

Any other formula belongs to the entrance of a B' if and only if the corresponding formula in P belongs to E, where E is the entrance of B.

Corollary. To each formula A' in P' there correspond one formula, say A, or two formulas, say A_1 and A_2, in P in a manner that A' belongs to a B' if and only if A belongs to B in the former case, and if and only if both A_1 and A_2 belong to B in the latter case.

Now we can proceed to the proof that P' is a proof with degree.

C1. As far as the key inferences are concerned, no new situation arises in P'. Therefore this follows from [C1,P] and the definition.

By Corollary above, for each formula C' in P' which belongs to a B' there is at least one corresponding formula C which belongs to B. Therefore C2 and C3 follow from [C,P].

B1.1. Suppose the descendent of $\tilde{D}$ in a S', say C', belongs to B'. Then the $\tilde{D}$ in S_1 and the corresponding C in S both belong to B. Hence by [B1.1,P] all ancestors of $\tilde{D}$ and C belong to B, from which follows that all ancestors of the C' in S' belong to B'. For any other formulas, this follows from [B1.1,P].

B1.2. Such D_1 and D_2 occur above (including) S_1' or in a thread which does not contain S_4'; hence there is no change from P.

B1.3, B2.1 and B2.2 follow directly from the conditions on P.

B2.3. Consider a C' which satisfies the condition. If such a C' is not a descendent of the $\tilde{D}$ in S_1', then this follows from [B2.3,P]. Suppose C' is a descendent of the $\tilde{D}$. Then there is an ancestor of the $\tilde{D}$ which is a key principal formula of dim i; hence there is an ancestor of the $\tilde{D}$ in S_1 of P which is a key principal formula of dim i. Therefore by [B2.3,P] the $\tilde{D}$ in S_1 (which is a cut formula) must lie outside any i-block. Then, by definition, C' lies outside any i-block. For any other formulas this follows from [B2.3,P].

B4. Let A',B' and F' satisfy the condition on A,B and F in B4 for P'. Since A' is not in the end piece, those A' and F' occur either above S_1' or in the threads which do not contain S_4'. Therefore there are corresponding formulas A (to A') and F (to F'), where A and A' are identical and F and F' are identical. Therefore B4 follows from [B4,P] immediately.

B5. We have defined the entrance E' for each (i-) block B'. From the definition it is evident that E' is a subset of B'.

B5.1. We should first remark that a key auxiliary formula (of dim i), say G, in P' occurs either above S_1' or in a thread which does not contain S_4'. Suppose G belongs to an i-block B' and let E' be its entrance. If G is an ancestor of C_1 or C_2 (see the definition of the entrances for C_1 and C_2), then C_1 or C_2 respectively does not belong to B' since C_1 and C_2 are the auxiliary formulas of a strong inference (cut) of P' and [B2.3,P'] has already been verified. Therefore if G is an ancestor of C_1 then only Case 1) can hold. If G is an ancestor of C_2, then either Case 1) or Subcase 2.2) holds. In any case, G belongs to E' by definition since G belongs to B'. For any other case, this follows from [B5.1,P].

B5.2. If C_1 or C_2, or an ancestor of one of them belongs to E', then in the definition of E' either Case 1) or Case 2.2) holds. In either case any ancestors of C_1 and C_2 as well as C_1 and C_2 themselves belongs to E' if it belongs to B'. From this fact and [B1.1,P'] B5.2 is easily proved for the formulas which belong to the formula bundles which contain C_1 and C_2 respectively. For any other formulas this follows from [B5.2,P].

B5.3. Suppose, for example, an ancestor of C_1, say C, belongs to B' - E'. This is possible only by Case 2.1). But Case 2.1) holds only if C_1 belongs to B'. Then by [B1.1,P'] all ancestors of C_1, hence all descendents of C, belong to B'. A similar argument goes through for C_2. For any other formula this follows from [B5.3,P].

B5.4. Let D_1 and D_2 be the C_1 and C_2 in the definition. C_1 belongs to B' - E' if and only if Case 2.1) holds, and C_2 belongs to B' - E' if and only if Case 2.1) holds. For any other cut formula, it belongs to B' - E' if and only if its corresponding formula in P belongs to B - E.

B5.5. Suppose, for example, a non-reducible formula C' is C_1 or an ancestor of C_1 and belongs to E'. Then Case 1) or Case 2.2) holds. Notice that in particular C' belongs to B'. By virtue of [B5.2,P'], all ancestors of C' belong to E'. Hence we may assume that C' occurs above S_4' since no logical inference applies to C' or to any of its descendents under S_4'.

Then $\tilde{D}$'s and C_1 are also non-reducible. The formula C in P, corresponding formula to C', belongs to B and C and C' are identical. If C belongs to B - E, then all its descendents, in particular the $\tilde{D}$ in S_1, belong to B - E. (See Corollary 2 of §4.) Then the $\tilde{D}$ in the antecedent of S_3, and hence the D in the antecedent of S_2 belong to B - E. (See [B5.4 and B5.2,P].) The D in the succedent of S_2 belongs to B by [B1.2,P].

If Case 1) holds, then C_1 in P' does not belong to B'. This means that in P (since the $\tilde{D}$ in S_1 belongs to B) C_1 does not belong to B. Hence in P any ancestor of C_1 which is in B does not belong to B - E ([B5.3,P]), i.e. it must belong to E. Then by [B5.5,P] it must occur in the antecedent of a sequent and has the required form. This is impossible, since, for example, the D in the succedent of S_2 belongs to E and is non-reducible, contradicting the condition. Therefore this case cannot happen. If Case 2.2) holds, then C_1 belongs to B'. This implies that in P C_1 and all the ancestors of C_1 belong to B. If C_1 belongs to B - E, then C_2 must belong to B - E by [B5.4,P], which in turn implies that in P' C_2 belongs to B', contradicting the condition of 2.2). Therefore C_1 must belong to E. But this contradicts [B5.5,P]. Therefore neither case holds; that is neither C_1 nor its ancestors can belong to E'.

Suppose, as another example, C is a non-reducible formula which is C_2 or its ancestor and which belongs to E'. Notice that the latter case implies that C_2 is non-reducible, and hence

C_1 as well as the $\tilde{D}$'s are non-reducible. By definition C belongs to B. If in P C belongs to E then in P C occurs in the antecedent of a sequent and has the required form by [B5.5,P], and hence C satisfies the same condition in P'. Suppose C does not belong to E. Then C belongs to B - E, and hence C_2 and C_1 as well as all their ancestors belong to B - E. (See Corollary 2) of §4, [B5.4,P] and [B5.3,P].) This implies that the D in the antecedent of S_2 belongs to B ([B1.2,P]). If this D belongs to E, then [B5.5,P] applies to it and, by the definition of C_1 and C_2, C_1 and C_2 have the form $\forall\varphi\neg\forall\psi\neg G(\varphi,\psi)$ where $\neg\forall\psi\neg G(\varphi,\psi)$ is reducible. Thus, C can be non-reducible if C is in the antecedent of a sequent and has the form $\forall\varphi\neg\forall\psi\neg G'(\varphi,\psi)$, where $\neg\forall\psi\neg G'(\varphi,\psi)$ is reducible. If the D in the antecedent of S_2 belongs to B - E, then so do the $\tilde{D}$ in the antecedents of S_3 and the $\tilde{D}$ in S_1 (see Corollary 2) of §4 and [B5.4,P].) Therefore both the $\tilde{D}$ in S_1 and C_1 belong to B. This implies that in P' both C_1 and C_2 belong to B', and hence by definition no ancestors of C_1 and C_2 belong to E', contradicting the assumption that C belongs to E'. Thus the latter case does not arise.

For other formulas, [B5.5,P'] follows from [B5.5,P] and the definition.

Now define d(J';P') = d(J;P) for every substitution J' in P', where J is the corresponding substitution in P. d(A';P') is defined as D1 - D3. Notice that if a substitution J' affects a formula A' in P', then the corresponding substitution J in P affects the corresponding formula A. Also A and A' are identical and hence $dp(J;A) = dp(J';A')$. Moreover, if there is a substitution above S_3, we may assume that the substitution variable is different from any other eigen variable. Therefore $d(A';P') = d(A;P)$. So D4 follows from [D4,P] and D5 follows from [D5,P]. If there is a key principal formula (of dim i) F' in an i-block B' and the eigen variable of a substitution J' occurs in a formula in B', say C', but not in F', then the corresponding formula F belongs to B and the eigen variable of the corresponding J occurs in the corresponding formula C in B but not in F. So follows D6 from [D6,P]. (Here by the corresponding formula of a descendent of $\tilde{D}$ in P', we mean the same formula in P, not the $\tilde{D}$ in S_1.)

All substitutions in P' are under any logical inference or induction since P satisfies the condition.

2°. (8.5.2*) Suppose P is of the form

$$\dfrac{\begin{matrix} S_2 & D \rightarrow D & & \\ & \vdots & & \vdots \\ S_3 & \Gamma_1, \tilde{D}, \Gamma_2 \rightarrow \Delta, \tilde{D} & S_1 & \tilde{D}, \Pi \rightarrow \Lambda \end{matrix}}{S_4 \quad \Gamma_1, \tilde{D}, \Gamma_2, \Pi \rightarrow \Delta, \Lambda}$$

$$\begin{matrix} & \vdots \\ (S) & \rightarrow \end{matrix}$$

Define P' as

$$\begin{matrix} & \vdots \\ S_1' & \tilde{D}, \Pi \rightarrow \Lambda \\ \hline\hline S_4' & \Gamma_1, \tilde{D}, \Gamma_2, \Pi \rightarrow \Delta, \Lambda \\ & \vdots \\ (S') & \rightarrow \end{matrix}$$

Define the blocks and their entrances just as in 1°. C_1 is now the last descendent of the $\tilde{D}$ which is a right cut formula and C_2 is its left cut formula. It is easily seen that the same Corollary as for 1° holds, and the same argument as in 1° goes through. The only conceivable trouble exists in B5.5. Suppose, for example, there is a non-reducible formula C' which is C_1 or an ancestor of C_1 and belongs to E'. Then Case 1) or Case 2.2) holds. Notice that in particular, C' belongs to B'. Since by [B5.2,P'] all ancestors of C' belong to E', we may assume that C' occurs above S_4'. (See 1°). The $\tilde{D}$'s, D and C_1 are also non-reducible. The formula C in P which corresponds to C' is identical with C' and belongs to B. If C belongs to B - E, then all its descendents, in particular the $\tilde{D}$ in S_1, belong to B - E. Then the $\tilde{D}$ in the succedent of S_3 and hence the D in the succedent of S_2 belong to B - E, which implies that the D in the antecedent of S_2 belongs to B.

If Case 1) holds, then C_1 in P' does not belong to B'. This means that C_1 in P does not belong to B, and hence any ancestor of C_1, in particular the D in the antecedent of S_2, does not belong to B - E, which implies that it belongs to E. Thus [B5.5,P] applies to the D in S_2. Therefore D is of the form $\forall \varphi \neg \forall \psi \neg G(\varphi,\psi)$, where $\neg \forall \psi \neg G(\varphi,\psi)$ is reducible; hence C as well as $\tilde{D}$ have the same form. As C is an ancestor of the $\tilde{D}$ in the antecedent of S_1, C must occur in the antecedent of a sequent, and the same applies to C' in P'.

If Case 2.2) holds, then C_1 belongs to B'. This implies that in P C_1 and all its ancestors belong to B. If C_1 belongs to B - E, then C_2 must belong to B - E, which implies that in P' C_2 belongs to B', contradicting the condition of Case 2.2). Therefore C_1 must belong to E. C_1 is non-reducible. So C_1 should satisfy the condition and hence C, its ancestor, also satisfies the condition.

Suppose, as another example, that C is a non-reducible formula which is C_2 or its ancestor and which belongs to E'. If in P C belongs to E, the entrance of B, then it satisfies the condition of [B5.5,P], and hence it satisfies the same condition in P'. Suppose C does not belong to E. Then C belongs to B - E, and hence C_2 and C_1 as well as all their ancestors belong to B - E. This implies that the D in the succedent of S_2 belongs to B. Since D is non-reducible, this D cannot belong to E by [B5.5,P]. Therefore this D belongs to B - E. Then the $\tilde{D}$ in the antecedent of S_1 belongs to B - E, which implies that the descendents of the $\tilde{D}$, in particular C_1, belong to B - E. This implies that C_2 belongs to B - E. Therefore by definition both C_1 and C_2 belong to B' in P'; then no ancestor of C_2, in particular C', can belong to E', contradicting the assumption. Thus this case does not arise.

For any other formula, this follows from [B5.5,P].

6.4 The elimination of the weakenings from the end piece (8.6*). Let Q be any subproof of P such that the end sequent of Q belongs to the end piece of P. Following 8.6*, define Q* by induction on the number of inferences (in the end piece) of Q, according to the last inference. P' is then defined as P*. We should remark here that we may assume that if the last substitution, say I, is eliminated, it means that all the ancestors of the substitution formulas of I are weakening formulas or the direct descendents of such. Therefore we may assume that in such a case I does not affect any other formulas in Q. This remark is useful when we later prove the condition D for P'.

Corollary. Q* is defined in a manner that for every formula C' in Q* there is a naturally corresponding formula C in Q, and P and P' (which is P*) differ only in that all the weakenings in the end piece of P are eliminated in P'; otherwise P' is a copy of P.

Define a block of P', say B' and its entrance E', corresponding to a block B of P and its entrance E as follows. A formula in P', say C', belongs to B' (E') if and only if its corresponding formula C in P belongs to B (E).

The conditions C1 - C3 and B1 - B5 follow directly from the above corollary and the definition of blocks of P' and those for P, since the properties and the relations of the formulas described in C and B do not change. For every substitution J' in P', define d(J') as d(J';P') = d(J;P), where J is the corresponding substitution in P. Define d(A';P') for every semi-formula A' in P' as D1 - D3. If a substitution J in P disappears in P', then its eigen variable does not occur in P'. Therefore for any formula A' in P', d(A;P') = d(A;P) holds, where A corresponds to A'. Thus D4 - D6 follow from [D,P].

Now we shall assume that the end piece of a proof with degree, P, contains one of the logical inferences, induction, beginning sequents other than the mathematical beginning sequents or weakenings. We also assume that the proof is different from its end piece. The existence of a suitable cut is proved as in 9* of Chapter 2, [5]. We can now proceed to the essential reductions. (Cf. 10* of Chapter 2, [5].) Let J be a lower-most suitable cut and let # stand for the outer-most logical connective of the cut formulas of J.

6.5 # is 'second order $\exists$'. Recall that the cut formula is of the form $\exists \varphi \forall \psi H(\varphi,\psi)$. Suppose P is of the following form:

$$
\dfrac{
\begin{array}{ll}
 & \vdots \\
S_0 & \Gamma_1 \to \Delta_1, \forall \psi H_1(V,\psi) \\
I_1 & \overline{\qquad\qquad\qquad\qquad\qquad} \\
S_1 & \Gamma_1 \to \Delta_1, \exists\varphi\forall\psi H_1(\varphi,\psi) \\
 & \vdots \\
S_2 & \Gamma_2 \to \Delta_2, \exists\varphi\forall\psi H(\varphi,\psi)
\end{array}
\qquad
\begin{array}{ll}
 & \vdots \\
S_3 & \forall \psi H_2(\alpha,\psi), \Pi_1 \to \Lambda_1 \\
I_2 & \overline{\qquad\qquad\qquad\qquad\qquad} \\
S_4 & \exists\varphi\forall\psi H_2(\varphi,\psi), \Pi_1 \to \Lambda_1 \\
 & \vdots \\
S_5 & \exists\varphi\forall\psi H(\varphi,\psi), \Pi_2 \to \Lambda_2
\end{array}
}{
S_6 \quad \Gamma_2, \Pi_2 \to \Delta_2, \Lambda_2
}\, J
$$

$$
\begin{array}{c}
\vdots \\
\to
\end{array}
$$

where H_1, H_2 and H are identical except some terms. This is because of [D3,P], since any substitution which influences H_1 (H_2)

affects $\exists\,\varphi\,\forall\,\psi H_1(\varphi,\psi)$ $(\exists\,\varphi\,\forall\,\psi H_2(\varphi,\psi))$. Therefore in the following we may omit the subscripts of H_1 and H_2.

Define P' as follows. Due to the limit of space, we first define subproofs P_1 and P_2, and then define P' in terms of P_1 and P_2.

P_1:

$$
\begin{array}{ll}
 & \vdots \\
S_3' & \forall\psi H(V,\psi),\Pi_1 \rightarrow \Lambda_1 \\
\hline\hline
S_4' & \exists\varphi\forall\psi H(\varphi,\psi),\Pi_1,\forall\psi H(V,\psi) \rightarrow \Lambda_1 \\
 & \vdots
\end{array}
$$

$$
\begin{array}{llll}
 & \vdots & & \\
S_2 & \Gamma_2 \rightarrow \Delta_2,\exists\varphi\forall\psi H(\varphi,\psi) & S_5' & \exists\varphi\forall\psi H(\varphi,\psi),\Pi_2,\forall\psi H(V,\psi) \rightarrow \Lambda_2 \\
\hline\hline
 & & S_6' & \forall\psi H(V,\psi),\Gamma_2,\Pi_2 \rightarrow \Delta_2,\Lambda_2
\end{array}
$$

P_2:

$$
\begin{array}{ll}
 & \vdots \\
S_0' & \Gamma_1 \rightarrow \Delta_1,\forall\psi H(V,\psi) \\
\hline\hline
S_1' & \Gamma_1 \rightarrow \forall\psi H(V,\psi),\Delta_1,\exists\varphi\forall\psi H(\varphi,\psi) \\
 & \vdots
\end{array}
$$

$$
\begin{array}{llll}
 & & & \vdots \\
S_2' & \Gamma_2 \rightarrow \forall\psi H(V,\psi),\Delta_2,\exists\varphi\forall\psi H(\varphi,\psi) & S_5 & \exists\varphi\forall\psi H(\varphi,\psi),\Pi_2 \rightarrow \Lambda_2 \\
\hline\hline
 & S_6'' & & \Gamma_2,\Pi_2 \rightarrow \Delta_2,\Lambda_2,\forall\psi H(V,\psi)
\end{array}
$$

P':

$$
\begin{array}{ll}
 & P_1 \qquad P_2 \\
\hline\hline
S_7 & \Gamma_2,\Pi_2,\Gamma_2,\Pi_2 \rightarrow \Delta_2,\Lambda_2,\Delta_2,\Lambda_2 \\
 & \vdots \\
 & \rightarrow
\end{array}
$$

Let B be an i-block of P and E be its entrance. Then define the corresponding i-block B' and its entrance E' as follows. Any one of the explicitly indicated $\exists \varphi \forall \psi H(\varphi,\psi)$ or $\forall \psi F(V,\psi)$ belongs to B' (E') if and only if the corresponding formula $\exists \varphi \forall \psi H(\varphi,\psi)$ in P belongs to B (E). (A sequent between S_3' and S_4' or S_o' and S_1' may be treated as S_4' or S_1' respectively, and the $\forall \psi H(V,\psi)$ in S_6' behaves like the $\exists \varphi \forall \psi H(\varphi,\psi)$ in S_5 and the $\forall \psi H(V,\psi)$ in S_6'' behaves like the $\exists \varphi \forall \varphi H(\varphi,\psi)$ in S_2.) Any other formula belongs to B' (E') if and only if its corresponding formula belongs to B (E).

Notice that, since V is reducible, the substitution of V for α does not change the reducibility of a formula. It should be also noticed that for every A' in P', either the corresponding A is identical with A' or A' is obtained from A by substituting V for α.

C1 - C3, B1 and B2 are direct consequences of [C,P], [B,P] and the definitions of P' and its blocks.

B2.3. Suppose, for example, the cut formula $\forall \psi H(V,\psi)$ in S_6', which is the auxiliary formula of a strong inference (cut), belongs to a formula bundle which contains a key principal formula of dim i. Then in P the $\forall \psi H_2(\alpha,\psi)$ in S_3 is such a formula, and therefore by [B2.3,P], it does not belong to any i-block. This implies that the $\exists \varphi \forall \psi F(\varphi,\psi)$ in S_5 does not belong to any i-block. Henceforth, by definition, the cut formula $\forall \psi F(V,\psi)$ in S_6' does not belong to any i-block. Similarly for the $\forall \psi H(V,\psi)$ in S_6''. As was mentioned at the beginning, there is no substitution which can apply to $\forall\psi H(V,\psi)$.

B3. Notice that V and $\forall\psi H(V,\psi)$ do not contain any eigen variable except those of substitution. Therefore the new occurrences of V and $\forall\psi H(V,\psi)$ do not cause any new situation.

B4. Let B' be an i-block and suppose A' and F' satisfy the conditions on A and F in B4 for B'. We should emphasize that A' does not occur in the end piece. Let B, A and F in P correspond to B', A' and F' respectively. Then F' is either F itself or $F\binom{\alpha}{V}$ and A' is either A itself or $A\binom{\alpha}{V}$. It is obvious that F and A satisfy the condition for B; in particular A does not occur in the end piece of P.

B4.1. If α does not occur in A, then this follows from [B4.1,P], since $\delta_j(F) \leq \delta_j(F')$ for every $j \leq n$. Suppose α occurs in A. It implies that in P α occurs in B.

Case 1) The $\forall\psi H(\alpha,\psi)$ in S_3 lies outside B. Then α occurs both inside and outside B, and α is a non-substitution eigen variable in P. Therefore by [B3,P] α occurs in F. By [B4,P], the following (a) and (b) hold.

(a) $\delta_i(A(\alpha)) < \delta_i(F(\alpha))$ and $\delta_j(A(\alpha)) \leq \delta_j(F(\alpha))$ if $j \geq i + 1$.

(b) $dp_k(\alpha;A) = 0$ for all $k \geq i$.

(b) and Proposition 1.6 for A and V yield

(c) $\delta_k(A(V)) = \max(\delta_k(A(\alpha)),\delta_k(V))$ for $k \geqq i$.

Since α actually occurs in F Proposition 1.6 applies to F and V, and hence

(d) $\delta_k(F(V)) = \max(\delta_k(F(\alpha)), \delta_k(V) + dp_k(\alpha;F(\alpha)))$ if $dp_j(\alpha;F(\alpha)) = 0$, for all $j \geq k + 1$.

Here we shall observe the following fact. Suppose F is of the form $\exists\, \eta \forall \chi F(\eta,\chi)$. Then $\forall\, \chi F(\eta,\chi)$, which may be called a semi-subformula of F satisfies the condition on A in P together with B and F. Therefore by [B4.3,P] applied to B,F and $\forall\, \chi F(\eta,\chi)$, $dp_k(\alpha;\forall\, \chi F(\eta,\chi)) = 0$ if $i \leq k$. This implies that

(e) $dp_i(\alpha;F) = 1$ and $dp_j(\alpha;F) = 0$ if $i + 1 \leq j$, since the type of F is of dim i.

From (a), (c), (d) and (e) follows

$$\delta_i(A(V)) < \delta_i(F(V)), \text{ or } \delta_i(A') < \delta_i(F'),$$

and

$$\delta_j(A(V)) \leq \delta_j(F(V)) \text{ for } j \geq i + 1.$$

Case 2) The $\forall\, \psi H(\alpha,\psi)$ in S_3 belongs to B. Then the dim of the type of $\exists\, \varphi \forall\, \psi H(\varphi,\psi)$, say j, is less than i ([B2.1,P]). Therefore

$$\delta_k(\exists\, \varphi \forall\, \psi H(\varphi,\psi)) = \delta_k(\forall\, \psi H(\alpha,\psi)) \text{ for } k \geq i.$$

By [B1.3,P] $\exists\, \varphi \forall\, \psi H(\varphi,\psi)$ in S_4 belongs to B. On the other hand, [C1,P] and [B1.3,P] require that the $\exists\, \varphi \forall\, \psi H_1(\varphi,\psi)$ in S_1 and $\forall\, \psi H_1(V,\psi)$ in S_o belong to some j-blocks $\tilde{B}$, and by [B4.1,P] for $\tilde{B}$, $\exists\, \varphi \forall\, \psi H(\varphi,\psi)$ and $\forall\, \psi H(V,\psi)$.

$$\delta_k(V) \leq \delta_k(\forall\ \psi H(V,\psi)) \leq \delta_k(\exists\ \varphi\ \forall\ \psi H(\varphi,\psi)) \text{ for } k \geq i,$$

where the first $\leq$ holds because $dp_k(\alpha;\forall\ \psi H(\alpha,\psi)) = 0$. (This follows from the fact that $\exists\ \varphi\ \forall\ \psi H(\varphi,\psi)$ is reducible.) In B, $\forall\ \psi H_2(\alpha,\psi)$ satisfies the condition on A, since this is of dim less than i, and hence it cannot be a descendent of a key principal formula of dim i by [B2.3,P]. Thus by [B4.1,P] for B,F and $\forall\ \psi\ H_2(\alpha,\psi)$,

$$\delta_i(\exists\ \varphi\ \forall\ \psi H(\varphi,\psi)) = \delta_i(\forall\ \psi H(\alpha,\psi)) < \delta_i(F),$$

and $\leq$ hold for δ_j for $j \geq i + 1$. Combining the above two results, we have

(f) $\delta_i(V) < \delta_i(F)$ and $\delta_j(V) \leq \delta_j(F)$ if $j \geq i + 1$.

(a), (b) and (c) in Case 1) are valid for Case 2) too. From (a), (c) and (f), we have $\delta_i(A) < \delta_i(F)$ and $\delta_i(V) < \delta_i(F)$, and hence $\delta_i(A(V)) < \delta_i(F) \leq \delta_i(F')$. Let $j \geq i + 1$. $\delta_j(A) \leq \delta_j(F)$ from (a) and $\delta_j(V) \leq \delta_j(F)$ from (f). Hence by (c) $\delta_j(A(V)) \leq \delta_j(F) \leq \delta_j(F')$.

B4.2. Suppose β is a substitution variable which affects F'. If β affects F, then from [B4.2,P],

(g) $dp_i(\beta;A) < dp_i(\beta;F)$ and $dp_j(\beta;A) \leq dp_j(\beta;F)$ for $j \geq i + 1$.

If β does not affect F (i.e. does not occur in F), then β occurs in V and α occurs in F. From [B4.3,P],

(g') $dp_j(\beta;A) = 0$ for $j \geq i$.

If α does not occur in A, or β does not occur in V, then either $dp_i(\beta;A') = dp_i(\beta;A) < dp_i(\beta;F) \leq dp_i(\beta;F')$ and $dp_j(\beta;A') = dp_j(\beta;A) \leq dp_j(\beta;F) \leq dp_j(\beta;F')$ for $j \geq i + 1$ by (g), or

$$dp_i(\beta;A') = dp_i(\beta;A) = 0 < 1 \leq dp_i(\beta;F')$$

and $dp_j(\beta;A') = 0 \leq dp_j(\beta;F')$ for $j \geq i + 1$ by (g'). So we are done. Therefore let us assume that α **actually** occurs in A and β occurs in V.

Case 1) $\forall \psi H_2(\alpha,\psi)$ does not belong to B. Then α occurs both inside and outside B. This together with [B3,P] implies that α occurs in F. On the other hand, by [B4.3,P] applied to α, $dp_k(\alpha;A) = 0$ for $k \geq i$. Therefore

$$\text{(h)} \qquad dp_k(\beta;A') = \max(dp_k(\beta;A), dp_k(\beta;V)) \text{ for } k \geq i,$$

by Proposition 1.7. (Recall that β occurs in V.) On the other hand (e) in the proof of B4.1 is valid here, and so $dp_i(\alpha;F) = 1$ and $dp_j(\alpha;F) = 0$ if $j \geq i+1$. Thus, again by Proposition 1.7,

$$\text{(i)} \qquad dp_k(\beta;F') = \max(dp_k(\beta;F), dp_k(\beta;V) + dp_k(\alpha;F))$$

if $k \geq i$. Now, by (g)-(i),

$$dp_i(\beta;A') < dp_i(\beta;F')$$

and

$$dp_j(\beta;A') \leq dp_j(\beta;F') \text{ for } j \geq i + 1.$$

Case 2) $\forall\ \psi H_2(\alpha,\psi)$ belongs to B. Then the dim of $\exists\ \varphi\ \forall\ \psi H(\varphi,\psi)$, say j, must be less than i ([B2.1,P]). By [B1.3,P], $\exists\ \varphi\ \forall\ \psi H_2(\varphi,\psi)$ in S_4 belongs to B. On the other hand, the $\exists\ \varphi\ \forall\ \psi H_1(\varphi,\psi)$ in S_1 and $\forall\ \psi H_1(V,\psi)$ in S_o belong to some j_o-block, say $\tilde{B}$, and hence by [B4.2,P] and [B4.3,P] applied to $\tilde{B}$, $\exists\ \varphi\ \forall\ \psi H_1(\varphi,\psi)$ and $\forall\ \psi H_1(V,\psi)$, we have

$$\begin{aligned} dp_k(\beta;V) \le dp_k(\beta;\forall\ \psi H_1(V,\psi)) &\le dp_k(\beta;\exists\ \varphi\ \forall\ \psi H_1(\varphi,\psi)) \\ &= dp_k(\beta;\exists\ \varphi\ \forall\ \psi H_2(\varphi,\psi)) \end{aligned}$$

for all $k \ge i > j_o$. (See the proof of B4.1). In B, $\forall\ \psi H_2(\alpha,\psi)$ satisfies the condition on A. (See the proof of B4.1.) Therefore by [B4.3,P] for B,F and $\forall\ \psi H_2(\alpha,\psi)$ (as β does not occur in F),

$$dp_k(\beta;\exists\ \varphi\ \forall\ \psi H_2(\varphi,\psi)) = dp_k(\beta;\forall\ \psi H_2(\alpha,\psi)) = 0$$

for all $k \ge j_o+1$ in particular for $k \ge i$. Hence

$$dp_k(\beta;V) \le dp_k(\beta;\exists\ \varphi\ \forall\ \psi H_2(\varphi,\psi)) = 0$$

if $k \ge i$.

Combining this with (h) (which is valid for Case 2) as well) we obtain

$$dp_k(\beta;A') = dp_k(\beta;A)$$

for all $k \ge i$. Therefore, if β occurs in F, then the desired inequalities follow from (g). If β does not occur in F, then α must occur in F. Therefore

$$dp_k(\beta;F') = dp_k(\beta;V) + dp_k(\alpha;F) = dp_k(\alpha;F),$$

and

$$dp_k(\beta;A') = dp_k(\alpha;A).$$

But $dp_i(\alpha;F) \geq 1$ and $dp_k(\alpha;A) = 0$ for all $k \geq i$ ([B4.3,P]). Therefore

$$dp_i(\beta;A') < dp_i(\beta;F')$$

and $\leq$ holds for dp_k, $k \geq i + 1$.

B4.3. Suppose β is a non-substitution eigen variable or a substitution variable which does not occur in F'. The latter case implies that in particular β does not occur in F. Therefore in either case $dp_k(\beta;A) = 0$ for $k \geq i$ by [B4.3,P]. If α does not actually occur in A, then $dp_k(\beta;A') = dp_k(\beta;A) = 0$, for $k \geq i$. Assume α occurs in A. If β is a non-substitution variable, then β does not occur in V. Therefore by [B4.3,P] $dp_k(\beta;A(V)) = dp_k(\beta;A(\alpha)) = 0$ for $k \geq i$. If β is a substitution variable but does not occur in V, then the same equations hold.

Now suppose that β is a substitution variable which occurs in V. If $\forall \psi H_2(\alpha,\psi)$ lies outside B, then α occurs both inside and outside B. So α occurs in F and hence F' must contain β, contradicting the hypothesis. Therefore $\forall \psi H_2(\alpha,\psi)$ belongs to B, and hence $\exists \varphi \forall \psi H_2(\varphi,\psi)$ is of dim, say j_0, less than i. $\exists \varphi \forall \psi H_2(\varphi,\psi)$ belongs to B ([B1.3,P]), and $\forall \psi H_1(V,\psi)$ and $\exists \varphi \forall \psi H_1(\varphi,\psi)$ belong to some j_0-section $\tilde{B}$. [B4,P] applies to $\tilde{B}$ and those formulas: in particular

$dp_j(\alpha;\forall\ \psi H_1(\alpha,\psi)) = 0$ if $j \geq j_o$, and by [B4.2 and B4.3,P]

$$dp_k(\beta;V) \leq dp_k(\beta;\forall\ \psi H_1(V,\psi)) \leq dp_k(\beta;\exists\ \varphi\ \forall\ \psi H_1(\varphi,\psi))$$
$$= dp_k(\beta;\forall\ \psi H_2(\alpha,\psi))$$

for $k \geq j_o + 1$; in particular $k \geq i$. The last term is 0 by [B4.3,P] with $\forall\ \psi H_2(\alpha,\psi)$ as A, since F does not contain β. So, $dp_k(\beta;A(V)) = dp_k(\beta;A(\alpha)) = 0$ if $k \geq i$.

B5. We have defined entrances, and B5.1-B5.5 follow from [B5,P] as obvious consequences of the definition.

For every substitution J' define d(J';P') = d(J;P), where J corresponds to J'. Then define the degrees of semi-formulas as D1-D3. It follows that if A and A' are identical, then

(j) $$d(A';P') = d(A;P).$$

In particular this holds for $\exists\ \varphi\ \forall\ \psi H(\varphi,\psi)$, and $d(\forall\ \psi H(V,\psi);P') = d(\forall\ \psi H_1(V,\psi H_1(V,\psi);P)$ for any occurence of $\forall\ \psi H(V,\psi)$. Notice that the $\forall\ \psi H(V,\psi)$ in S_o and the $\exists\ \varphi\ \forall\ \psi H(\varphi,\psi)$ in S_1 are the auxiliary formula and the principal formula of a key inference in P. So, by Corollary 7) in §4,

(k) $$d(\forall\ \psi H(V,\psi);P) < d(\exists\ \varphi\ \forall\ \psi H(\varphi,\psi);P).$$

D4. Such a formula A' is either identical with its corresponding formula A or is $\forall\ \psi H(V,\psi)$. In the former case by (j), [D4,P] and the definition, d(A';P') = d(A;P) < d(J;P) = d(J';P'). For the latter case, (k),(j) and [D4,P] imply $d(A';P') = d(A';P) < d(\exists\ \psi\ \forall\ \psi H(\varphi,\psi);P) =$ $d(\exists\ \varphi\ \forall\ \psi H(\varphi,\psi);P') < d(j;P) = d(J';P')$.

D5 follows from [D5,P] and the definition.

D6. Suppose B', F' and J' satisfy the condition in D6 for P' and J' affects a formula A' in B'. Then the corresponding A belongs to B. Let F and J correspond to F' and J' respectively. Since the eigen variable of J does not occur in F', it does not occur in F either. If the eigen variable of J occurs in A (in P), then by [D6,P] $d(J';P') = d(J;P) < d(F;P) \leq d(F';P')$. Suppose the eigen variable of J does not occur in A. Then the eigen variable of J', say β, (and hence of J) occurs in V. There are two cases: Case 1) α occurs in A and A' is $A\binom{\alpha}{V}$ and Case 2) A' is one of the indicated $\forall\ \psi H(V,\psi)$'s and A is the corresponding $\exists\ \varphi\ \forall\ \psi H(\varphi,\psi)$.

Case 1) We first observe that the $\forall\ \psi H_2(\alpha,\psi)$ in S_3 must belong to B, for otherwise α occurs both inside and outside B, and hence [B3,P] implies that α occurs in F; which in turn means that in P' β occurs in F', contradicting our assumption. The fact that $\forall\ \psi H_2(\alpha,\psi)$ belongs to B implies together with [B2.1,P] that the dim of $\forall\ \psi H_2(\alpha,\psi)$, say j, is less than i. On the other hand [C1,P] requires that the $\exists\ \varphi\ \forall\ \psi H_1(\varphi,\psi)$ in S_1 belongs to some j-block, say $\tilde{B}$.

Suppose first that β does not occur in $\exists\ \varphi\ \forall\ \psi H_1(\varphi,\psi)$. Then, as β occurs in $\forall\ \psi H_1(V,\psi)$, [D6,P] applied to $\tilde{B}$, $\exists\ \varphi\ \forall\ \psi H_1(\varphi,\psi)$ and $\forall\ \psi H_1(V,\psi)$ implies

(ℓ) $\quad d(J;P) < d(\exists\ \varphi\ \forall\ \psi H_1(\varphi,\psi);P) = d(\exists\ \varphi\ \forall\ \psi H_2(\varphi,\psi);P)$.

[B4,P] applied to B,F and $\forall\ \psi H_2(\alpha,\psi)$ implies the following:

$$\delta_i(\forall\ \psi H_2(\alpha,\psi)) < \delta_i(F)$$

and

$$\delta_k(\forall\ \psi H_2(\alpha,\psi)) \leq \delta_k(F) \quad \text{if} \quad k \geq i + 1.$$

This implies

(m) $$\delta_i(\exists\ \varphi\ \forall\ \psi H_2(\varphi,\psi)) < \delta_i(F)$$

and

$$\delta_k(\exists\ \varphi\ \forall\ \psi H_2(\varphi,\psi)) \leq \delta_k(F) \quad \text{if} \quad k \geq i + 1,$$

since $\exists\ \varphi\ \forall\ \psi H_2(\varphi,\psi)$ is of dim less than i. If γ is a substitution variable which occurs in F, then by [B4,P]

$$dp_i(\gamma;\forall\ \psi H_2(\alpha,\psi)) < dp_i(\gamma;F)$$

and

$$dp_k(\gamma;\forall\ \psi H_2(\alpha,\psi)) \leq dp_k(\gamma;F) \quad \text{for} \quad k \geq i + 1.$$

Therefore for any such γ

(n) $$dp_i(\gamma;\exists\ \varphi\ \forall\ \psi H_2(\varphi,\psi)) < dp_i(\gamma;F)$$

and

$$dp_k(\gamma;\exists\ \varphi\ \forall\ \psi H_2(\varphi,\psi)) \leq dp_k(\gamma;F) \quad \text{if} \quad i + 1 \leq k.$$

Suppose γ is a substitution variable which does not occur in F. Then

$$dp_k(\gamma;\forall\ \psi H_2(\alpha,\psi)) = 0 \quad \text{for all} \quad k \geq i,$$

which implies

(o) $$dp_k(\gamma;\exists\ \varphi\ \forall\ \psi H_2(\varphi,\psi)) = 0 \quad \text{for all} \quad k \geq i.$$

If γ is the eigen variable of a substitution J_o such that γ occurs in B but not in F, then by [D6,P]

(p) $$d(J_o;P) < d(F;P).$$

(o),(p),(n) and (m) satisfy the conditions $1^o)-4^o)$ in Lemma for Corollary 7) in §4 for B,F and $\exists\varphi\forall\psi H_2(\varphi,\psi)$. Therefore by Lemma, $d(\exists\varphi\forall\psi H_2(\varphi,\psi);P) < d(F;P)$. Combining this with ($\ell$), we obtain

$$d(J';P) = d(J;P) < d(\exists\varphi\forall\psi H_2(\varphi,\psi);P) < d(F;P) \leq d(F';P').$$

Suppose next that β occurs in $\exists\varphi\forall\psi H_1(\varphi,\psi)$, and hence in $\forall\psi H_1(\alpha,\psi)$. Then taking this as A in [D6,P], we have

$$d(J';P') = d(J;P) < d(J;F) \leq d(J';F').$$

Case 2) A is $\exists\varphi\forall\psi H(\varphi,\psi)$ and A' is $\forall\psi H(V,\psi)$ (and β occurs in V).

Case 2.1) In P, A is a descendent of $\forall\psi H_1(V,\psi)$. Then, since A' belongs to B', $\forall\psi H_1(V,\psi)$ belongs to B by definition. Therefore D6 follows from [D6,P] applied to B,F,J.

Case 2.2) In P, A is a descendent of $\forall\psi H_2(\alpha,\psi)$. If β occurs in $\forall\psi H_2(\alpha,\psi)$, then this follows from [D6,P]. If β does not occur in $\forall\psi H_2(\alpha,\psi)$ but occurs in V, then the same argument as in Case 1) goes through.

In the following 6.6-6.9, we consider various cases where # is 'second order $\forall$'. (See (10.1*).)

6.6. # is 'second order $\forall$', the cut formula is reducible and the auxiliary formula of the boundary inference which introduces the cut formula under concern does not belong to any block.

Suppose that P has the following form.

$$
\dfrac{
\begin{array}{ll}
 & \vdots \\
S_o & \dfrac{\Gamma_1 \rightarrow \Delta_1, H_1(\alpha)}{\Gamma_1 \rightarrow \Delta_1, \forall\varphi H_1(\varphi)}\, J_o \\
S_1 & \\
 & \vdots \\
S_2 & \Gamma_2 \rightarrow \Delta_2, \forall\varphi H(\varphi)
\end{array}
\qquad
\begin{array}{ll}
 & \vdots \\
S_3 & \dfrac{H_2(V), \Pi_1 \rightarrow \Lambda_1}{\forall\varphi H_2(\varphi), \Pi_1 \rightarrow \Lambda_1} \\
S_4 & \\
 & \vdots \\
S_5 & \forall\varphi H(\varphi), \Pi_2 \rightarrow \Lambda_2
\end{array}
}{
\begin{array}{ll}
S_6 & \Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2 \\
 & \vdots \\
S_7 & \Gamma_3 \rightarrow \Delta_3 \\
 & \vdots \\
 & \rightarrow
\end{array}
}
$$

where $\Gamma_3 \rightarrow \Delta_3 (S_7)$ is the i-loader of S_6. It is obvious from [D,P] that there is no substitution between S_1 and S_2 and between S_4 and S_5 which influences $\forall\varphi H_1(\varphi)$ and $\forall\varphi H_2(\varphi)$ respectively. Therefore we may omit the subscripts 1 and 2. Define P' as follows. (See (10.1.1*).)

$$\vdots$$

$$S'_0 \qquad \Gamma_1 \to \Delta_1, H(\alpha)$$

$$\overline{\overline{}}$$

$$S'_1 \qquad \Gamma_1 \to H(\alpha), \Delta_1, \forall\varphi H(\varphi)$$

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots$$

$$S'_2 \qquad \Gamma_2 \to H(\alpha), \Delta_2, \forall\varphi H(\varphi) \qquad\qquad S_5 \qquad \forall\varphi H(\varphi), \Pi_2 \to \Lambda_2$$

$$S'_6 \qquad \Gamma_2, \Pi_2 \to H(\alpha), \Delta_2, \Lambda_2$$

$$\vdots$$

$$S'_7 \qquad \Gamma_3 \to H(\alpha), \Delta_3$$

$$\overline{\overline{}}$$

$$\Gamma_3 \to \Delta_3, H(\alpha)$$

$$\overline{}\ J_1 \qquad\qquad \vdots$$

$$S_8 \qquad \Gamma_3 \to \Delta_3, H(V) \qquad\qquad S'_3 \qquad H(V), \Pi_1 \to \Lambda_1$$

$$\Gamma_3, \Pi_1 \to \Delta_3, \Lambda_1$$

$$\overline{\overline{}}$$

$$S_9 \qquad \forall\varphi H(\varphi), \Pi_1, \Gamma_3 \to \Delta_3, \Lambda_1$$

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots$$

$$S_2 \qquad \Gamma_2 \to \Delta_2, \forall\varphi H(\varphi) \qquad S_{10} \qquad \forall\varphi H(\varphi), \Pi_2, \Gamma_3 \to \Delta_3, \Lambda_2$$

$$\Gamma_2, \Pi_2, \Gamma_3 \to \Delta_2, \Delta_3, \Lambda_2$$

$$\overline{\overline{}}$$

$$S_{11} \qquad \Gamma_2, \Pi_2, \Gamma_3 \to \Delta_3, \Delta_2, \Lambda_2$$

$$\vdots$$

$$\Gamma_3, \Gamma_3 \to \Delta_3, \Delta_3$$

$$\overline{\overline{}}$$

$$S_{12} \qquad \Gamma_3 \to \Delta_3$$

$$\vdots$$

$$\to$$

Notice that J_1 is the new substitution and that no substitution applies to $H(\alpha)$ except J_1.

Define a block B' and its entrance E' corresponding to each block B of P and its entrance E as follows. None of the indicated H(α)'s belongs to any block. The principal formula of the substitution J_1, i.e. H(V), does not belong to any block. Any other formula belongs to B' if and only if its corresponding formula in P belongs to B. A formula in P' which belongs to a block B' belongs to its entrance E' if and only if its corresponding formula in P belongs to the entrance E of B. Recall that H(α)'s and the H(V) in S_8 do not belong to any block. In particular a formula in Γ_3 or Δ_3 in one of the indicated places belongs to B' (E') if and only if its corresponding formula in S_7 belongs to B' (E').

Now we shall prove that all the conditions are satisfied. We omit all the easy consequences of the conditions on P and the definitions.

C2 and C3 follow from the fact that for every formula A' in P' there is a corresponding formula A in P such that A' belongs to B' if and only if A belongs to B.

B1.1. Since the new H(α)'s and the H(V) in S_8 do not belong to any block, we do not have to worry about them. If, for example, a formula Γ_3 in S_{12} belongs to B', then by definition the corresponding formula in S_7 belongs to B. Therefore, again by definition the same formula of Γ_3 in any other indicated sequent in P' belongs to B'; also any ancestor of it above S_7' belongs to B' since any ancestor of the corresponding formula in S_7 belongs to B.

B1.2 - B2.2 and B3. The new formulas H(α)'s and H(V) are irrelevant to those conditions.

B2.3. The H(V) in S_8 lies outside any block by definition. Notice that to the H(V) in S_3' corresponds the auxiliary formula of a strong inference in P. For any other strong inference this follows from [B2.3,P].

B4. If a formula A' which is not in the end piece of P' belongs to an i-block B', then the corresponding formula A belongs to B in P, and A and A' are identical. Therefore if B',A' and F' satisfy the condition in B4 for P', then the corresponding B,A, and F satisfy the same condition for P.

B4.2. α is a new substitution variable in P'. However, it is a non-substitution variable in P and hence by [B4.3,P] $dp_k(\alpha;A) = 0$ $(= dp_k(\alpha;A'))$ for all $k \geq i$. But if α affects a key principal formula F' of dim i in B', then $dp_i(\alpha;F') \geq 1$. Thus $dp_i(\alpha;A') < dp_i(\alpha;F')$ and $dp_k(\alpha;A') \leq dp_k(\alpha;F')$ if $k \geq i + 1$. For any other variable, this follows from [B4.2,P] and the definition.

B4.3. If β is not α and satisfies the condition for P', then it satisfies the same condition for P; hence by [B4.3,P] $dp_k(\beta;A') = 0 = dp_k(\beta;A)$ if $k \geq i$. $dp_k(\alpha;A') = dp_k(\alpha;A) = 0$ as in B4.2.

B5. We have defined the entrances.

B5.4. The cut formula H(V) in S_8 does not belong to any block. If the cut formula H(V) in S_3' belonged to B' - E', then the corresponding formula in S_3 would belong to B - E by definition. Therefore by Corollary 2) the $\forall \varphi H(\varphi)$ in S_5 would belong to B - E. Hence by [B5.4,P] the $\forall \varphi H(\varphi)$ in S_2

must belong to B - E, which implies that $H_1(\alpha)$ must belong to B ([B5.2,P]), contradicting our major assumption. Thus, the H(V) in S_3' cannot belong to a B' - E'. For any other cut formulas, this follows from [B5.4,P] and the definition.

Define $d(J_1;P') = d(\forall\varphi H(\varphi);P)$ and $d(J';P') = d(J;P)$ for any other substitution J' in P', where J is the corresponding substitution in P. Let ι be $d(J_1;P')$. Define degrees for semi-formulas in P' as D1 - D3. Then $d(A';P') = d(A;P)$ if J_1 does not affect A'. Otherwise $\geq$ holds.

D4. No formula under S_1' and S_5 except $H(\alpha)$'s is influenced by J_1. J_1 does not affect $H(\alpha)$ since $\forall\varphi H(\varphi)$ is reducible. Therefore by definition

$$d(H(\alpha);P') = d(H(\alpha);P) < d(\forall\varphi H(\varphi);P) = d(J_1;P') \ (= \iota).$$

If there is a substitution J' between S_1' and S_2', then $\forall\varphi H(\varphi)$ is in the upper sequent of the corresponding J between S_1 and S_2 in P. So, by [D4,P],

$$d(H(\alpha);P') < d(\forall\varphi H(\varphi);P) < d(J;P) = d(J';P').$$

Suppose there is a substitution J' between S_6' and S_7'. Then there is a corresponding substitution J between S_6 and S_7 in P and so its degree is greater than ι, since S_7 is the ι-loader of S_6. Therefore

$$d(H(\alpha);P') < d(\forall\varphi H(\varphi);P) = \iota < d(J;P) = d(J';P').$$

Let A be a formula in Γ_3 or Δ_3. Then $d(A;P') = d(A;P) < \iota$ by [D,P], since S_7 is the ι-loader of S_6.

Suppose there is a substitution J' between S_9 and S_{10}. Then there is a corresponding substitution J in P between S_4 and S_5. Let A' be any formula in the upper sequent of J' and A be its corresponding formula in P. If A occurs between S_4 and S_5, then by [D4,P]

$$d(A';P') = d(A;P) < d(J;P) = d(J';P').$$

If A is in Γ_3 or Δ_3, then, since $\Gamma_3 \rightarrow \Delta_3$ (i.e. S_7) is an ι-loader os S_6,

$$d(A';P') = d(A;P) < \iota = d(\forall\varphi H(\varphi);P) < d(J;P) = d(J';P').$$

Suppose there is a substitution J' between S_{11} and S_{12}. Then there is a corresponding substitution J between S_6 and S_7 and, since S_7 is the ι-loader of S_6, $d(J;P) > \iota$. Therefore if A' is any formula in Γ_3 or Δ_3, then

$$d(A';P') = d(A;P) < \iota < d(J;P) = d(J';P').$$

For any other substitution, D4 follows from the above remark and [D4,P].

D5. $d(J_1;P') = d(\forall\ \varphi H(\varphi);P) = \iota$ and $0 < \iota < \omega^{n+1}$ by definition.

D6. Suppose F' is a key principal formula in an i-block B'. Then the corresponding formula F belongs to B in P. Suppose the eigen variable of J_1, i.e. α occurs in a formula A' in B' but not in F'. Then α does not occur in F and A belongs to B, and hence α occurs in B. On the other hand, $H_1(\alpha)$ does not belong to B by our assumption. Therefore, by [B3,P], α occurs in F, contradicting our hypothesis. Therefore J_1 does not apply to this case.

Suppose the eigen variable of some other substitution J' occurs in an A' in B' but not F'. Then the eigen variable of the corresponding J occurs in A in B but not in F. So by [D6,P] $d(J';P') = d(J;P) < d(F;P) \leq d(F^1;P')$.

6.7. # is second order $\forall$, the cut formula $\forall\, \varphi H(\varphi)$ is reducible, the auxiliary formula of the boundary inference which introduces the cut formula under concern belongs to a block, and, if i_o is the smallest number such that $H_1(\alpha)$ belongs to an i_o-block B_o, then $H_1(\alpha)$ does not belong to its entrance. (See 6.6 for P.)

The entrance of B_o will be called E_o. P' is defined exactly as in 6.6. The blocks B' and their entrances E' are defined as follows, corresponding to the blocks B of P and their entrances E. The $H(\alpha)$ in one of $S'_1, \ldots, S'_2$ belongs to an i-block B' (its entrance E') if and only if the corresponding $\forall\, \varphi H(\varphi)$ belongs to B (E). The $H(\alpha)$ in one of $S'_6, \ldots, S'_7$ and the H(V) in S_8 belong to B' (E') if and only if the $\forall\, \varphi H(\varphi)$ in S_2 belongs to B (E). It follows that, in particular, all the $H(\alpha)$'s in $S'_1, \ldots, S'_7 \ldots$ and the H(V) in S_8 belong to $B'_o - E'_o$ (cf. Corollary 2) of §4). Any other formula belongs to a k-block B' (its entrance E') if and only if the corresponding formula in P belongs to B (E).

As for other cases, for every formula A' in P' there is a corresponding formula A in P such that A' belongs to a B' (E') if and only if A belongs to B (E). Therefore C1-C3 follow directly from [C,P].

B2.3. It is our assumption that in P, $H_1(\alpha)$ belongs to $B_o - E_o$, which implies that the cut formula $\forall \varphi H(\varphi)$ in S_2 belongs to $B_o - E_o$ (cf. Corollary 2)), and so the cut formula $\forall \varphi H(\varphi)$ in S_5 belongs to $B_o - E_o$ ([B5.4,P]). This in turn implies that $H_2(V)$ in S_3 belongs to $B_o - E_o$ ([B1.1 and B5.2,P]). Therefore by [B2.3,P] there cannot be a key subformula of dim i_o as an ancestor of $H_1(\alpha)$ or $H_2(V)$, since these are the auxiliary formulas of strong inferences. So, there is no problem about the auxiliary formula of J_1, i.e. $H(\alpha)$ and the cut formulas $H(V)$ in S_8 and S_3'. Suppose there is a key auxiliary formula of dim i ($i \neq i_o$) as an ancestor of the $H(V)$ in one of S_8 and in S_3'. Suppose one of them belongs to an i-block, say B'. Then by definition $H_1(\alpha)$ in S_o or $H_2(V)$ in S_3 respectively belongs to B in P. But those are auxiliary formulas of strong inferences. Therefore in either case, if ($H_1(\alpha)$ or $H_2(V)$) must lie outside any i-block ([B2.3,P]). So, from the above argument, neither $H(V)$ in S_3' nor the one in S_8 can belong to B'. For any other formulas this follows from [B2.3,P].

B4. Suppose A' and F' belong to an i-block B' and satisfy the condition B4. Recall that it is assumed that A' does not occur in the end piece of P', and hence the new $H(\alpha)$'s and $H(V)$'s are irrelevant. Also A' and A are identical unless A' is in the end piece and B,A and F satisfy the condition in B4.

B4.2. α is a substitution variable in P' but is a non-substitution eigen variable in P. Therefore by [B4.3,P],

$$dp_j(\alpha;A) = 0 = dp_j(\alpha;A') \text{ if } j \geq i.$$

If α affects F, then $dp_i(\alpha;F) = dp_i(\alpha;F') \geq 1$. Thus $dp_i(\alpha;A') < dp_i(\alpha;F')$ and $\leq$ holds for $j = i + 1$. For any other variable, this follows immediately from [B4.2,P].

B5.4. Suppose the H(V) in S_8 belongs to a B' - E'. Then by definition the cut formula $\forall \varphi H(\varphi)$ in S_2 belongs to B - E; hence by [B5.4,P] the $\forall \varphi H(\varphi)$ in S_5 belongs to B - E. Therefore by [B5.2,P] and [B1.1,P] $H_2(V)$ belongs to B - E, which implies that the cut formula H(V) in S_3' belongs to B' - E'.

For any other formula, this follows from the definition and [B5.4,P].

Define $d(J_1;P') = d(\forall \varphi H(\varphi);P)$ (= ι) and $d(J';P') = d(J;P)$ for any other substitution J' in P', where J is the corresponding substitution in P. Define degrees for semi-formulas in P' as D1 - D3.

D6. Let us consider J as the substitution in question. Suppose B' is an i-block, F' is a key principal formula of dim i in B' and α, the eigen variable of J_1, occurs in a formula $A'(\alpha)$ in B' but not in F'. Then the corresponding formula of F',F, is a key principal formula (of dim i) in B and is identical with F', and hence α does not occur in F.

Case 1) $i = i_o$ and B' is B_o'. Then A,F and $\forall \varphi H_1(\varphi)$ belong to B_o and

(*) B_o, F and $\forall\ \varphi H_1(\varphi)$ satisfy the conditions on B,F, and A in Lemma of §4.

(*) is proved below. Applying Lemma of §4, we have

$$d(J_1;P') = d(\forall\ \varphi H_1(\varphi);P) < d(F;P) = (F';P').$$

Proof of (*). We must show that $1^o) - 4^o)$ in the lemma hold.

$1^o)$ Let γ be a substitution variable (of J) in P. If γ does not occur in F, then since $H_1(\alpha)$ satisfies the condition on A in B4 ($H_1(\alpha)$ is not a descendent of a key principal formula of dim i_o), B4.3 implies $dp_j(\gamma;H_1(\alpha)) = 0$, and hence $dp_j(\gamma;\forall\ \varphi H_1(\varphi)) = 0$ if $j \geq i_o$.

$2^o)$ If γ as above occurs in B_o but not in F, then $d(J;P) < d(F;P)$ by [D6,P].

$3^o)$ If γ as above occurs in F, then $dp_{i_o}(\gamma;\forall\ \varphi H_1(\varphi)) = dp_{i_o}(\gamma;H_1(\alpha)) < dp_{i_o}(\gamma;F)$, and $\leq$ holds for $j \geq i_o$ by [B4.2,P].

$4^o)$ $\delta_{i_o}(\forall\ \varphi H_1(\alpha)) < \delta_{i_o}(F)$ and $\leq$ holds for $j \geq i_o$ by [B4.1,P].

Case 2) B is an i_o-block, but not B_o. Then α occurs both inside and outside B in P, since $H_1(\alpha)$ belongs to B_o. Therefore by [B3,P] α occurs in F, contradicting the assumption. Therefore this case does not arise.

Case 3) B is an i-block, where $i > i_o$, and B_o is included by B. Then A, F, $\forall\ \varphi H_1(\varphi)$ belong to B. Similarly to Case 1) we can show that the four conditions in Lemma are satisfied for B,F and $\forall\ \varphi H_1(\varphi)$. Therefore

$$d(J_1;P') = d(\forall\ \varphi H_1(\varphi);P) < d(F;P) = d(F';P').$$

Case 4) B is an i-block where $i > i_o$ and B_o is not included by B. This means that $B_o \cap B$ is empty by [C3,P]. Then $H_1(\alpha)$ lies outside B, which implies as Case 2), this case cannot happen.

Case 5) $i < i_o$ and B is included by B. $H_1(\alpha)$ does not belong to B by major assumption on i_o. Therefore α occurs both inside and outside B; hence by [B3,P] α occurs in F, contradicting the assumption. Therefore this case does not arise either. Consider some other substitution, say J'. Let B' be an i-block, F' be a key principal formula of dim i in B' and A' be a formula in B' such that the eigen variable of J' occurs in A'.

Case 1') Suppose A' is the H(V) in S_8.

Case 1.1') $i = i_o$. Then B is B_o and hence, as was proved in the proof of B2.3, the $H_2(V)$ in S_3 belongs to B', which means that the eigen variable of J occurs in B. Therefore by [D6,P]

$$d(J';P') = d(J;P) < d(F;P) \leq d(F';P').$$

Case 1.2') $i > i_o$. Since H(V) in S_8 belongs to B', $B_o' \cap B'$ is not empty; this implies that $B_o \cap B$ is not empty, and hence B_o is included by B due to [C3,P]. The $H_2(V)$ in S_3 belongs to B_o, and hence to B. Therefore like in Case 1.1'), $d(J';P') < d(F';P)$.

Case 2') A' is any other formula. Then the corresponding formula A and A' have exactly the same substitution variables.

Therefore by [D6,P] $d(J';P) = d(J;P) < d(F;P) \leq d(F';P')$.

6.8. # is second order $\forall$, the cut formula is reducible, the auxiliary formula of the boundary inference which introduces the cut formula under concern belongs to some block, and, if i_o is the smallest number such that $H_1(\alpha)$ belongs to an i_o-block, say B_o, then $H_1(\alpha)$ belongs to the entrance of B_o, say E_o. (See 6.6 for P.)

In order to define a suitable reduct of P, we need the following lemma, which is originally due to Kleene.

Lemma. (Kleene's Basis Theorem.) Let $H(\alpha)$ be a semi-isolated formula in which α is not tied by any second order quantifier (cf. [5] for 'tied') and no second order $\exists$ occurs. Then there exists a semi-isolated abstract, say $V_{H(\alpha)}$, such that the second order variable of it are only those which occur in $H(\alpha)$ and distinct from α, and $H(V_{H(\alpha)}) \rightarrow \forall\, \varphi H(\varphi)$ is SINN-provable.

Recall that $\forall\, \varphi\, H_1(\varphi)$ in P is reducible. Therefore there is a semi-isolated formula $\tilde{H}(\alpha, \beta_1, \ldots, \beta_m)$ (without second order $\exists$) and reducible abstracts $V_1, \ldots, V_m$ which start with second order $\exists$ such that $H_1(\alpha)$ is $\tilde{H}(\alpha, V_1, \ldots, V_m)$. Notice that α is not tied by any second order quantifier. Therefore by the above Lemma, there exists a semi-isolated abstract, say $V_{H_1}(\beta_1, \ldots, \beta_m)$ and an SINN-proof, say $Q(\beta_1, \ldots, \beta_m)$ (without introductions of second order $\exists$, such that

$$\tilde{H}(V_{H_1}(\beta_1, \ldots, \beta_m), \beta_1, \ldots, \beta_m) \rightarrow \forall\, \varphi \tilde{H}(\varphi, \beta_1, \ldots, \beta_m)$$

is the end sequent of $Q(\beta_1,\ldots,\beta_m)$. Then $Q(V_1,\ldots,V_m)$, or, for short, Q, which is obtained from $Q(\beta_1,\ldots,\beta_m)$ by substituting $V_1,\ldots,V_m$ for $\beta_1,\ldots,\beta_m$ respectively, is an $\mathfrak{F}$-admissible proof in which there is not use of the rules, which introduce second order $\exists$, where we assume that P is F-admissible.

Now define P' by using the above Q.

$$
\begin{array}{ll}
S'_o & \dfrac{\Gamma_1 \rightarrow \Delta_1, H_1(V_{H_1}) \qquad Q \left\{ S \quad H_1(V_{H_1}) \rightarrow \forall\, \varphi H_1(\varphi) \right.}{S'_1 \quad \Gamma_1 \rightarrow \Delta_1, \forall\, \varphi H_1(\varphi)} J_o \\
 & \dfrac{S'_2 \quad \Gamma_2 \rightarrow \Delta_2, \forall\, \varphi H(\varphi) \qquad S_5 \quad \forall\, \varphi H(\varphi), \Pi_2 \rightarrow \Lambda_2}{S'_6 \quad \Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2} \\
 & \qquad\qquad \vdots \\
 & \qquad\qquad \rightarrow
\end{array}
$$

The blocks and the entrances are defined as follows. No formula in Q belongs to any k-block if $k \leq i_o$. The $\forall\, \varphi H(\varphi)$'s in $S'_1,\ldots,S'_2$ do not belong to any k-block if $k \leq i_o$. Any other formula belongs to a k-block B' (its entrance E'), where $k \leq i_o$, if and only if its corresponding formula in P belongs to B (E).

Let $k > i_o$. If in P the $H_1(\alpha)$ in S_o belongs to a k-block B, then all formulas in Q belong to B'. If the $H_1(\alpha)$ does not belong to any k-block, then no formula in Q belongs to any k-block in P'. Any other formula belongs to a B' if and only if its corresponding formula belongs to B.

Suppose the $H_1(\alpha)$ in S_o belongs to a B. Then in Q all formulas except the $\forall\ \varphi H_1(\varphi)$ in S and its ancestors belong to B' - E' and the $\forall\ \varphi H_1(\varphi)$ in S and its ancestors belong to E' if and only if the $H_1(\alpha)$ in S_o (in P) belongs to E. If $H_1(\alpha)$ in S_o belongs to B, then the $H_1(V_{H_1})$ in S'_o belongs to B' - E'. For any other formula A',A' belongs to B' (E') if and only if A belongs to B (E).

It is easily seen that if a formula in Q belongs to a k-block B', then the $H_1(\alpha)$ in S_o of P belongs to B. For any other formula A' in P', there is a corresponding formula A such that A' belongs to B' if and only if A belongs to B. According to the definition, Q does not contain any key inference. Therefore C1 - C3 follows from [C,P].

B1.1. Suppose $\forall\ \varphi H_1(\varphi)$ in one of $S'_1,\ldots,S'_2$ belongs to a k-block B'. Then by definition $k > i_o$ and the corresponding $\forall\ \varphi H_1(\varphi)$ and its ancestors in P belong to B, and hence by [B1.1,P] $H_1(\alpha)$ belongs to B. So, by definition all formulas in Q belong to B', which means that in P' all ancestors of $\forall\ \varphi H_1(\varphi)$ belong to B'. If the $H_1(V_{H_1})$ in S belongs to a k-block B', then by definition $k > i_o$ and all formulas in Q belong to B'; in particular all the ancestors of the $H_1(V_{H_1})$ belong to B'. Similarly for the $\forall\ \varphi H_1(\varphi)$ in S. For any other formula this follows from [B1.1,P].

All formulas in Q do or do not belong to a block simultaneously. If the auxiliary formula of a logical inference belongs to Q, then so does its principal formula. Therefore B1.2 and B1.3 are easily proved.

Since Q has no introduction of second order $\exists$, B2.1 and B2.2 are easily proved.

B2.3. The $\forall \varphi H(\varphi)$ in S_2' is the auxiliary formula of a strong inference. If it is the descendent of the auxiliary formula of a key inference of dim k, then the $\forall \varphi H(\varphi)$ in S_2 (in P) satisfies the same condition. Therefore by [B2.3,P] it does not belong to any k-block; hence it does not belong to any k-block in P' either.

B3. We may assume that a free variable in Q which may occur somewhere else occurs in S or in $H(\varphi)$, since we may assume that V_H has only free variables which occur in $H(\varphi)$. Therefore we may exclude Q from the consideration altogether, since all formulas of Q do or do not belong to a block simultaneously. Therefore this follows from [B3,P].

B4. Suppose B' is a k-block and B',A' and F' satisfy the condition on B,A, and F in B4. Since A' does not belong to the endpiece of P,A' is either A itself, $A(\overset{\alpha}{V_{H_1}})$ or a formula in Q.

First we deal with the case where A' is A or $A(\overset{\alpha}{V_{H_1}})$. The corresponding formulas A and F belong to B (by definition), and F' is F or $F(\overset{\alpha}{V_{H_1}})$. By [B4.1,P],

(a) $\delta_k(A) < \delta_k(F)$ and $\delta_j(A) \leq \delta_j(F)$ if $k + 1 \leq j \leq n$.

B4.1. If α does not occur in A, then A' is A and hence $\delta_k(A') = \delta_k(A) < \delta_k(F) \leq \delta_k(F')$, and $\leq$ holds for $k + 1 \leq j \leq n$.

Suppose α occurs in A. Then by [B4.3,P] $dp_j(\alpha;A) = 0$ if $k \leq j \leq n$. Therefore by Proposition 1.6

$$\text{(b)} \qquad \delta_j(A') = \delta_j(A(V_{H_1})) = \max(\delta_j(A), \delta_j(V_{H_1}))$$

$$= \max(\delta_j(A), \delta_j(H_1(\alpha))) \text{ if } k \leq j \leq n.$$

If $H_1(\alpha)$ also belongs to the same B, then [B4.1,P] applies to F and $H_1(\alpha)$ (cf. [B2.3,P]; hence $\delta_k(H_1(\alpha)) < \delta_k(F)$ and $\leq$ holds if $k + 1 \leq j \leq n$. Therefore (a) and (b) imply $\delta_k(A') < \delta_k(F) \leq \delta_k(F')$ and $\leq$ holds if $k + 1 \leq j \leq n$. If $H_1(\alpha)$ does not belong to B, then α occurs both inside and outside B in P. Therefore by [B3,P] α occurs in F. By [B4.3,P] applied to any sub-semi-formula of F, say G, $dp_j(\alpha;G) = 0$ for all j if $k \leq j \leq n$. This implies that

(c) $dp_j(\alpha;F) = 0$ if $k + 1 \leq j \leq n$ and $dp_k(\alpha;F) = 1$.

Therefore by Proposition 1.6

$$\text{(d)} \qquad \delta_k(F') = \max(\delta_k(F), \delta_k(V_H) + 1)$$

and

$$\delta_j(F') = \max(\delta_j(F), \delta_j(V_H)) \text{ if } k + 1 \leq j \leq n.$$

From (a), (b) and (d)

$$\delta_k(A') < \delta_k(F')$$

and $\leq$ holds if $k + 1 \leq j \leq n$.

B4.2. Suppose β is a substitution variable which affects F'.

Case 1) β occurs in F but not in $H_1(\alpha)$. Then by [B4.2,P]

$$dp_j(\beta;A') = dp_j(\beta;A(V_{H_1})) = dp_j(\beta;A(\alpha)) < dp_j(\beta;F) \leq dp_j(\beta;F')$$

if $j = k$ and $\leq$ holds if $k + 1 \leq j \leq n$.

Case 2) β does not occur in F but α occurs in F and β occurs in $H_1(\alpha)$, $dp_j(\alpha;A) = 0$ if $j \geq k$ as in B4.1, which together with Proposition 1.7 implies

(e) $$dp_j(\beta;A') = \max(dp_j(\beta;A),dp_j(\beta;H_1)) = dp_j(\beta;H_1) = dp_j(\beta;V_{H_1}),$$

since $dp_j(\alpha;H_1) = 0$. From (c) and Proposition 1.7 we have

$$dp_j(\beta;F') = \max(dp_j(\beta;F),dp_j(\beta;H_1) + 1) = dp_j(\beta;H_1) + 1$$

if $j = k$ and $= dp_j(\beta;H_1)$ if $k + 1 \leq j \leq n$. Hence $dp_k(\beta;A') < dp_k(\beta;F')$ and $\leq$ holds if $j \geq k + 1$.

Case 3) β occurs both in F and $H_1(\alpha)$. Then by [B4.2,P]

(f) $$dp_k(\beta;A) < dp_k(\beta;F)$$

and $\leq$ holds if $k + 1 \leq j \leq n$. If A does not contain α, then $dp_j(\beta;A') = dp_j(\beta;A)$. So we assume α occurs in A. Then $dp_j(\beta;A') = \max(dp_j(\beta;A),dp_j(\beta;V_{H_1}))$ as in (e) above. First suppose α occurs in F. Then the right hand side of the above equation is $< dp_j(\beta;F')$ if $j = k$ and $\leq$ holds if $k + 1 \leq j \leq n$. (See the argument in B4.1 above.) Next suppose α does not occur in F.

Case 3.1) $H_1(\alpha)$ belongs to B. Then by [B4.2,P] applied to F and $H_1(\alpha)$

$$dp_j(\beta;H_1(\alpha)) < dp_j(\beta;F) = dp_j(\beta;F')$$

if $j = k$ and $\leq$ holds if $k + 1 \leq j \leq n$. Combining this with (f) and the equality in Case 3), B4.2 holds.

Case 3.2) $H_1(\alpha)$ does not belong to B. Then α occurs both inside and outside B. Therefore by [B3,P] α must occur in F, contradicting the assumption.

B4.3. If β is a non-substitution eigen variable, then it does not occur in V_{H_1} and hence $dp_j(\beta;A') = dp_j(\beta;A) = 0$ by [B4.3,P]. Suppose β is a substitution variable which does not occur in F'. This implies that β does not occur in F. [B4.3,P] applied to α implies $dp_j(\alpha;A) = 0$ if $k \leq j \leq n$. Also $dp_j(\beta;A) = 0$ by [B4.3,P] for such j. If α does not actually occur in A, then $dp_j(\beta;A') = dp_j(\beta;A) = 0$. Suppose α occurs in A.

Case 1) $H_1(\alpha)$ does not belong to B. Then α occurs both inside and outside B. Therefore by [B3,P] α occurs in F. So the assumption that F' (which is $F\binom{\alpha}{V_{H_1}}$) does not contain β means that V_{H_1} does not contain β. Thus $dp_j(\beta;A') = dp_j(\beta;A) = 0$ follows from [B4.3,P] trivially.

Case 2) $H_1(\alpha)$ belongs to B. Then $H_1(\alpha)$ satisfies the same condition as A. Therefore by [B4.3,P] applied to $H_1(\alpha)$, $dp_j(\beta;H_1(\alpha)) = 0 = dp_j(\beta;V_{H_1})$ and hence $dp_j(\beta;A') = dp_j(\beta;A) = 0$ for all $j \geq k$.

Second case. (This can happen only if $k > i_o$.) A' is a formula in Q. Recall that $H_1(\alpha) \equiv \tilde{H}(\alpha,\beta_1,\ldots,\beta_m)\binom{\beta_1,\ldots,\beta_m}{V_1,\ldots,V_m}$, where $\tilde{H}$ is semi-isolated and $V_1,\ldots,V_m$ are reducible; if $\tilde{Q}(\beta_1,\ldots,\beta_m)$ is a SINN-proof of $\tilde{H}(V_{\tilde{H}_1},\beta_1,\ldots,\beta_m) \rightarrow \forall\, \varphi\tilde{H}(\varphi,\beta_1,\ldots,\beta_m)$, then Q is obtained from $\tilde{Q}(\beta_1,\ldots,\beta_m)$ by substituting $V_1,\ldots,V_m$ for $\beta_1,\ldots,\beta_m$ respectively. So, the δ_j of the formulas of Q are determined by $H_1(\alpha)$.

(*) $\delta_j(A') \leq \delta_j(H_1(\alpha))$ and $dp_j(\beta;A') \leq dp_j(\beta;H_1(\alpha))$ for any j.

Since we are assuming that A' belongs to B', $H_1(\alpha)$ belongs to B by definition. So F, $H(\alpha)$ and B satisfies the condition of B4. Hence by [B4,P] and (*) above we obtain the following.

B4.1. $\delta_k(A') \leq \delta_k(H_1(\alpha)) < \delta_k(F) \leq \delta_k(F')$ and $\leq$ holds if $k + 1 \leq j \leq n$.

B4.2. $dp_k(\beta;A') = dp_k(\beta;H_1(\alpha)) < dp_k(\beta;F) \leq dp_k(\beta;F')$ and $\leq$ holds if $k + 1 \leq j \leq n$.

B4.3. $dp_j(\beta;A') \leq dp_j(\beta;H_1(\alpha)) = 0$ if $k \leq j \leq n$.

B5. We have defined entrances. Let B' be a k-block of P' and E' be its entrance and let B be the corresponding k-block of P and E be its entrance.

B5.2. For $k \leq i_o$ this is obvious. Suppose $k > i_o$.

First part: Suppose $\forall\, \varphi H(\varphi)$ (call it D) in one of $S_1',\ldots,S_2'$ belongs to E'. Then by definition and [B5.2,P] the corresponding $\forall\, \varphi H(\varphi)$ and its ancestors belong to E, which implies that $H_1(\alpha)$ belongs to E. So, by definition $\forall\, \varphi H(\varphi)$ in S and all its ancestors belong to E'; therefore all ancestors of D belong to E'. If we started with a formula in Q, then the entrance is so defined that all its ancestors belong to E'. For any other formula, this follows from [B5.2,P].

Second part: Suppose an ancestor of the $\forall \varphi H(\varphi)$ (in S) in Q belongs to E'. Then by definition $H_1(\alpha)$ belongs to E in P and so all its descendents in Q belong to E'. By [B5.2,P] any descendent of $H_1(\alpha)$ which belongs to B belongs to E. So any descendent of the $\forall \varphi H(\varphi)$ in S which belongs to B' belongs to E'. For any other formula, this follows from [B5.2,P] and the definition.

B5.3. If $k \leq i_o$, then Q is irrelevant. For $k = i_o$, $H_1(V_H)$ in S'_o belongs to E'_o, since $H_1(\alpha)$ belongs to E_o by assumption. If $k < i_o$, then $H_1(V_{H_1})$ does not belong to any block since $H_1(\alpha)$ does not. So this is also irrelevant. Suppose $k > i_o$.

Case 1) The $H_1(\alpha)$ in S_o belongs to E. Then $\forall \varphi H_1(\varphi)$ in S and all its ancestors belong to E' by definition; hence those formulas are irrelevant. If a formula in Q belongs to B' - E' then it disappears within Q, which implies that all its descendents belong to B' - E'. $H_1(V_{H_1})$ in S' as well as all its ancestors belong to B' - E'.

Case 2) $H_1(\alpha)$ in S_o belongs to B - E. Then all the formulas of Q belong to B' - E'. Also $\forall \varphi H(\varphi)$'s in $S'_1, \ldots, S'_2$ belong to B' - E' since the corresponding formulas in P belong to B - E (cf. [B5.3,P] applied to $H_1(\alpha)$).

Case 3) The $H_1(\alpha)$ does not belong to any block. Then all formulas in Q lie outside any block and so do the descendents of $\forall \varphi H_1(\varphi)$.

B5.4. Suppose the $H_1(V_{H_1})$ in S_o' belongs to B' - E'. This originates in the fact that $H_1(\alpha)$ in S_o belongs to B. Then by definition the $H_1(V_{H_1})$ in S belongs to B' - E'. Conversely, if the $H_1(V_{H_1})$ in S belongs to B' - E', then the only possibility is that $H_1(\alpha)$ belongs to B; hence the $H_1(V_{H_1})$ in S_o' belongs to B' - E'.

For the cut formulas within Q, this holds by definition.

Define d(J';P') = d(J;P) for every substitution J' of P', where P is the corresponding substitution of P. Notice that P and P' have exactly the same kinds of substitutions since Q is substitution-free.

Define d(A;P') as D1 - D3. If A' is identical with its corresponding formula A in P, then d(A';P') = d(A;P). If α actually occurs in A and A' is $A(\frac{\alpha}{V_{H_1}})$ then d(A;P) $\leq$ d(A';P').

D6. First suppose $k \leq i_o$. Let B' be a k-block of P' and F' is a key principal formula of dim k which belongs to B'. Suppose the eigen variable of a substitution J' occurs in a formula, say A', in B' but not in F'. Let A' be $A(\frac{\alpha}{V_{H_1}})$ where A is the corresponding formula of A' in P and F' be $F(\frac{\alpha}{V_{H_1}})$. Then by definition A and F belong to B. Let β be the eigen variable of J' (and so of J). The assumption implies that

1^o. β does not occur in F, and

2^o. Either α does not occur in F, or α does occur in F but β does not occur in V_{H_1}.

1^o implies that J does not affect F.

Case 1) The eigen variable of J occurs in A (in P). Then by 1^o above F,B,J and A satisfy the condition D6 for P. Therefore by [D6,P]

$$d(J';P') = d(J;P) < d(F;P) \leq d(F(V_{H_1});P').$$

Case 2) The eigen variable of J does not occur in A in P. Then α occurs in A and β occurs in V_{H_1} since the eigen variable of J' occurs in A'. This implies that β occurs in $\forall\varphi H_1(\varphi)$, and hence J affects $\forall\varphi H_1(\varphi)$ in P.

Case 2.1) $H_1(\alpha)$ belongs to B in P. Then the eigen variable of J occurs in B but not in F; thus by [D6,P]

$$d(J';P') = d(J;P) < d(F;P) \leq d(F';P').$$

Case 2.2) $H_1(\alpha)$ does not belong to B. Since α actually occurs in A, α occurs both inside and outside B. Therefore by B3 α occurs in F. Then by 2^o β does not occur in V_{H_1}, contradicting the assumption of Case 2). Therefore this case is impossible.

Next suppose $k > i_o$. If the eigen variable of a substitution J' occurs in a formula A' in Q and Q is included by a k-block B', then, as $H_1(\alpha)$ belongs to B and we may assume that a substitution variable β occurs in Q only if it occurs in S, in P we may take $H_1(\alpha)$ as A. Hence by [D6,P] $d(J';P') = d(J;P) < d(F;P) \leq d(F';P')$. (Recall that F' cannot belong to Q.)

6.9. # is second order $\forall$ and the cut formula is non-reducible. Let P be of the following form:

$$
\begin{array}{llcll}
 & & \vdots & & \vdots \\
S_o & & \Gamma_1 \rightarrow \Delta_1, H_1(\alpha) & S_3 & H_2(V), \Pi_1 \rightarrow \Lambda_1 \\
 & J_o & \overline{\hphantom{\Gamma_1 \rightarrow \Delta_1, \forall\varphi H_1(\varphi)}} & & \overline{\hphantom{\forall\varphi H_2(\varphi), \Pi_1 \rightarrow \Lambda_1}} \\
S_1 & & \Gamma_1 \rightarrow \Delta_1, \forall\varphi H_1(\varphi) & S_4 & \forall\varphi H_2(\varphi), \Pi_1 \rightarrow \Lambda_1 \\
 & & \vdots & & \vdots \\
S_2 & & \Gamma_2 \rightarrow \Delta_2, \forall\varphi H(\varphi) & S_5 & \forall\varphi H(\varphi), \Pi_2 \rightarrow \Lambda_2 \\
 & J & \multicolumn{3}{c}{\overline{\hphantom{\Gamma_2 \rightarrow \Delta_2, \forall\varphi H(\varphi) \qquad\qquad \forall\varphi H(\varphi), \Pi_2 \rightarrow \Lambda_2}}} \\
 & & \multicolumn{3}{c}{S_6 \quad \Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2} \\
 & & \multicolumn{3}{c}{\vdots} \\
 & & \multicolumn{3}{c}{\rightarrow}
\end{array}
$$

Notice that there is no substitution between S_1 and S_2 and between S_4 and S_5.

Define P' as follows in terms of the following subproofs P_1 and P_2.

P_1:

$$\begin{array}{ll} & \vdots \\ S_0' & \Gamma_1 \to \Delta_1, H_1(V) \\ \hline\hline S_1' & \Gamma_1 \to H_1(V), \Delta_1, \forall\varphi H_1(\varphi) \\ & \vdots \\ S_2' & \Gamma_2 \to H(V), \Delta_2, \forall\varphi H(\varphi) \end{array} \qquad \begin{array}{ll} & \vdots \\ S_4 & \forall\varphi H(\varphi), \Pi_2 \to \Lambda_2 \end{array}$$

$$\begin{array}{ll} S_6' & \Gamma_2, \Pi_2 \to H(V), \Delta_2, \Lambda_2 \\ \hline\hline & \Gamma_2, \Pi_2 \to \Delta_2, \Lambda_2, H(V) \end{array}$$

P_2:

$$\begin{array}{ll} & \vdots \\ S_2 & \Gamma_2 \to \Delta_2, \forall\varphi H(\varphi) \end{array} \qquad \begin{array}{ll} & \vdots \\ S_3' & H_2(V), \Pi_1 \to \Lambda_1 \\ \hline\hline S_4' & \forall\varphi H_2(\varphi), \Pi_1, H_2(V) \to \Lambda_1 \\ & \vdots \\ S_5' & \forall\varphi H(\varphi), \Pi_2, H(V) \to \Lambda_2 \end{array}$$

$$\begin{array}{ll} S_6'' & \Gamma_2, \Pi_2, H(V) \to \Delta_2, \Lambda_2 \\ \hline\hline & H(V), \Gamma_2, \Pi_2 \to \Delta_2, \Lambda_2 \end{array}$$

P':

$$\begin{array}{ll} & P_1 \quad P_2 \\ \hline S_7 & \Gamma_2, \Pi_2, \Gamma_2, \Pi_2 \to \Delta_2, \Lambda_2, \Delta_2, \Lambda_2 \\ \hline\hline & \Gamma_2, \Pi_2 \to \Delta_2, \Lambda_2 \\ & \vdots \\ & \to \end{array}$$

For every i-block B (its entrance E) of P, the corresponding i-block B' (its entrance E') of P' is defined as follows. The H(V) in one of $S_1', \ldots, S_2'$ and of $S_4', \ldots, S_5'$ belongs to B' (E') if and only if its corresponding $\forall\, \varphi H(\varphi)$ belongs to B (E). The H(V)'s in one of $S_6', \ldots$ and $S_6'', \ldots$ belongs to B' (E') if and only if the $\forall\, \varphi H(\varphi)$ in S_2' and S_4' respectively belongs to B(E). Any other formula belongs to B' (E') if and only if its corresponding formula belongs to B (E).

B3. We may assume that neither V nor $H_1(\varphi)$ contains any non-substitution eigen variable. Therefore a non-substitution eigen variable of P' occurs in a formula A' in P' if and only if it occurs in its corresponding formula A in P.

B4. Suppose B' is an i-block and suppose A',B' and F' satisfy the condition in B4 for P'. Let A and F correspond to A' and F' respectively (in P). Then A' is A or $A\binom{\alpha}{V}$ and F' is F or $F\binom{\alpha}{V}$. B,A,F satisfy the same condition. Hence [B4,P] holds; in particular,

(a) $dp_k(\alpha;A) = 0$ if $i \leq k \leq n$; $\delta_i(A) < \delta_i(F)$ and $\leq$ holds for $j \geq i + 1$.

B4.1. We only have to deal with the case where α actually occurs in A.

Case 1) $H_1(\alpha)$ belongs to B in P. Then $\forall\, \varphi H_1(\varphi)$ in S_1 belongs to B by [B1.3,P]. This implies that it belongs to B - E, since it is non-reducible (cf. [B5.5,P]). Therefore all $\forall\, \varphi H(\varphi)$'s under S_1 belong to B - E by [B5.3,P], in particular the one in S_2, which implies together with [B5.4,P] that the $\forall \varphi H(\varphi)$ in S_4 belongs to B - E; hence the $H_2(V)$ belongs to B ([B1.1,P]). Due to [B2.3,P] $H_2(V)$ cannot be a descendent of a key principal formula of dim i. Therefore $H_2(V)$ satisfies the condition of A in B4 for P; thus, by [B4.1,P] applied to $H_2(V)$,

(b) $\delta_i(V) \leq \delta_i(H_2(V)) < \delta_i(F)$, and $\leq$ holds for $i + 1 \leq k \leq n$.

(a) and Proposition 1.6 yield

(c) $$\delta_k(A') = \max(\delta_k(A), \delta_k(V)).$$

From (a),(b) and (c) we obtain

$$\delta_i(A') < \delta_i(F') \text{ and } \delta_j(A') \leq \delta_j(F')$$

if $i + 1 \leq j \leq n$.

Case 2) $H_1(\alpha)$ does not belong to B. Then [B3,P], [B4,P] and Proposition 1.6 applied to A and F as well as [B4,P] applied to a subformula of F and F imply B4.1.

Suppose β is the concerning eigen variable in B4.2 and B4.3. Then B4.2 and B4.3 are proved for β by applying [B4,P] for α and β as well as [B3,P] for α.

B5. For every block B' (of dim i) we have defined its entrance E', corresponding to B and its entrance E.

B5.2. The only crucial fact for this case is that, due to [B5.5,P], $\forall \varphi H_1(\varphi)$ in S_1 does not belong to any entrance.

B5.4. Suppose the cut formula $H(V)$, in S_6' belongs to a $B' - E'$. This means that the $\forall \varphi H(\varphi)$ in S_2 belongs to $B - E$ by [B5.3,P]; hence the $\forall \varphi H(\varphi)$ in S_4 belongs to $B - E$ by [B5.4,P].

B5.5. Notice that if $H(V)$ is non-reducible then so is $\forall \varphi H(\varphi)$.

Assign to every substitution J' of P' the degree of the corresponding substitution J of P. Define $d(A';P')$ for all semi-formulas in P' as D1 - D3. $d(A;P) = d(A';P')$ if A is identical with A', since, in that case, a substitution J' affects A' if and only if J affects A. $\leq$ holds if A' is $A(\substack{\alpha \\ V})$.

D4. There is no substitution between S_1 and S_2, and between S_4 and S_5, since $\forall \varphi H(\varphi)$ is non-reducible. (See [D2,D4 and D5,P].)

6.10. # is $\neg$. Suppose P is of the following form.

P:

$$
\begin{array}{llll}
 & \vdots & & \vdots \\
S_1 & A_1, \Gamma_1 \to \Delta_1 & S_4 & \Pi_1 \to \Lambda_1 \\
\hline
S_2 & \Gamma_1 \to \Delta_1, \ A_1 & S_5 & A_2, \Pi_1 \to \Lambda_1 \\
 & \vdots & & \vdots \\
S_3 & \Gamma_2 \to \Delta_2, \ A & S_6 & A, \Pi_2 \to \Lambda_2 \\
\hline
\end{array}
\quad J
$$

$$
\begin{array}{c}
S_7 \quad \Gamma_2, \Pi_2 \to \Delta_2, \Lambda_2 \\
\vdots \\
\to
\end{array}
$$

Define P' as follows.

P_1:

$$
\begin{array}{ll}
 & \vdots \\
S_1 & A_1, \Gamma_1 \rightarrow \Delta_1 \\
\hline\hline
S_2' & \Gamma_1, A_1 \rightarrow \Delta_1, \neg A_1 \\
 & \vdots \\
S_3' & \Gamma_2, A \rightarrow \Delta_2, \neg A \qquad\qquad S_6 \quad \neg A, \Pi_2 \rightarrow \Lambda_2 \\
\hline
S_7' & \Gamma_2, A, \Pi_2 \rightarrow \Delta_2, \Lambda_2 \\
\hline\hline
S_7'' & A, \Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2
\end{array}
$$

P_2:

$$
\begin{array}{ll}
 & \vdots \\
S_4' & \Pi_1 \rightarrow \Lambda_1, A_2 \\
\hline\hline
S_5' & \neg A_2, \Pi_1 \rightarrow A_2, \Lambda_1 \\
 & \vdots \\
S_6' & \neg A, \Pi_2 \rightarrow A, \Lambda_2
\end{array}
$$

$$
\begin{array}{ll}
 & \vdots \\
S_3 & \Gamma_2 \rightarrow \Delta_2, \neg A \qquad\qquad S_6' \quad \neg A, \Pi_2 \rightarrow A, \Lambda_2 \\
\hline
S_7''' & \Gamma_2, \Pi_2 \rightarrow A, \Delta_2, \Lambda_2 \\
\hline\hline
S_7'''' & \Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2, A
\end{array}
$$

P':

$$
\begin{array}{ll}
 & P_1 \quad P_2 \\
\hline
S_8 & \Gamma_2, \Pi_2, P_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2, \Delta_2, \Lambda_2 \\
\hline\hline
S_9 & P_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2 \\
 & \vdots \\
 & \rightarrow
\end{array}
$$

Define the blocks and the entrances of P' in a natural way.

B5.5. Suppose, for example, the A_1 in S_2' is non-reducible and belongs to E'. This A_1 is in the left hand side of a sequent. This means that the $\neg A_1$ in S_2 belongs to E. By assumption $\neg A$ is non-reducible. [B5.5,P] requires that $\neg A_1$ must have a form $\forall \varphi \neg \forall \psi \neg G$, which is impossible. So, the A_1 in S_2' does not belong to any entrance if it is non-reducible.

To each substitution in P' assign the same degree as the corresponding substitution in P, and define $d(A;P')$ as in D1 - D3. Then $d(A';P') = d(A;P)$ or $d(A';P') = d(A;P) - 1$ in case A is $\neg A'$.

D4. If an A_1 or its descendent $\tilde{A}$ is in the upper sequent of a substitution in P', then the corresponding $\neg A_1$ or $\neg \tilde{A}$ is in the upper sequent of a substitution in P. Therefore

$$d(\tilde{A};P') < d(\neg \tilde{A};P) < d(J;P) = d(J';P') \text{ by } [D4,P].$$

6.11. # is $\wedge$. See 10.2*. Define P' as in 10.2* and define its blocks and entrances naturally.

6.12. # is first order $\forall$. Similarly to 6.11.

§7. The consistency proof. In order to complete the proof of Theorem 1.3 in §5, we now assign the o.d.'s of the system $O(\omega^{n+1}+1, \omega^{2(n+1)} + 1)$ to the proofs with degree of dim n, where $n \geq 1$.

Note. We do not include $n = 0$ (namely the SINN-proofs) here.

Definition 1.19. Let P be a proof with degree of dim n. The o.d.s are assigned to the sequents in P in the same manner as 6.1 - 6.8 of Chapter 2 in [5] by reading ω^{n+1} in place of ω except the cases where the concerning inferences are the reducible, second order $\forall$ in the succedent and the second order $\exists$ in both sides. We shall give a precise definition, although most part is overlapping with [5].

1) The o.d. of an initial sequent is 0.

2) If S_1 and S_2 are the upper sequent and the lower sequent respectively of a structural inference, then the o.d. of S_2 is equal to that of S_1.

3) If S_1 and S_2 are the upper sequent and the lower sequent respectively of one of the inferences $\neg, \wedge$ in the antecedent, first order $\forall$, non-reducible second order $\forall$ in the succedent, second order $\exists$ in the antecedent, explicit, second order $\exists$ in the succedent, or explicit, second order $\forall$ in the antecedent, then the o.d. of S_2 is $(\omega^{n+1};0,\sigma)$, where σ is the o.d. of S_1.

4) If S_1 and S_2 are the upper sequents and S is the lower sequent of an inference $\wedge$ in the succedent, then the o.d. of S is $(\omega^{n+1};0,\sigma_1 \# \sigma_2)$, where σ_1 and σ_2 are the o.d.'s of S_1 and S_2 respectively.

5) If S_1 and S_2 are the upper sequent and the lower sequent respectively of an implicit second order $\forall$ in the antecedent, then the o.d. of S_2 is $(\omega^{n+1},a + 2,\sigma)$, where a is the grade of the auxiliary formula and σ is the o.d. of S_1.

6) If S_1 and S_2 are the upper sequents and S is the lower sequent of a cut, then the o.d. of S is $(\omega^{n+1}; a+1, \sigma_1 \# \sigma_2)$, where a is the grade of the cut formula and σ_1 and σ_2 are the o.d.'s of S_1 and S_2 respectively.

7) If S_1 and S_2 are the upper sequent and the lower sequent respectively of a substitution with the degree i, then the o.d. of S is $(i; 0, \sigma)$, where σ is the o.d. of S_1.

8) If S_1 and S_2 are the upper sequent and the lower sequent respectively of an induction, then the o.d. of S_2 is $(\omega^{n+1}; a + 2, \sigma)$, where a is the grade of the induction formula and σ is the o.d. of S_1.

9) Let S_1 and S_2 be the upper sequent and the lower sequent of a reducible, second order $\forall$ in the succedent.

9.1) The auxiliary formula of the concerning inference belongs to a block. Let i be the smallest number such that the auxiliary formula belongs to an i-block. Then the o.d. of S_2 is $(\omega^{n+1}; \omega^{(n+3)+(n-i)}, \sigma)$, where σ is the o.d. of S_1.

9.2) The auxiliary formula of the inference does not belong to any block. Then the o.d. of S_2 is $(\omega^{n+1}; 0, \sigma)$, where σ is the o.d. of S_1.

10) Let S_1 and S_2 be the upper sequent and the lower sequent of an implicit second order $\exists$ in the succedent. The o.d. of S_2 is $(\omega^{n+1}; a + 2, \sigma)$, where a is the grade of the auxiliary formula and σ is the o.d. of S_1.

Notice that the grade of any formula is less than ω^{n+3}, and hence is less that $\omega^{(n+3)+(n-i)}$ for all $1 \leq i \leq n$ (cf. 9) above). The o.d. of a sequent S in P may be denoted by $o(S;P)$, or for short $o(S)$. The o.d. of P is defined as the o.d. of the end sequent of P. In passing, we call the m in (i,m,α), where (i,m,α) is any part of an o.d. β, a second element of β.

Concerning the γ-degree, Proposition 1 and its corollary in Chapter 2 of [5] can be easily proved for our present version of γ. If we read ω^{n+1} instead of ω in the lemmas in Appendix to 10.1.1.2 of §4 of [5] (i.e. 10.1.1.2*), then all arguments there go through for the modified version of those lemmas. We shall distinguish those modified lemmas by putting *; for example Lemma 1* corresponds to Lemma 1 in Appendix to 10.1.1.2*.

Now we must show that the o.d. of the proof decreases when a reduction as §6 is performed. Since the proof is basically the same as that for SINN in [5], we shall only note some crucial points. We quote the numbering in §6. In most cases it is a direct consequence of the definition of blocks that 9.1) in Definition 1.19 does not arise anew after reduction; hence the second element of o.d.'s do not increase.

6.5. Put $o(S_o) = \mu$, $o(S_2) = \rho$, $o(S_3) = \lambda$, $o(S_5) = \tau$, $o(S_6) = \nu$ and $o(\rightarrow) = \sigma$ in P. Similarly in P' put $o(S'_o) = \mu'$, $o(S'_2) = \rho'$, $o(S'_3) = \lambda'$, $o(S'_5) = \tau'$, $o(S_7) = \nu'$ and $o(\rightarrow) = \sigma'$.

Notice that $\gamma(A(V);P') = \gamma(A(\alpha);P)$ and $v(A(V);P') < v(A(\alpha);P)$ for any formula $A(\alpha)$ above S_3 which contains α. Therefore $g(A(V);P') < g(A(\alpha))$ for such formulas, and $g(A';P') = g(A;P)$ for any other formulas A'. Thus follows $\lambda' \leq_j \lambda$ for all j. Similarly we can show that $\mu' = \mu$.

In order to show $\sigma' <_o \sigma$ it suffices to prove $\nu' <_j \nu$ for all j.

$$\nu = (\omega^{n+1}; m+1, \rho \# \tau)$$

and

$$\nu' = (\omega^{n+1}; m'+1, (\omega^{n+1}; m+1, \rho\#\tau) \# (\omega^{n+1}; m+1, \rho'\#\tau)),$$

where $m' = g(\forall\ \psi H(V,\psi); P)$ $(= g(\forall\ \psi H(V;\psi); P')$ and

$$m = g(\exists\ \varphi\ \forall\ \psi H(\varphi,\psi); P)\ (= g(\exists\ \varphi\ \forall\ \psi H(\varphi,\psi); P')).$$

The crucial fact is $m' < m$, which follows from Corollary 6) in §4 (i.e. $\ell(\forall\ \psi H(V,\psi)) < \ell(\exists\ \varphi\ \forall\ \psi H(\varphi,\psi))$), for both formulas take value 0 for Υ and v. From this and Lemma 1* follows $\nu' <_j \nu$ (cf. 10.1.2*).

6.8. As in 6.5, we can easily prove that for any A' in P', $g(A';P) \leq g(A;P)$, where A corresponds to A'. Since Q is not included by any k-block if $k \leq i_o$, every second element of the o.d.s of any sequent in Q is less than $\omega^{(n+3)+(n-i_o)}$. We should also recall that the grade is less than ω^{n+3}. Let q be $o(S;P')$. Then

$$\begin{aligned} o(S_1';P') &= (\omega^{n+1}; g(H(V_H))+1, \lambda'\#q) <_j (\omega^{n+1}; \omega^{(n+3)+(n-i_o)}, \lambda) \\ &= o(S_1;P) \end{aligned}$$

for all j (cf. Lemma 1) in 2.6 of [7]). (This is the most crucial point.) Therefore Lemma 1* applies and $\sigma' <_j \sigma$ is proved as in 6.5.

§8. The well ordering of O(I,A). In §7 we carried out the consistency proof of D_n by the help of transfinite induction along $<_0$, which is the ordering of $O(\omega^n,\omega^n)$ with respect to 0. Therefore now the problem is to see what is necessary in order to prove the well ordering of $O(\omega^n,\omega^n)$ for each n. We shall, however, state a more general theorem first.

Theorem 1.5. Let I and A be primitive recursive sets (of natural numbers) with primitive recursive well orderings $<_I$ and $<_A$ respectively, and O(I,A) be the system of ordinal diagrams (o.d.) based on I and A. Then the well ordering of O(I,A) for each member of I or the maximal element is proved in the system which is obtained from SINN by adding to it the principles of transfinite induction along $<_I$ and $<_A$ and the semi-isolated inductive definitions along $<_I$.

As for the systems with inductive definitions, one should refer to Chapter 4 of [5]. Since the proof of the theorem is similar to the argument in [4], we shall not present the detailed computation, which is routine and straightforward, but shall only discuss the theorem in a more precise manner. Let us introduce two new predicate constants A_1 and A_2, where $A_1(i,a,\beta)$ is to be interpreted as "a is an i-fan with respect to β" and $A_2(i,a)$ is to be interpreted as "a is i-accessible" (cf. [4]). By simply arithmetizing the theory of o.d.s, we can easily define two semi-isolated formulas G_1 and G_2 which express the intended meanings of A_1 and A_2, where G_1 contains neither A_1 nor A_2, while G_2 may contain

A_1. (One might see a hint of this in [2].) Thus we add the following two inductive definitions to SINN as the initial sequents:

$$A_1(i,a,A_2) \leftrightarrow G_1(i,a,A_2,\{x,y\}(A_1(x,y,A_2)) \wedge x <_I i))$$

and

$$A_2(i,a) \leftrightarrow G_2(i,a,\{x,y\}(A_2(x,y) \wedge x <_I i)).$$

Furthermore permit TI(I) and TI(A), which read 'the transfinite induction along $<_I$' and 'the transfinite induction along $<_A$' respectively, as the initial sequents. Then the accessibility of O(I,A) with respect to $<_i$ for each i, where i is a member of I or i is ∞, is formulated in a second order formula and is proved in the above presented system.

We should remark that if the transfinite induction along $<_I$ and $<_A$ are provable in SINN, then the latter two initial sequents can be eliminated, and thus it should be emphasized that for the case of our concern, viz., the case where $I = \omega^n$ and $A = \omega^n$, it suffices to assume the system which is obtained from SINN by adding to it the semi-isolated inductive definitions along ω^n.

§9. The semi-isolated inductive definitions along ω^n. We begin this section with the following two remarks. Let F be the set of provably-Δ^1_2-abstracts of dim m. Then an F-abstract or an F-formula (cf. Definition 1.6) is called essentially provably-Δ^1_2 of dim m. If in the definition of D_n

in §1 we permit essentially provably-Δ^1_2 abstracts of dim $< n$ as the comprehension abstracts, then the resulting system is actually equivalent to D_n. Therefore in the following we shall identify those two systems.

Although the following theorem is concerned with the semi-isolated inductive definitions, we only have to show the definability of Π^1_1-inductive definitions in the under mentioned system, since the semi-isolated inductive definitions can be obtained from Π^1_1 (and arithmetical inductive definitions by substitutions).

Theorem 1.6. Let $n \geq 1$ and $(\omega^n, \prec)$ be the standard well ordering of natural numbers whose order type is ω^n. Then the semi-isolated inductive definitions along $(\omega^n, \prec)$ (cf. Chapter 4 of [5]) can be defined in the system $P\Delta^1_2$ (cf. §1).

Proof. We shall prove that for each $n \geq 1$, the Π^1_1-inductive definitions along $(\omega^n, \prec)$ can be defined in the system D_n (cf. §1). Notice that $(\omega^n, \prec)$ can be regarded as the lexicographical ordering of ordered n tuples of numbers. Let us fix n and first introduce some notations.

Notation. 1) $x^n, \ldots$ and $a^n, \ldots$ stand for series of n bounded and free variables respectively, and hence $(x^n), \ldots$ and $(a^n), \ldots$ stand for ordered n tuples of such variables.

2) Let $\prec$ denote the lexicographical ordering of ordered n tuples of numbers. Then $\alpha^{(x_n)}$ denotes an abstract of the form $\{u^n\}((u^n) \prec (x^n) \wedge \alpha[u^n])$.

3) $G(b^n, \beta)$ be an arbitrary semi-isolated formula with the indicated occurrences of b^n and β. Then $F(\alpha, a^n)$ is the

abbreviation of a formula of the form $\forall x^n((x^n) \prec (a^n) \supset \alpha[x^n] \equiv G(x^n, \alpha^{(x^n)}))$, where $\forall x^n$ stands for $\forall x_1 \ldots \forall x_n$.

We are concerned with the inductive definition along $(\omega^n, \prec)$, where the basis for the inductive definition is a formula like G above. The argument goes as follows. For every i such that $1 \leq i \leq n$, the following are provable in D_{i-1}.

(3.i) $$F(\alpha, a^{n-i}, a, 0^{i-1}) \rightarrow \exists \varphi F(\varphi, a^{n-i}, a+1, 0^{i-1}),$$

where 0^i stands for $\underbrace{0,\ldots,0}_{i}$.

(4.i) $$F(\alpha, a^{n-i}, 0^i) \rightarrow \forall y^i \exists \varphi F(\varphi, a^{n-i}, y^i).$$

Then from (3.n), applying induction on b,

(5) $$\forall y \exists \varphi F(\varphi, y, 0^{n-1})$$

and, from (4.n),

(6) $$F(\alpha, 0^n) \rightarrow \forall y^n \exists \varphi F(\varphi, y^n).$$

(5) and (6) yield

(7) $$\forall y^n \exists \varphi F(\varphi, y^n)$$

in D_{n-1}.

Define $A(a^{n-1}, b)$ as $\exists \varphi(F(\varphi, a^{n-1}, b+1) \wedge \varphi[a^{n-1}, b])$.

Our last task will be to show that

(8) A is essentially provably-Δ_2^1

and

(9) $$A(a^n) \leftrightarrow G(a^n, A^{(a^n)})$$

in D_n from (7). This completes the proof of the theorem.

Now we proceed to the detailed argument.

(1.1) $$F(\alpha, a^n), F(\beta, a^n), (b^n) \prec (a^n) \rightarrow \alpha[b^n] \equiv \beta[b^n]$$

and

(1.2) $$F(\alpha, a^n), (b^n) \prec (a^n) \rightarrow F(\alpha, b^n)$$

are provable in some elementary, second order arithmetic.

Define $E(\alpha, \beta, a^{n-1}, b)$ as

$$\forall\ y^n \{\beta[y^n] \equiv [((y^n) \quad (a^{n-1}, b) \wedge \alpha[y^n])$$

$$\vee((y^n) = (a^{n-1}, b) \wedge G(y^n, \alpha^{(y^n)}))]\}.$$

Then

(2.1) $$F(\alpha, a^{n-1}, b), E(\alpha, \beta, a^{n-1}, b) \rightarrow F(\beta, a^{n-1}, b+1)$$

and

(2.2) $$\forall\ \varphi\ \exists\ \psi\ E(\varphi, \psi, a^{n-1}, b)$$

are provable in SINN (or D_o).

(3.i) and (4.i) are proved together by induction on i.

(3.1) $$F(\alpha, a^{n-1}, b) \rightarrow \exists\ \varphi F(\varphi, a^{n-1}, b+1)$$

follows from (2.1) and (2.2) (in D_o).

(4.1) $$F(\alpha, a^{n-1}, 0) \rightarrow \forall\ x\ \exists\ \varphi F(\varphi, a^{n-1}, x)$$

follows from (2.1) and (2.2) (in D_o), using induction on b applied to $\exists\ \varphi F(\varphi, a^{n-1}, b)$.

Assume now that (3.i) and (4.i) have been proved in D_{i-1}. We must deduce (3.i+1) and (4.i+1) in D_i.

(3.i+1) $$F(\alpha, a^{n-(i+1)}, a, 0^i) \rightarrow \exists \varphi F(\varphi, a^{n-(i+1)}, a+1, 0^i).$$

(3.i+1) is proved by the following procedure. Consider two abstracts U and U' which are defined as follows.

$$U : \{y^n\}[F(\alpha,a^{n-(i+1)},a,0^i) \wedge (y^n) \prec (a^{n-(i+1)},a+1,0^i) \supset \exists\, \varphi(F(\varphi,y^{n-(i+1)},x,y+1,0^{i-1}) \wedge \varphi[y^{n-(i+1)},x,y,0^{i-1}])]$$

and

$$U' : \{y^n\}[F(\alpha,a^{n-(i+1)},a,0^i) \wedge (y^n) \prec (a^{n-(i+1)},a+1,0^i) \supset \forall\, \varphi(F(\varphi,y^{n-(i+1)},x,y+1,0^{i-1}) \supset \varphi[y^{n-(i+1)},x,y,0^{i-1}])],$$

where x and y are the (n - i)th and (n - i + 1)th variables in y^n. As a consequence of (1.1), (1.2) and (4.i),

1^o. $\forall\, x^n(U(x^n) \equiv U'(x^n))$ is provable in D_{i-1}.

From 1^o we can show that

2^o. there is an essentially provably-Δ^1_2 abstract of dim i - 1, say V, such that $\forall\, x^n(U(x^n) \equiv V(x^n) \equiv U'(x^n))$ is D_{i-1}-provable.

In order to prove (3.i+1) it suffices to show

3^o. $F(\alpha,a^{n-(i+1)},a,0^i) \rightarrow F(U,a^{n-(i+1)},a+1,0^i)$

in D_i, since then U can be replaced by V, which is the essentially provably-Δ^1_2 abstract of dim i - 1 obtained in 2^o. 3^o is proved by using (1.1),(1.2) and (4.i).

(4.i+1) $\quad F(\alpha,a^{n-(i+1)},0^{i+1}) \rightarrow \forall\, x^{i+1}\exists\, \varphi\, F(\varphi,a^{n-(i+1)},x^{i+1})$.

From (3.i+1) we have

$$\exists\, \varphi F(\varphi,a^{n-(i+1)},a,0^i) \rightarrow \exists\, \varphi F(\varphi,a^{n-(i+1)},a+1,0^i),$$

is D_i-provable, from which follows

4^o. $\exists\ \varphi F(\varphi, a^{n-(i+1)}, a, 0^i)$.

(4.i) implies that

5^o. $\exists\ \varphi F(\varphi, a^{n-(i+1)}, a, 0^i) \rightarrow \forall\ x^i \exists\ \varphi F(\varphi, a^{n-(i+1)}, a, x^i)$

is D_{i-1}-provable. 4^o and 5^o yield (4.i+1).

Finally we shall deduce (8) and (9) from (7). It is easily seen that (1.1) and (7) imply

$$A(a^{n-1}, b) \leftrightarrow \forall\ \varphi(F(\varphi, a^{n-1}, b+1) \supset \varphi[a^{n-1}, b]),$$

and hence, similarly to 2^o, we can show that

(8) there is an essentially provably-Δ_2^1 formula of dim n - 1, say $\tilde{A}$, such that $A(a^{n-1}, b) \leftrightarrow \tilde{A}(a^{n-1}, b)$ is D_{n-1}-provable.

(7) also implies

6^o. $A(0^n) \leftrightarrow G(0^n, A^{(0^n)})$.

On the other hand, (1.1) reinforced with the comprehension axiom applied to $\tilde{A}$, which is an essentially provably-formula obtained in (8), implies

7^o. $F(A, a^{n-1}, b), F(\beta, a^{n-1}, b+1) \rightarrow \forall\ x^n(\beta^{(a^{n-1}, b)}[x^n]$

$\equiv A^{(a^{n-1}, b)}(x^n))$,

which is provable in D_n. It is a matter of routine to deduce

8^o. $\forall\ x^n((x^n) \prec (a^n) \supset [A(x^n) \equiv G(x^n, A^{(a^n)})])$

$\rightarrow [A(a^n) \equiv G(a^n, A^{(a^n)})]$

from 7^o. 6^o and 8^o enable us to apply n-induction on (a^n) to the formula $A(a^n) \equiv G(a^n, A^{(a^n)})$, thus yielding (9) and completing the proof of the theorem.

§10. Conclusion.

Theorem 1.7. Let ϑ_n be the order type of $O(\omega^n, \omega^n)$ with respect to its ordering $<_o$. Then the ordinal of the system $P\Delta^1_2$, i.e. second order arithmetic with the provably-Δ^1_2 comprehension axiom, is the limit of ϑ_n for all $n < \omega$.

Proof. From Theorem 1.4, the remark after Theorem 1.5 and Theorem 1.6.

CHAPTER II

The Ordinal of Second Order Arithmetic with the Δ_2^1-Comprehension Axiom

§11. A revised version of the consistency proof of second order arithmetic with the Π_1^1-comprehension axiom and the extended inductive definitions.

In Chapter 4 of [5], Takeuti presented a consistency proof of second order arithmetic with the Π_1^1-comprehension axiom and the extended inductive definitions, using a system of o.d.s $O(O(\{0\},I_\infty) \cup \{\xi\},N(I_\infty))$ (cf. 8 in Chapter 4 of [5]). This system of o.d.'s is, however, unnecessarily large and we can improve the result: the consistency of the system is proved by using the system of o.d.s $O(\omega^{I_\infty} + 1,\omega^{I_\infty},\omega\cdot\omega^{I_\infty})$. Since most of the definitions in [5] may be taken over, we shall only demonstrate how to modify the original method. We shall quote the item numbers in [5] by adding asterisque. Thus, for example, 4.3* denotes 4.3 in Chapter 4 of [5]: Proposition 2* denotes Proposition 2 in Chapter 4 of [5]. We denote the ordering ω^{I_∞} by simply $<$.

11.1. The rank is defined as in 3*.

Corollary. Let B and C be arbitrary two formulas in which A_m and A_n occur respectively. Then $r(A_m:B) <_\infty r(A_n:C)$ if $m < n$.

11.2. The γ-degree of a quasi-formula is defined as a number less than ω^{I_∞}. The definition is like in 4*, replacing $\#\tilde{\sigma}$

by + 1, except the following cases.

4.3*. If A is of the form $A_n(s,t,V) \wedge s <^* i$, then $\gamma(A)$ is $\gamma(V) + \omega^{r(A_n:A)} + 1$.

4.6*. If A is of the form $A_n(s,t,V)$, then $\gamma(A)$ is $\gamma(V) + \omega^{r(A_n:A)}$.

Corollary. Let $\{x_1,\ldots,x_n\}H(x_1,\ldots,x_n)$ be an abstract and $s_1,\ldots,s_n$ be arbitrary terms. Then

$$\gamma(H(s_1,\ldots,s_n)) \leq \gamma(\{x_1,\ldots,x_n\}H(x_1,\ldots,x_n)).$$

Lemma 1. If $G(\beta,\alpha)$ is a semi-isolated quasi-formula (allowing other free second order variables as well) which contains none of $A_n, A_{n+1}, \ldots,$ s is a constant for which $I(s)$ is provable, and V is an arbitrary abstract which is not semi-isolated, then

$$\gamma(G(V;A_n^s(V))) \leq \gamma(V) + \sum_{\ell=1}^{k} \omega^{r(A_{j_\ell}:B_\ell)} + m$$

for some $j_1,\ldots,j_k \leq n$, some formulas $B_1,\ldots,B_k$, and for a number m, where $A_n^s(V)$ is an abbreviation of $\{x,y\}(A_n(x,y,V) \wedge x <^* s)$, and $r(A_{j_\ell}:B_\ell) < r(A_n:A_n)$ for $\ell \leq k$. We omit the proof of this lemma as well as of any subsequent lemma in this section, since it is all a routine computation.

Proposition 2*. If s is a constant for which $I(s)$ is provable, V is not semi-isolated and $G_n(a,b,\alpha,\beta)$ is as in 1.2.2*, then

$$\gamma(G_n(s,t,V,A_n^s(V))) < \gamma(A_n(s,t,V)).$$

Proof. As a special case of Lemma 1,

$$\gamma(G(V,A_n^s(V))) \leq \gamma(V) + \sum_{\ell} \omega^{r(A_{j_\ell}:B_\ell)} + m,$$

where $r(A_{j_\ell}:B_\ell) < r(A_n:A_n)$ and $m < \omega$. On the other hand $\gamma(A_n(s,t,V)) = \gamma(V) + \omega^{r(A_n:A_n)}$. Thus follows the proposition.

11.3. The conditions on the degree are given as in 5* with the following modifications. The degrees are assigned from $\omega^{I_\infty} + 1$.

5.2.2*. If A is implicit and not semi-isolated, then $d(A)$ is ω^{I_∞}.

5.2.3.3*. If A is of the form $A_n(s,t,V) \wedge s <^* i$, then $d(A)$ is

$$\max_J(d(V),d(J)) + \omega^{r(A_n:A)} + 1,$$

where J ranges over all the substitutions which affect A.

5.2.3.6*. If A is of the form $A_n(s,t,V)$, then $d(A)$ is $\max(d(V),d(J)) + \omega^{r(A_n:A_n)}$, where J ranges over all the substitutions which affect A.

Lemma 2. Suppose $G(\beta,\alpha)$ is a semi-isolated quasi-formula whose only free f-variables are β and α, and which contains none of $A_n, A_{n+1}, \ldots$. Assume also that i is a constant for which $I(i)$ is provable. If V is semi-isolated, then

$$d(G(V,A_n^i(V))) \leq \max_J(d(V),d(J)) + \sum_{\ell=1}^{k} \omega^{r(A_{j_\ell}:B_\ell)} + m,$$

for some $j_1,\ldots,j_k \leq n$, some $B_1,\ldots,B_k$, and a number m, where $j \leq m$, $r(A_{j_\ell}:B_\ell) <_{oo} r(A_n:A_n)$ and $m < \omega$, and J ranges over all substitutions which influence V.

As a special case of Lemma 2, we have

Proposition 4*. Suppose $A_n(i,t,V)$ is semi-isolated (i.e. V is semi-isolated), and i is a constant for which $I(i)$ is provable. If either

$$I(i), A_n(i,t,V) \rightarrow G_n(i,t,V,A_n^i(V))$$

or

$$I(i), G_n(i,t,V,A_n^i(V)) \rightarrow A_n(i,t,V)$$

is an initial sequent in a proof with degree, in which $A_n(i,t,V)$ is implicit, then

$$d(G_n(i,t,V,A_n^i(V))) < d(A_n(i,t,V)).$$

11.4. The norm of a quasi-formula is assigned from ω^{I_∞}, in the same manner as in 6*, although $\#\eth$ in 6* is replaced by $+1$ here and 6.3* and 6.6* are slightly changed.

6.3*. If A is of the form $A_n(s,t,V) \wedge s <^* i$, then $n(A)$ is $n(V) + \omega^{r(A_n:A)} + 1$.

6.6*. If A is of the form $A_n(s,t,V)$, then $n(A) = n(V) + \omega^{r(A_n:A)}$.

Lemma 3. If $G(\beta,\alpha)$ contains none of $A_n, A_{n+1}, \ldots$, i is a constant for which $I(i)$ is provable and V is an arbitrary variety, then

$$n(G(V,A_n^i(V))) \leq n(V) + \sum_{\ell=1}^{k} \omega^{r(A_{j_\ell}:B_\ell)} + m,$$

where $j_\ell \leq n$, $r(A_{j_\ell}:B_\ell) < r(A_n:A_n)$ and $m < \omega$.

The following proposition is a special case of Lemma 3.

Proposition 5*. If $I(i), G_n(i,t,V,A_n^i(V)) \rightarrow A_n(i,t,V)$ or $I(i), A_n(i,t,V) \rightarrow G_n(i,t,V,A_n^i(V))$ is an initial sequent of our system, and i is a constant for which $I(i)$ is provable, then

$$n(G_n(i,t,V,A_n^i(V))) < n(A_n(i,t,V)).$$

11.5. The grade is assigned to a formula from $\omega^{I_\infty} \times \omega \times \omega^{I_\infty}$. Let $N(I_\infty)$ denote $\omega^{I_\infty} \times \omega \times \omega^{I_\infty}$ and $\prec$ be the lexicographical ordering of $N(I_\infty)$. The grade of a formula A, $g(A)$, is given by $<\gamma(A),a,n(A)>$, which is a member of $N(I_\infty)$. (See 7*.)

Proposition 6*. If $I(i), A_n(i,t,V) \rightarrow G_n(i,t,V,A_n^i(V))$ or $I(i), G_n(i,t,V,A_n^i(V)) \rightarrow A_n(i,t,V)$ is an initial sequent of a proof with degree, and i is a constant for which $I(i)$ is provable, then

$$g(G_n(i,t,V,A_n^i(V))) \prec g(A_n(i,t,V)).$$

11.6. The o.d.s of $O(\omega^{I_\infty} + 1, \omega^{I_\infty} \times \omega \times \omega^{I_\infty})$ are assigned to the sequents of a proof with degree as in 8*. We might remark here that, although in the original work the initial sequents are given $g(D)$ for some D, it is not necessary. It suffices to assign $<0,0,0>$ to an initial sequent.

11.7. Having finished the definitions, the consistency proof of the system may be carried out exactly as in 9*. Proposition 4* and Proposition 6* are essentially used.

11.8. Now we shall exploit the above general result to our special concern.

Definition 2.1. Let w_n denote $\left.\omega^{\omega^{\cdot^{\cdot^{\omega}}}}\right\} n$ with its canonical ordering. Then second order arithmetic with the semi-isolated comprehension axiom and the semi-isolated inductive definitions along w_n shall be called ID_n.

Note. We can replace "semi-isolated" by "Π^1_1" .

Theorem 2.1. The consistency of ID_n is proved by the system of o.d.s $O(w^{\omega}_{n+1} + 1, w^{\omega\cdot 2}_{n+1})$.

Proof. From the result in 11.6, the consistency proof for ID_n can be carried out by the o.d.s. of $O(\omega^{I_\infty} + 1, \omega^{I_\infty} \times \omega \times \omega^{I_\infty})$, where I is, in this case, w_n, and hence $I_\infty = (2.|I| + 1)\cdot\omega = (w_n + 1)\cdot\omega$, which implies that $\omega^{I_\infty} = w^{\omega}_{n+1}$ and $w^{\omega}_{n+1} \times \omega \times w^{\omega}_{n+1} = w^{\omega\cdot 2}_{n+1}$.

§12. The Δ^1_2-comprehension axiom and the semi-isolated inductive definitions. In this section we shall establish the proof-theoretical equivalence between second order arithmetic with the Δ^1_2-comprehension axiom and the system $\bigcup_{n<\omega} ID_n$.

Friedman has shown the following result in his [1] as a corollary of a theorem:

Σ^1_2-AC and $ID^{<\varepsilon_0}$ have the same theorems in the common part of their language, where $ID^{<\varepsilon_0}$ is the theory of iterated inductive definitions.

He also mentions a result of R. Mansfield:

Δ^1_2-CA and Σ^1_2-AC have the same theorems. On the other hand it is a simple matter to show that $ID^{<\varepsilon_0}$ can be embedded in $\bigcup_{n<\omega} ID_n$ (cf. Definition 2.1). Thus, in virtue of the above results by Friedman and Mansfield, holds the following.

Proposition 2.1. Second order arithmetic with the Δ^1_2-comprehension axiom is a subsystem of $\bigcup_{n<\omega} ID_n$.

The opposite direction of equivalence is stated as follows.

Proposition 2.2. The semi-isolated inductive definitions along w_n (cf. Definition 2.1) are defined in second order arithmetic with the Δ^1_2-comprehension axiom.

Proof. The proof is similar to that of Theorem 1.6 except that it is simpler this time.

Let $\prec$ denote the canonical well-ordering with the order type ε_0 and $\prec_n$ be its restriction to w_n. Let $G(b,\alpha)$ be in a Π^1_1-formula with the indicated occurrences of b and α. Define $F(\alpha,a)$ as

$$\forall x \prec_n a(\alpha[x] \equiv G(x,\alpha^x))$$

where α^x is the abbreviation of $\{y\}(y \prec x \wedge \alpha[y])$. Suppose the following 1° and 2° are provable with the Δ^1_2-comprehension axiom:

1°. $\forall y \exists \varphi F(\varphi, y)$.

2°. $\exists \varphi(F(\varphi, a) \wedge \varphi[a]) \equiv \forall \varphi(F(\varphi, a) \supset \varphi[a])$.

Then define $A(a)$ as

$$\exists \varphi(F(\varphi, a) \wedge \varphi[a])$$

and show that

3. $a(a) \leftrightarrow G(a, A^a)$ is provable with the Δ_2^1-comprehension axiom, thus completing the proof. It should be noted that according to Gentzen's result the principal of transfinite induction along $\prec_n$ (for each $n > 0$) is provable without comprehension axioms.

3° (under 1° and 2°) is proved by transfinite induction along $\prec_n$ as follows. The argument is similar to that of (9) in the proof of Theorem 1.6. The crucial step is to deduce

$$F(A, a),\ F(\alpha, a) \rightarrow \forall y(\alpha^a[y] \equiv A^a(y)),$$

where A is $\{x\}A(x)$, by a use of the Δ_2^1-comprehension axiom (applied to A) and 2°. Otherwise 1° and 2° are used as in the proof of Theorem 1.6.

1° and 2° are proved simultaneously by transfinite induction along $\prec_n$. Let $D(y)$ stand for

$$\exists \varphi(F(\varphi, y) \wedge \varphi[y]) \equiv \forall \varphi(F(\varphi, y) \supset \varphi[y]).$$

Then what must be shown is

$$\forall y \prec_n a(D(y) \wedge \exists \varphi F(\varphi, y)) \rightarrow D(a) \wedge \exists \varphi F(\varphi, a).$$

Let U denote the abstract $\{y\}(x \prec_n a \wedge \exists \varphi(F(\varphi,x) \wedge \varphi[x]))$.

4^o. $\forall y \prec_n a(D(y) \wedge \exists \varphi F(\varphi,y)) \rightarrow \forall y \prec_n aF(U,y)$ is proved by transfinite induction on y, by some uses of the Δ^1_2-comprehension axiom applied to U. 4^o together with the Δ^1_2-comprehension axiom implies

5^o. $\forall y \prec_n a(D(y) \wedge \exists \varphi F(\varphi,y)) \rightarrow \exists \varphi \forall y \prec_n aF(\varphi,y))$.

5^o implies immediately

6^o. $\forall y \prec_n a(D(y) \wedge \exists \varphi F(\varphi,y)) \rightarrow \exists \varphi F(\varphi,a)$.

6^o and the uniqueness of α satisfying $F(\alpha,a)$ yield $\forall y \prec_n a(D(y) \wedge \exists \varphi F(\varphi,y)) \rightarrow D(a)$. This completes the proof.

We have now established the following

Theorem 2.2. Second order arithmetic with the Δ^1_2-comprehension axiom is proof-theoretically equivalent to $\bigcup_{n<\omega} ID_n$.

§13. Conclusion.

Theorem 2.3. Let η_n be the order type of $O(w_n,w_n)$ with respect to its ordering $<_o$. Then the ordinal of second order arithmetic with the Δ^1_2-comprehension axiom is the limit of η_n for all positive $n < \omega$.

Proof. From Theorem 2.1, the remark after Theorem 1.5, and Theorem 2.2.

Additional remark. The evaluation of the o.d.s of the semi-isolated iductive definitions (§11) does not necessarily give the least upper bound of the ordinals. As an exemplary case for this fact, let us take the inductive definitions along ω. Theorem 2.1 for $n = 1$ gives the corresponding ordinal $O({}_{\omega}\omega^{\omega} + 1, {}_{\omega}\omega^{\omega\cdot 2})$, while we have shown in Chapter I that $O(\omega^2 + 1, \omega^4 + 1)$ suffices in proving the consistency of the semi-isolated inductive definitions along ω.

Chapter III. APPLICATIONS OF THE REDUCTION METHOD

The reduction method which was used in proving the consistency of the system of proofs with degree can be exploited in investigating various structural aspects of some systems with the ω-rule. In most cases the arguments go parallel to those in [6] and [7]. Therefore we shall only state the results and sketch the proofs for a few exemplary cases.

§14. ω-proofs and cut elimination.

Let us first define a system of second order arithmetic which is in substance the system with the provably-Δ_2^1 comprehension axiom and the construction ω-rule and will be called Z_n.

Definition 3.1. A system Z_n' is defined similarly to the system of proofs with degree of dim n (cf. Definition 1.17) with the following modifications.

(1) Only the formulas which do not contain any first order free variables are involved. (Such a formula may be called t-closed, meaning that it is closed with respect to terms.)

(2) The constructive ω-rule is added. (cf. Introduction of [6] for the definition of the constructive ω-rule.)

The system Z_n is then defined as the subsystem of Z_n' which does not involve the rule "substitution" .

Note. 1) The condition (1) in Definition 3.1 implies that no induction for first order $\forall$ in the succedent is involved in a proof of Z_n'.

2) It can be easily shown that " substitution" is actually redundant in Z_n.

3) For any proof with blocks which has no substitution the condition on degree is automatically satisfied. Therefore for Z_n-proofs it suffices to require the conditions on blocks only.

Theorem 3.1. Let n be an arbitrary, but fixed, positive integer. Let $\prec$ denote the well ordering of $O(\omega^{n+1} + 1, \omega^{2(n+1)} + 1)$ with reference to the element 0 (which is normally denoted by $<_o$). The $\prec$ -recursive functions are defined as in Introduction of [6]. Then there exists a $\prec$ -recursive function f such that for every proof with degree of dim n whose sequent, say S, consists of t-closed formulas only, $f(\ulcorner P \urcorner)$ $(= \ulcorner P' \urcorner)$ is Gödel number of a Z_n-proof of S. Furthermore if a formula A in S belongs to an i-block (its entrance) of P' if and only if A belongs to an i-block of P, and two formulas in S belong to a same block of (its entrance) P' if and only if they belong to a same block (its entrance) of P.

Proof. For the proof of the theorem, we can closely follow the proof of Theorem 1 in Chapter I of [6] as well as the reduction argument in §6 of this article. We only have to worry about the definition of blocks and entrances. In most cases, however, the conditions on blocks are either easily taken care of or dealt with as in §6. We shall explain the situation with one example. Suppose the end piece of P does not contain any first order free variable not used as an eigen variable but does contain a key inference of dim i as a lowermost, explicit, non-structural inference in the end piece of P;

$$\begin{array}{ll} & \vdots \\ S_o & \Gamma \to \Delta, \forall \psi F(V,\psi) \\ \hline & \Gamma \to \Delta, \exists \varphi \forall \psi F(\varphi,\psi) \\ & \vdots \\ & \Gamma_o \to \Delta_o \end{array} .$$

Define r(P) as:

$$\begin{array}{l} \vdots \\ \Gamma \to \Delta, \forall \psi F(V,\psi) \\ \hline\hline \Gamma \to \forall \psi F(V,\psi), \Delta \\ \vdots \\ \Gamma_o \to \forall \psi F(V,\psi), \Delta_o \end{array} .$$

For every block B (its entrance E) of P, a block B' (its entrance E') of r(P) is induced as follows: all the explicitly indicated $\forall \psi F(V,\psi)$ belong to B' (E') if and only if the $\forall \psi F(V,\psi)$ in S_o belongs to B (E), and any other formula belongs to B' (E') if and only if its corresponding formula in P belongs to B (E).

r(P)$\prec$ P is easily shown, and hence by induction hypothesis f(r(P)) has been defined and satisfies the conditions in the theorem. We define f(P) as:

$$\begin{array}{lll} f(r(P)) \left\{ \begin{array}{l} \\ S_o' \end{array} \right. & & \begin{array}{l} \vdots \\ \Gamma_o \to \forall \psi F(V,\psi), \Delta \end{array} \\ \hline\hline & S_1 & \Gamma_o \to \Delta_o, \forall \psi F(V,\psi) \\ \hline & S_2 & \Gamma_o \to \Delta_o, \exists \varphi \forall \psi F(\varphi,\psi) \\ & & \vdots \\ & & \Gamma_o \to \Delta_o \end{array} .$$

Enlarge each block of $f(r(P))$ and its entrance, say $\tilde{B}$ and $\tilde{E}$, in order to define the corresponding block and its entrance, say $\tilde{B}'$ and $\tilde{E}'$, of $f(P)$, as follows: the $\forall \psi F(V,\psi)$ and $\exists \varphi \forall \psi F(\varphi,\psi)$ in S_1 and S_2 respectively belong to an i-block $\tilde{B}'$ if and only if the $\forall \psi F(V,\psi)$ in S_o' belongs to $\tilde{B}$; any descendent of $\exists \varphi \forall \psi F(\varphi,\psi)$ belongs to $\tilde{B}'$ if and only if the corresponding formula in Δ_o (in S_o') belongs to $\tilde{B}$; a formula in Γ_o or Δ_o belongs to $\tilde{B}'$ if and only if the same formula in S_o' belongs to $\tilde{B}$. The entrance $\tilde{E}'$ is defined similarly from $\tilde{E}$.

It is only a matter of routine to confirm that $f(P)$ is a proof with blocks.

We should also note that if P is reduced to more than one proof, then in defining the blocks of $f(P)$ from those of the proofs which have been defined by induction hypothesis we take the unions of corresponding blocks.

§15. A system with a function symbol.

Definition 3.2. A system U_n is defined as the system of proofs with degree of dim n augmented by a function symbol as well as the related rules of inference (cf. §2, Chapter I, of [6]). The system $\tilde{Z}_n$ is obtained from Z_n (cf. Definition 3.1) by adding to it the f-ω-rule (cf. §2, Chapter I of [6]).

Theorem 3.2. There is a $\prec$-recursive function g such that for every proof of U_n, say P, $g(\ulcorner P \urcorner)$ is Gödel number of a $\tilde{Z}_n$-proof of the same end sequent as P, satisfying the same conditions

on blocks and entrances as stated for the $f(\ulcorner P \urcorner)$ in Theorem 3.1.
Proof. The proof in [6] can be strictly followed. We shall deal with one case as an example. Suppose that the end piece of P does not contain any first order free variable other than eigen variables but does contain an induction as a lowermost explicit, non-structural inference. Suppose furthermore that $A(s)$ is the principal formula in the succedent of the concerning induction and s contains a function symbol f. Then define $r(P)$ as in 1.2.1.2, Chapter I of [6]. The descendents of $A(a)$ and $A(a + 1)$ above $A(0),\Gamma,f(m) = n \rightarrow \Delta,A(s)$ are defined to belong to a block B' if and only if $A(a)$ and $A(a + 1)$ belong to the corresponding B. $f(m) = n$ are defined not to belong to any block. Now follow the arguments in [6] and Theorem 3.1.

§16. Cut elimination theorem of the system with the ω-rule.
Theorem 3.3. Let Z_n and Z_n' be as in Definition 3.1. For any Z_n'-proof there exists a cut free Z_n-proof of the same sequent. Furthermore, this is proved by using the system of o.d.s of $O(\omega^{n+1} + 1, \omega_1 \times (\omega^{2(n+1)} + 1))$ where ω_1 is the first non-constructive ordinal.

This theorem is proved similarly to the theorem in 3.3 of [7] except that here we must define blocks and entrances. The technique is, however, similar to that of Theorem 3.1 of this article.

REFERENCES

[1] H. Friedman, Generalized inductive definitions and Δ^1_2-CA, Mimeographed notes for Buffalo Meeting, 1968.

[2] A. Kino, Formalization of the theory of ordinal diagrams of infinite order, Mimeographed notes for Buffalo Meeting, 1968.

[3] W. Tait, Applications of the cut elimination theorem to some subsystems of classical analysis, Mimeographed notes for Buffalo Meeting, 1968.

[4] G. Takeuti, On the formal theory of the ordinal diagrams, Annals of the Japan Association for Philosophy of Science, Vol. 1(1958), pp. 151-170.

[5] G. Takeuti, Consistency proofs of subsystems of classical analysis, Annals of Mathematics, Vol. 86(1967), pp. 299-348.

[6] G. Takeuti, The Π^1_1-comprehension schema and ω-rules, Proceedings of the Summer School in Logic, Lecture Notes in Mathematics, Vol. 70(1968), pp. 303-330.

[7] M. Yasugi, Cut elimination theorem for second order arithmetic with the Π^1_1-comprehension axiom and the ω-rule, to appear.