ON SPLIT EXACT SEQUENCES

HOPF ALGEBRAS

by

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Report 70-8

## 0. INTRODUCTION

Let $k \to M \xrightarrow{i} H \xrightarrow{\rho} J$ be an exact sequence of Hopf algebras (with antipode) over the ring $k$. Let $\sigma : J \to H$ be a Hopf algebra map such that $\rho\, \sigma = 1_J$. The main result in this paper (see section 2) gives a decomposition of H as a semi-direct product of M and J with J acting on M via inner Hopf algebra automorphisms. This is an immediate consequence of a more general result about semi-direct products of group objects in an arbitrary category with finite products, discussed in section 1. In section 3 we use this main result to recover some results of Konstant on cocommutative Hopf algebras.

It has been brought to the author's attention that P. Gabriel has obtained essentially the same results for formal groups. Since a reference is not readily available and our setting is slightly different we believe the reader may find this paper useful. At this point we wish to thank Professor Stephen U. Chase for many useful conversations concerning the material discussed here.

Throughout this paper, if $\underline{C}$ is a category, we denote by $\underline{C}(A,B)$ the set of $\underline{C}$ homomorphisms from A to B for A and B in $|\underline{C}|$.

## 1. SEMI-DIRECT PRODUCTS

Let $\underline{C}$ be a category with finite products. Denote by $\hat{\underline{C}}$ the category $\underline{\mathrm{Hom}}\ (\underline{C}^{o}, \mathrm{Sets})$ of contravariant functors from $\underline{C}$

to Sets. $\hat{\underline{C}}$ has arbitrary projective limits, thus in particular fiber products and a terminal object. For X in $|\underline{C}|$ we denote by $\hat{X}$ the functor $\underline{C}(\ ,X)$ in $\hat{\underline{C}}$. We recover structures from the categories of sets and ordinary groups by first defining them functorially in $\hat{\underline{C}}$ and then using the full embedding $\underline{C} \to \hat{\underline{C}}$: $X \to \hat{X}$ to go over to $\underline{C}$. Since the major portion of the material in this section is well known--see [SGAD]--we shall content ourselves with giving brief definitions and indications of proof as in the following examples.

<u>EXAMPLE 1.1</u>. Let G be a group functor in $\hat{\underline{C}}$. A functor H in $\hat{\underline{C}}$ is a left G object if H(S) is a left G(S) object functorial in S for every S in $|\underline{C}|$. In particular this implies a functor morphism $G \times H \to H$ satisfying the obvious commutative diagrams. If H is a group functor and G(S) operates on H(S) via group automorphisms functorial in S, then H is a left G group object. If in addition H is an abelian group functor, then H is a left G module in the category $\hat{\underline{C}}$.

Now let H be a left G group object in $\hat{\underline{C}}$. We can define on the product $H \times G$ a unique group structure such that for every S in $|\underline{C}|$, $(H \times G)(S)$ is the semi-direct product of H(S) and G(S) relative to the action of G(S) on H(S) and functorial in S.

Given elements $(h,g)$, $(h',g')$ in $(H\times G)(S) = H(S) \times G(S)$, their product is defined by $(h\cdot(g \rightharpoonup h'), g\circ g')$ where $g \rightharpoonup h'$ denotes the action of $G(S)$ on $H(S)$. This construction is clearly functorial in $S$. We denote this group functor by $H \# G$ and call it the semi-direct product of $H$ with $G$. This group structure can be carried over to objects in $\underline{C}$ by considering now representable functors $\hat{G}$ and $\hat{H}$. In this setting $G$ is a group object in $\underline{C}$ and $H$ is a left $G$ group object--$H$ being a left $G$ group object if $\hat{H}$ is a left $\hat{G}$ group object or equivalently, if there exist morphisms $\alpha : G \times H \to H$, $\omega : H \times H \to H$ satisfying the obvious commutative diagrams. See [Ch ]. Since $\hat{H} \times \hat{G}$ is representable by $H \times G$ as a set functor via an isomorphism of functors $\varphi : \hat{H} \times \hat{G} \to \widehat{H \times G}$, it is readily seen that the semi-direct product structure on $\hat{H} \times \hat{G}$ endows $H \times G$ with a unique group structure such that $\varphi$ is a group functor isomorphism. If we denote $H \times G$ with this group structure by $H \# G$ then $\varphi : \hat{H} \# \hat{G} \to \widehat{H \# G}$ gives us the representability of the group functor $\hat{H} \# \hat{G}$. Using the full embedding of $\underline{C}$ in $\hat{\underline{C}}$, ie. Yoneda's lemma, it is not hard to describe explicitly the group operations on $H \# G$. For instance the multiplication on $H \# G$ is given by a morphism $\nu$ in $\underline{C}$ defined by the composition

$$H \times G \times H \times G \xrightarrow{1\times\Delta_G\times 1\times 1} H \times G \times G \times H \times G \xrightarrow{1\times 1\times\tau\times 1} H \times G \times H \times G \times G \xrightarrow{1\times\alpha\times\mu} H \times H \times G \xrightarrow{\omega\times 1} H \times G$$

where $\Delta_G : G \to G \times G$ is the diagonal map, $\tau : G \times H \to H \times G$ is the twist map, $\alpha : G \times H \to H$ is the $G$ structure on $H$, $\mu$ and $\omega$ are the group structures on $G$ and $H$ respectively. Note that here we have used the fact that $H \# G \cong H \times G$ as an object of $\underline{C}$.

EXAMPLE 1.3. Let $1 \to H \xrightarrow{i} F \xrightarrow{f} G$ be an exact sequence of group functors, that is for every $S$ in $|\underline{C}|$ the sequence of groups and group homomorphisms $1 \to H(S) \xrightarrow{i(S)} F(S) \xrightarrow{f(S)} G(S)$ is exact. Assume $f : F \to G$ has a group section, ie. a group morphism $\rho : G \to F$ such that $f \cdot \rho = 1_G$. It is then easy to see that for each $S$ in $|\underline{C}|$, $G(S)$ acts on $H(S)$ via inner automorphisms. Evidently this action endows $H$ with a left $G$ group functor structure. Moreover for every $S$ in $|\underline{C}|$ there exist set isomorphisms

$$F(S) \xrightarrow{\varphi} H(S) \times G(S) \xrightarrow{\xi} F(S)$$

given by $\varphi(x) = (x \bullet (\rho(S)f(S)(x))^{-1}, f(S)(x))$ ; $\xi(h,g) = h \bullet \rho(S)(g)$ where $x$ is in $F(S)$, $g$ in $G(S)$, $h$ in $H(S)$ and $\bullet$ denotes the group operation in $F(S)$. Identifying $F(S)$ with $H(S) \times G(S)$ via $\varphi$, it is easy to show that the group structure on $F(S)$ is precisely that of the semi-direct product of $H(S)$ with $G(S)$. This is strictly an exercise in the theory of groups. Since $S$ was an arbitrary element of $\underline{C}$ we conclude that $F \cong H \# G$ as group functors. Now assume that both $H$ and $G$ are representable, that is if the form $\hat{H}$ and $\hat{G}$ for $H$ and $G$ group objects in $\underline{C}$.

Then it follows from the above discussion and example 1.3 that $F$ is representable and in fact $F \cong H \# G$. Thus we have shown the following.

PROPOSITION 1.4. Let $1 \to H \xrightarrow{i} F \xrightarrow{f} G$ be an exact sequence of group functors in $\hat{\underline{C}}$ with $H, G$ representable. Assume $f : F \to G$ has a group section $\rho : G \to F$ such that $f \cdot \rho = 1_{\underline{G}}$. Then $F$ is representable and $F \cong H \# G$, the semi-direct product of $H$ and $G$.

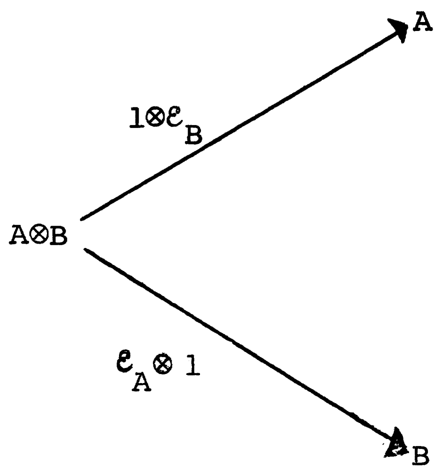
REMARKS 1.5. a) The above proposition is a special case of a general result about extensions of $G$ by $H$ where the section $\rho$ is not necessarily a group section. As might be expected a group cohomology is involved in their classification as in the classical case.

b) Let $1 \to H \to F \to G$ be an exact sequence of group objects in $\underline{C}$, exactness meaning, as usual, exactness of the group functors they represent. Let $\rho : G \to F$ be a group section. Then Proposition 1.4 implies $F \cong H \# G$.

We now turn to the category $\underline{C}$ of cocommutative coalgebras for an interesting application of the above results. We will need a few results on coalgebras which we briefly resume in the next section.

## 2. Cocommutative Coalgebras.

In this section we will work over a ring $k$. Unadorned $\otimes$ shall stand for $\otimes_k$. We refer the reader to [Sw] or [Sw]" for the notation and terminology we will use relevant to coalgebras over $k$. In particular we denote by $\underline{C}$ the category of cocommutative $k$ coalgebras. For a coalgebra $A$ in $\underline{C}$, we will often write $\Delta_A$ (or $\Delta$) $: A \to A \otimes A$ for the comultiplication, and $\varepsilon_A$ (or $\varepsilon$) $: A \to k$ for its counit. $\underline{C}$ has a terminal object namely the ring $k$ itself. Finite products exist in $\underline{C}$; for $A$ and $B$ in $|\underline{C}|$, $A \otimes B$ has a natural coalgebra structure and the diagram




is a direct product diagram where we have viewed the natural isomorphisms $A \otimes k \cong A$ and $k \otimes B \cong B$ as identifications. Finite coproducts exist in $\underline{C}$, if $A$ and $B$ are in $|\underline{C}|$, then $A \oplus B$ the vector space direct sum is their coproduct with diagonal map given by

$$\Delta_{A \oplus B} : A \oplus B \xrightarrow{\Delta_A \oplus \Delta_B} A \otimes A \oplus B \otimes B \hookrightarrow (A \oplus B) \otimes (A \oplus B)$$

and counit

$$\varepsilon_{A \oplus B} : A \oplus B \xrightarrow{<\varepsilon_A,\ \varepsilon_B>} k$$

where $<\varepsilon_A, \varepsilon_B>$ denotes the induced morphism from the direct sum. Under this coalgebra structure the natural morphisms

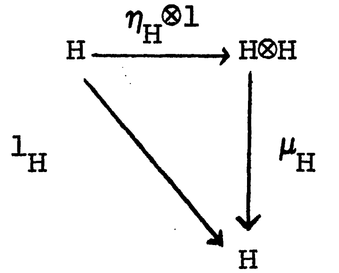
$$\begin{array}{ccc} A & & \\ & \searrow & \\ & & A \quad B \\ & \nearrow & \\ B & & \end{array}$$

are coalgebra morphisms and define a coproduct diagram.

Recall now that a group object in $\underline{C}$ is a coalgebra $H$ with coalgebra morphisms $\mu_H : H \otimes H \to H$; $\gamma_H : H \to H$ and $\eta_H : k \to H$ rendering the following diagrams commutative

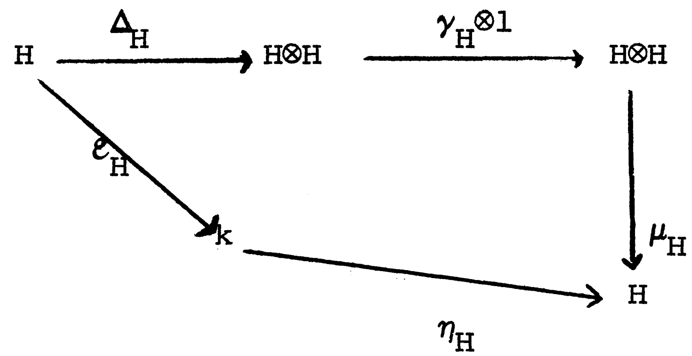
$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{1 \otimes \mu_H} & H \otimes H \\ {\scriptstyle \mu_H \otimes 1} \downarrow & & \downarrow {\scriptstyle \mu_H} \\ H \otimes H & \xrightarrow[\mu_H]{} & H \end{array} \qquad \text{(associativity);}$$

2.1




(left identity)

and




(left inverse)

Note that $\eta_H$ and $\mu_H$ give H an algebra structure; we shall write the action of $\mu_H$ as ordinary multiplication and we will identify k with a subalgebra of H via $\eta_H$. We will use the term <u>bialgebra</u> for monoids in the category <u>C</u>, that is coalgebras H with an algebra structure induced by coalgebra morphisms $\mu_H$ and $\eta_H$ as above satisfying the relevant diagrams in 2.1. We will reserve the term <u>Hopf</u> <u>algebra</u> for those bialgebras H with an inverse morphism $\gamma_H$; in other words group objects in <u>C</u>. Note that for x in H,

$$\gamma_H^2(x) = \sum_{(x)} \gamma_H^2(x_{(1)}) \cdot \gamma_H(x_{(2)}) \cdot x_{(3)} = x$$

so that $\gamma_H$ is also a right inverse for H. Similarly $\eta_H$ is a right unit and both $\gamma_H$ and $\eta_H$ are uniquely determined. Furthermore $\gamma_H$ is an algebra antiendomorphism under the algebra structure for H and the comultiplication and counit are both algebra morphisms for a bialgebra (Hopf algebra) H. We will denote the category of Hopf algebras by $\underline{H}$.

We can now carry over to our setting all of the formalism for group objects in a category discussed in the previous section. Given a Hopf algebra H, a <u>left H coalgebra</u> is a coalgebra N with a coalgebra morphism $\alpha : H \otimes N \to N$ which is associative and unitary. This endows N with an H module structure. For h in H, n in N, we will write $\alpha(h \otimes n)$ as $h \rightharpoonup n$. N is a left H object in $\underline{C}$ and we denote the category of left H objects by $\underline{C}^H$. Products exist in $\underline{C}^H$; given N and M in $\underline{C}^H$, the H module structure on their product $N \otimes M$ is described notationally by $h \rightharpoonup (n \otimes m) = \sum_{(h)} (h_{(1)} \rightharpoonup n) \otimes (h_{(2)} \rightharpoonup m)$ for h in H, n in N, m in M. A left H <u>Hopf algebra</u> is a group in the category $\underline{C}^H$; in other words a Hopf algebra M such that its multiplication, inverse and unit morphisms are H-module morphisms. In particular for h in H, r and s in M, $h \rightharpoonup (r \cdot s) = \sum_{(h)} (h_{(1)} \rightharpoonup r) \cdot (h_{(2)} \rightharpoonup s)$. If M is abelian, i.e. has commutative multiplication, then M will be called a left H <u>Hopf algebra module</u> or H <u>Hopf module</u> for short.

Let H be a Hopf algebra, M a left H Hopf algebra. In section 1, Example 1.1 we described the semi-direct product of two group objects one acting on the other. In our context then, the semi-direct product of M and H is the Hopf algebra $M \# H$ such that:

i) $M \# H = M \otimes H$ as a coalgebra. This allows us to write the elements of $M \# H$ as $\Sigma\, m \otimes h$, m in M, h in H.

ii) the algebra structure is defined by the commutative diagram 1.2 in section 1. Using Sweedler's notation, this turns out to be

$$(m \otimes h)\cdot(m' \otimes h') = \sum_{(h)} m\cdot(h_{(1)} \rightharpoonup m') \otimes h_{(2)}\cdot h'$$

for m and m' in M, h and h' in H.

iii) The unit map of $M \# H$ is trivially defined using the product structure of $M \# H$ as a coalgebra. To define the inverse map $\gamma_{M \# H} : M \# H \to M \# H$ define $\xi : M \# H \to M$ by $\xi(m \otimes h) = \gamma_H(h) \rightharpoonup \gamma_M(m)$ where $\gamma_H$ and $\gamma_M$ are the inverse maps of H and M respectively, m in M, h in H. $\xi$ is trivially a coalgebra morphism. Let $\gamma_{M\#H} = (\xi, \varepsilon_M \otimes \gamma_H) : M \otimes H \to M \otimes H$. Claim $\gamma_{M\#H}$ is an inverse for $M \# H$. We will show it is a left inverse. For m in M, h in H,

$$\mu_{M \# H}\cdot(\gamma_{M\#H}\otimes 1)\cdot \Delta_{M\#H}(m\otimes h)$$

$$= \mu_{M\#H}\cdot(\gamma_{M\#H}\otimes 1)\Big(\sum_{(m),(h)} m_{(1)}\otimes h_{(1)}\otimes m_{(2)}\otimes h_{(2)}\Big)$$

$$= \mu_{M\#H}\Big(\sum_{(m),(h)} (\gamma_H(h_{(1)}) \rightharpoonup \gamma_M(m_{(1)}))\otimes\gamma_H(h_{(2)}) \otimes m_{(2)}\otimes h_{(3)}\Big)$$

2.2 $$= \sum_{(m),(h)} (\gamma_H(h_{(1)}) \rightharpoonup \gamma_M(m_{(1)}))\cdot(\gamma_H(h_{(2)}) \rightharpoonup m_{(2)}) \otimes \gamma_H(h_{(3)})\cdot h_{(4)}$$

$$= \sum_{(m),(h)} \gamma_H(h_{(1)}) \rightharpoonup (\gamma_M(m_{(1)})\cdot m_{(2)}) \otimes \gamma_H(h_{(2)})\cdot h_{(3)}$$

$$= \sum_{(h)} \gamma_H(h_{(1)}) \rightharpoonup \varepsilon_M(m)\otimes\varepsilon_H(h_{(2)})$$

$$= \varepsilon_M(m)\otimes\varepsilon_H(h)$$

$$= \varepsilon_{M\#H}(m\otimes h).$$

Thus $M\#H$ is the semi direct product of $M$ and $H$ in the category $\underline{H}$.

In section 1, Example 1.3 we mentioned the action of a group functor on another via inner automorphisms functorial in $S$ for $S$ in $|\underline{C}|$. In particular a group acts on itself by such automorphisms. In our setting, the action is described in the following:

<u>Definition 2.3</u>. Let $H$ be a Hopf algebra. We define an inner left action of $H$ on itself by

$$h \rightharpoonup h' = \sum_{(h)} h_{(1)} \cdot h' \cdot \gamma_H(h_{(2)})$$

for h and h' in H. It is easy to see that this gives H an H Hopf algebra structure. It is also easily checked using Yoneda's lemma that this action gives rise to an action of $\underline{C}(\ ,H)$ on itself which is precisely the functorial inner automorphism action described in section 1, Example 1.3.

In this setting, Proposition 1.4 of section 1 becomes

Proposition 2.4. Let $k \to M \xrightarrow{i} H \xrightarrow{\rho} J$ be an exact sequence of Hopf algebras with a Hopf algebra map $\sigma : J \to H$ such that $\rho\cdot\sigma = 1_J$. Then M is a left J Hopf algebra via $j \rightharpoonup m = \sum_{(J)} \sigma(j_m)\cdot m\cdot\gamma_H(\sigma(j_{(2)}))$ for j in J, m in M. Note that we are using i to identify M with a sub-Hopf algebra of H. Furthermore $H \cong M \# J$ the semi-direct product of M with J.

We will use Proposition 2.4 to extend a theorem of Konstant on the structure of Hopf algebras over a field k. For this we will need some further results on coalgebras and Hopf algebras.

## 3. Structure of Cocommutative Coalgebras Over a Field k.

In what follows all of our algebras (coalgebras) will be assumed commutative (cocommutative) except when otherwise stated. Most of the results hold without these assumptions, but we won't need them in that generality.

We start out by listing without proofs some elementary properties of coalgebras and coalgebra morphisms. As in the previous section, $\underline{C}$ will denote the category of cocommutative coalgebras, and $\underline{H}$ that of Hopf algebras in $\underline{C}$; this time over a field k.

3.1. If C is a coalgebra over k, and $\bar{k}$ is a field extension of k, then $\bar{k} \otimes C$ is a $\bar{k}$ coalgebra with diagonal map

$$\bar{k} \otimes C \xrightarrow{1\otimes\Delta_C} \bar{k} \otimes C \otimes C \cong \bar{k} \otimes C \otimes_{\bar{k}} \bar{k} \otimes C$$

and the obvious counit.

3.2 Let $f : C \to D$ be a morphism of coalgebras. Then Imf is a subcoalgebra of D. See [Sw], Proposition 1.4.4, p. 13.

3.3 Finite fiber products exist in $\underline{C}$. A proof of this statement can be found in [MM]. As a direct consequence we have the following.

<u>Proposition 3.4</u>. Kernels exist in $\underline{H}$; if $f : A \to B$ is a morphism of Hopf algebras, then there exist a Hopf algebra M such that the sequence

3.5 $$k \to M \to A \xrightarrow{f} B$$

is exact as a sequence of Hopf algebras. Moreover

$$M = \{a \text{ in } A \,|\, (1\otimes f)\, \Delta_A(a) = a \otimes 1\}$$

Proof: See [O.S.], [Sw].

3.6 Let $A$ be a finite dimensional algebra over the field $k$. Setting $C = A^*$, $C$ has a natural coalgebra structure where

$$\Delta_C : A^* \to (A \otimes A)^* \cong A^* \otimes A^*, \quad \varepsilon_C = \eta_A^* : A^* \to k.$$

Conversely if $C$ is a coalgebra, then $C^*$ has a natural algebra structure via the adjoint morphisms. In fact the functor $A \to A^*$ gives an anti-equivalence between the category of finite dimensional algebras and that of finite dimensional coalgebras. If $A$ is not a finite dimensional algebra then $(A \otimes A)^* \ncong A^* \otimes A^*$ in general and $A^*$ does not have a natural coalgebra structure. However we have a functor $(\ )^o$ from the category of algebras to that of coalgebras such that the following adjointness relation holds.

3.7 $$\mathrm{Alg}(A,B^*) \cong \mathrm{Coalg}(B,A^o)$$

where $A$ is an algebra, $B$ a coalgebra. $A^o$ is defined by

$$A^o = \mu_A^{*-1}(A^* \otimes A^*) = \{x \text{ in } A^* \mid \ker x \text{ contains an ideal } I \text{ of finite codimension}\},$$

where $\mu_A : A \otimes A \to A$ is the multiplication on $A$. Clearly for $A$ finite dimensional, $A^o = A^*$. If $H$ is a Hopf algebra then $H^o$ is also a Hopf algebra which we call the Hopf algebra dual of $H$. See [Sw], [Sw]' for further comments.

3.8 If $C$ is a finite dimensional coalgebra, then there exists a $1-1$ inclusion inverting correspondence between ideals in $C^*$ and subcoalgebras of $C$. An ideal $I$ in $C^*$ corresponds to $I^\perp = \{x \text{ in } C \mid \langle i^*, x\rangle = 0 \text{ for every } i^* \text{ in } I\} = (C^*/I)^*$, a subcoalgebra of $C$. A subcoalgebra $D$ of $C$ corresponds to the ideal $D^\perp = \{f \text{ in } C^* \mid \langle f, 1\rangle = 0$ for every $d$ in $D\}$ in $C^*$.

The following result will be very useful to us in the ensuing discussion. A proof of it can be found in [Sw], Theorem 2.2.1, p. 31 or [SHS], Exposé 11, Lemma 1.4.

Proposition 3.9. Any coalgebra $C$ is the union of finite dimensional subcoalgebras.

Definition 3.10. Given a set $E$, we define on the vector space $k(E)$ generated by $E$, a coalgebra structure by means of the $k$ linear maps

$$\Delta_{k(E)} : k(E) \to k(E)\otimes k(E) : e \to e \otimes e$$

$$\varepsilon_{k(E)} : k(E) \to k : e \to 1 , \quad e \text{ in } E .$$

A coalgebra of this form is called a constant coalgebra. If $E$ is finite, $k(E)$ is the dual of $\prod_E k$, the algebra of functions from $E$ to $k$ with pointwise operations. For any coalgebra $C$ the set of elements $g$ in $C$ such that $\Delta_C(g) = g \otimes g$, $\varepsilon_C(g) = 1$ are called the group like elements of $C$ and we will use the notation $G(C)$ to denote this set. It is easy to see that $G(\ )$ is a product preserving functor from coalgebras to sets, represented by k. The set G(C) is linearly independent, and is a basis if C is a constant coalgebra. Let $\bar{k}$ be the separable closure of k. A coalgebra C will be called <u>separable</u> if $\bar{k} \otimes C$ is a constant $\bar{k}$ coalgebra. If C is finite dimensional, this is equivalent to C* being a separable algebra. A coalgebra D is called <u>simple</u> if it contains no proper subcoalgebras; D is called <u>irreducible</u> if it contains a unique simple subcoalgebra $\neq 0$; D is called <u>coconnected</u> if D is both irreducible and its unique simple subcoalgebra is 1-dimensional. Finally a Hopf algebra is irreducible (coconnected) if it is irreducible (coconnected) as a coalgebra.

<u>Remarks 3.11</u>. a) It follows immediately from Proposition 3.9 that every simple subcoalgebra is finite dimensional. Moreover, under the correspondence cited in 3.8 above, the simple subcoalgebras of C correspond to the maximal ideals of C* and if D is a simple subcoalgebra of C then $D^* \cong C^*/D^{\perp}$ is a field.

b) By going over to the separable closure, reducing to the finite case and dualizing, it is easily seen that a coalgebra C is separable if and only if it is the union of finite separable subcoalgebras, in which case every subcoalgebra is separable. It is evident that the sum of separable coalgebras is separable.

c) Let C be a separable coalgebra over k, $\bar{k}$ the separable closure of k. Then

$$\overline{k} \otimes C \cong \overline{k}(G(\overline{k} \otimes C)),$$

$G(\overline{k} \otimes C)$ as before the set of grouplike elements of $\overline{k} \otimes C$. Let $\Gamma$ be the Galois group of $\overline{k}$ over k. Then $\Gamma$ is a profinite group in the Krull topology and acts on the elements of $\overline{k} \otimes C$ via $\gamma(k \otimes c) = \gamma k \otimes c$ for $\gamma$ in $\Gamma$, $k \otimes c$ in $\overline{k} \otimes C$. In particular it acts continuously on the set $G(\overline{k} \otimes C)$ endowed with the discrete topology. More generally if E is a $\Gamma$ set on which $\Gamma$ acts continuously, E with the discrete topology, $\Gamma$ acts on $\overline{k}(E)$ in the obvious manner. Setting $k\langle E\rangle = \overline{k}(E)^{\Gamma}$ = fixed points under $\Gamma$, we get that $k\langle E\rangle$ is a k coalgebra such that $\overline{k} \otimes k\langle E\rangle \cong \overline{k}(E)$. See [SHS] Exposé 11, for further details. In particular for C a separable coalgebra,

$$C \cong (\overline{k} \otimes C)^{\Gamma} = k\langle G(\overline{k} \otimes C)\rangle$$

is an isomorphism of coalgebras and this establishes an equivalence of categories between the category of separable coalgebras and the category of sets E with continuous $\Gamma$ action; E with the discrete topology. Clearly Hopf algebras which are separable as coalgebras correspond to $\Gamma$ groups under this equivalence.

Proposition 3.12. Let C be a coalgebra in $\underline{C}$. Then

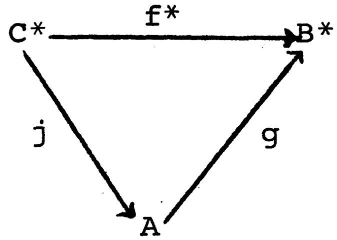
a) There exists a unique maximal separable subcoalgebra $C_s$ of C, with an inclusion morphism $i : C_s \to C$, such that if B is a separable coalgebra and $f : B \to C$ a morphism of coalgebras, f factors uniquely through $i$.

b) If $C$ is such that its simple subcoalgebras are separable, then there exists a unique coalgebra morphism $\rho : C \to C_s$ such that $\rho\iota = 1_{C_s}$, and any coalgebra morphism $g : C \to B$ for $B$ a separable coalgebra factors uniquely through $\rho$. Both $\iota$ and $\rho$ are natural in $C$.

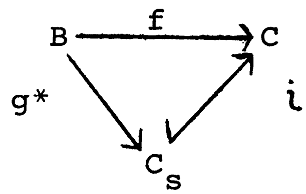
Proof: By Remark 3.11 (b) above it suffices to restrict ourselves to finite dimensional coalgebras, construct $C_s$ and by a limit argument extend our results to arbitrary coalgebras. Thus assume $C$ is finite dimensional. Then $C^*$ is a finite dimensional commutative algebra and as such the product of local rings. Hence $C^* = \prod_{i \in I} R_i$, $R_i$ local with maximal ideal $m_i$, $I$ finite. Set $C_s = A^*$ where $A$ is the product of those $R_i/m_i$ which are separable field extensions of $k$. $A$ is the maximal separable quotient algebra of $C^*$ with the obvious surjection $j : C^* \to A$. Clearly

$$\iota = J^* : C_s \hookrightarrow C$$

is a separable subcoalgebra of $C$. Any coalgebra morphism $f : B \to C$ from a separable coalgebra $B$ to $C$ gives rise to an algebra morphism $f^* : C^* \to B^*$ where $B^*$ is separable. (Recall that all coalgebras are assumed to be finite dimensional now.) It follows easily from the theory of separable algebras that $f^*$ factors uniquely through $A$, and so we get a commutative diagram




Dualizing we obtain a unique morphism $g^*$ such that the following diagram commutes

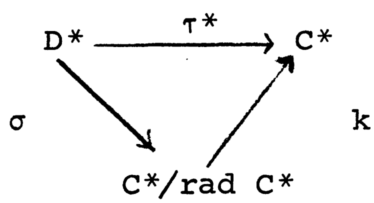



Thus $f$ factors uniquely through $i$. Note that this implies $C_s$ is the unique maximal separable subcoalgebra of $C$ - the sum of all the separable subcoalgebras of $C$.
To prove part (b) we observe that by Remark 3.11 (a) the simple subcoalgebras of $C$ are of the form $(R_i/m_i)^*$ for all $i$ in $I$, $I$ the finite index set introduced above. Hence $R_i/m_i$ is a finite separable field extension of $k$ for all $i$ in $I$ and

$$A = \prod_{i \in I} R_i/m_i \cong C^*/\text{rad } C^* ,$$

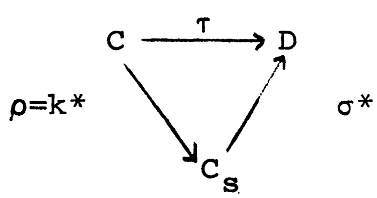
that is $C_s \cong (C^*/\text{rad } C^*)^* \cong (\text{rad } C^*)^{\perp}$ . In particular this implies that $C^*$ is a commutative finite dimensional algebra with $C^*/\text{rad } C^*$ separable. By the Wedderburn Malcev Theorem - see [CR] Theorem 72.19, p. 491 - there exists a splitting

$k : C^*/\text{rad } C^* \to C^*$ such that $\pi k : C^*/\text{rad } C^* \to C^*/\text{rad } C^*$ is the identity. Moreover any two splittings are conjugate, [CR] ibid; hence since $C^*$ is commutative, splitting is unique. Going over to the dual situation we get a unique coalgebra morphism $\rho = k^* : C \to C_s$ such that $\rho \cdot i = 1_{C_s}$. If $D$ is a separable coalgebra and $\tau : C \to D$ a coalgebra morphism, we get an algebra morphism $\tau^* : D^* \to C^*$ where $D^*$ is a separable algebra. The splitting $k : C^*/\text{rad } C^* \to C^*$ gives rise to a vector space decomposition of $C^*$ as $C^* \cong \text{rad } C^* \oplus S$ where $S =$ image of $k$ and thus is a subalgebra of $C^*$ isomorphic to $C^*/\text{rad } C^*$. Since $D^*$ is a separable algebra, $\text{Im } \tau^* \subseteq S$ and hence $\tau^*$ factors uniquely through $C^*/\text{rad } C^*$ as in the following commutative diagram




where $\sigma$ is uniquely determined.

Dualizing we arrive at a commutative diagram




and $\tau$ factors uniquely through $\rho$.

Finally we prove that $i : C_s \hookrightarrow C$ and, whenever it exists, $\rho : C_s \to C$ with $\rho \cdot i = 1_{C_s}$, are both natural in $C$. This is an easy consequence of the factorizations shown above. If $f : C \to C'$ is a morphism of coalgebras, then the composite $f \cdot i : C_s \to C'$ is a coalgebra morphism with $C_s$ separable and therefore by above there exists a unique coalgebra morphism $f_s : C_s \to C'_s$ such that the following diagram commutes

$$\begin{array}{ccc} C_s & \xrightarrow{\;i\;} & C \\ \downarrow{\scriptstyle f_s} & & \downarrow{\scriptstyle f} \\ C'_s & \xrightarrow{\;i'\;} & C' \end{array}$$

Similarly, in the case of coalgebras with a unique $\rho : C \to C_s$ satisfying the above conditions, there exists a unique coalgebra morphism $f^s : C_s \to C'_s$ such that the following diagram commutes

$$\begin{array}{ccc} C & \xrightarrow{\;\rho\;} & C_s \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^s} \\ C' & \xrightarrow{\;\rho'\;} & C'_s \end{array}$$

Moreover $f^s = f^s \cdot \rho \cdot i = \rho' \cdot f \cdot i = \rho' \cdot i \cdot f_s = f_s$ so the two coincide. We will use $f_s$ to denote the induced morphism from $C_s$ to $C'_s$ and we have shown both $i$ and $\rho$ are natural in $C$.

Remark 3.13. (a) The proof of part (a) above appears in [SHS] Exposé 11. A much simpler proof can be obtained by defining $C_s$ to be the maximal separable subcoalgebra of $C$ and showing directly that any coalgebra morphism from a separable coalgebra to $C$ factors uniquely through $C_s$. We have preferred, however, the more explicit description of $C_s$ given above for the purpose of the proof of part (b).

(b) Proposition 3.12 shows that $(\ )_s$ is a functor from the category of coalgebras $\underline{C}$ to the category of separable coalgebras right adjoint to the forgetful functor. As such, it preserves products and the natural morphism $(A \otimes B)_s \to A_s \otimes B_s$ is a coalgebra isomorphism for coalgebras $A$ and $B$ in $|\underline{C}|$. If $\underline{C}'$ is the full subcategory of $\underline{C}$, consisting of coalgebras whose simple subcoalgebras are separable, then the natural inclusion $i : C_s \to C$ for any $C$ in $|\underline{C}'|$ is such that $\rho \cdot i = 1_{C_s}$ with $\rho : C \to C_s$ uniquely determined and natural in $C$. In particular, let $H$ be a Hopf algebra such that its simple subcoalgebras are separable. Applying the product preserving functor $(\ )_s$ to the structure maps of $H$ gives a Hopf algebra structure on $H_s$ with $i : H_s \to H$ and $\rho : H \to H_s$, Hopf algebra maps since both are natural in $H$. $H_s$ will be called the separable component of $H$.

(c) Let $K$ be a field extension of $k$. It is easy to see that the natural morphism $(K \otimes A)_s \to K \otimes A_s$ is an isomorphism of $K$ coalgebras. In particular set $K = \overline{k}$ = the separable closure of $k$. The separable coalgebras over $\overline{k}$ are the constant coalgebras. Thus for $C$ a $k$ coalgebra, $(\bar{k} \otimes C)_s = \bar{k} \otimes C_s = \bar{k}(G(\bar{k} \otimes C))$ the largest constant $\bar{k}$ coalgebra of $\bar{k} \otimes C$, where $G(\bar{k} \otimes C)$ denotes as usual the group like elements of $\bar{k} \otimes C$. This implies $C_s = k\langle G(\bar{k} \otimes C)\rangle$. If $H$ is a Hopf algebra, then $H_s = k\langle G(\bar{k} \otimes H)\rangle$ where now $G(\bar{k} \otimes H)$ is a $\Gamma$ group with continuous $\Gamma$ action, $\Gamma$ the Galois group of $\bar{k}$ over $k$, $G(\bar{k} \otimes H)$ endowed with the discrete topology.

Proposition 3.14. Let $H$ be a Hopf algebra whose simple subcoalgebras are separable. Then $H$ is the semi-direct product of a coconnected Hopf algebra and the separable component $H_s$ of $H$.

Proof: By above we have obtained a unique Hopf algebra morphism $H \xrightarrow{\rho} H_s$ such that $\rho \cdot i = 1_{H_s}$, $i : H_s \hookrightarrow H$ the inclusion morphism. Let $M$ be the kernel of $\rho : H \to H_s$. We have an exact sequence of Hopf algebras

$$k \to M \underset{\sigma}{\to} H \underset{\rho}{\overset{i}{\rightleftarrows}} H_s$$

with a Hopf algebra section $i$. By proposition 2.4, $H \cong M \# H_s$, the semi-direct product of $M$ and $H_s$ with $H_s$ acting on $M$ via inner automorphisms and this is a Hopf algebra isomorphism. Let $L$ be a simple subcoalgebra of the Hopf algebra $M$. We can consider $M$ as a sub-Hopf algebra of $H$. By hypothesis, $L$ is separable, hence by 3.12 above there exists a subcoalgebra $L'$

in $H_s$ such that $L = \dot{\iota}(L')$. But then

$$L' = \rho\cdot\dot{\iota}(L') = \rho(L) = \varepsilon_H(L) \subset k \subset H_s$$

which implies $L = k \subset M$. Thus $M$ has a unique simple subcoalgebra, namely $k$; hence is coconnected and the result follows.

Definition 3.15. Let $H$ be a Hopf algebra. An element $x$ in $H$ is called primitive if $\Delta_H(x) = x \otimes 1 + 1 \otimes x$. Denote by $\mathcal{P}(H)$ the vector space of primitive elements in $H$. Then it is easy to see that $\mathcal{P}(H)$ has a Lie algebra structure given by $[x,y] = xy - yx$ for $x,y$ in $\mathcal{P}(H)$. This is the Lie algebra associated to the Hopf algebra $H$.

Corollary 3.16. Let **k** be a perfect field. If $H$ is a Hopf algebra over $k$, then $H$ is the semi-direct product of a coconnected Hopf algebra and the separable component $H_s$ of $H$. Furthermore, if $k$ has characteristic zero, $H$ is the semi-direct product of the universal enveloping algebra of $\mathcal{P}(H)$ and the separable component $H_s = k\langle G(\bar{k} \otimes H)\rangle$ of $H$ where $\bar{k}$ is the separable (algebraic) closure of $k$.

Proof: Let $D$ be a simple subcoalgebra of $H$. Then $D^*$ is a finite field extension of the perfect field $k$, thus separable and hence $D$ is a separable coalgebra. By Proposition 3.14, $H$ decomposes into a semi-direct product of a coconnected Hopf algebra $M$ and the separable component $H_s = k\langle G(\bar{k} \otimes H)\rangle$ of $H$.

If $k$ has characteristic zero, a coconnected Hopf algebra $M$ over $k$ is the universal enveloping algebra of the Lie algebra of its primitive elements. See [Sw], Theorem 13.0.1, p. 169. Let $x$ be a primitive element of $H$. We now use the notation of Proposition 3.14 and consider the exact sequence

$$k \to M \underset{\sigma}{\to} H \underset{\rho}{\to} H_s .$$

For $x$ in $\mathcal{P}(H)$, $\rho(x)$ is a primitive element of $H_s$ and remains so under change of ground field. By going over to the separable closure we can assume $\rho(x)$ is a primitive element in a constant coalgebra $k(E)$ say. But then it is easily shown by means of a simple basis argument that $\rho(x) = 0$. This implies that

$$(1 \otimes \rho)\Delta_H(x) = (1 \otimes \rho)(x \otimes 1 + 1 \otimes x) = x \otimes 1$$

or equivalently $x$ lies in $M$. See Proposition 1.5. Thus $M$ contains all the primitives of $H$ and is the universal enveloping algebra of the Lie algebra $\mathcal{P}(H)$.

Remarks 3.17. a) Note that if $H$ is commutative as an algebra, that is an abelian group object in $\underline{C}$, the action of $H_s$ on $M$ is trivial and the semi-direct product becomes the ordinary tensor product of the abelian Hopf algebras $M$ and $H_s$ with the induced abelian Hopf algebra structure. These remarks and Corollary 3.16

have connections with the theory of algebraic groups. Over a field of characteristic zero the algebraic group analogue is the decomposition of a commutative algebraic group into a product of unipotent and reducible components. See [SHS], Expose 11, p. 11 for more details.

b) M. Sweedler has constructed a Hopf algebra morphism $\rho' : H \to k(G(H))$ for Hopf algebras $H$ whose simple subcoalgebras are one-dimensional, such that $\rho' = 1_{k(G(H))}$ where $k(G(H))$ is the group ring on the group like elements of $H$ and $i : k(G(H)) \to H$ is the inclusion mapping. From this he gets a decomposition $H \cong M \# k(G(H))$ where $M$ is the kernel of $\rho'$ and is the maximal coconnected sub-Hopf algebra of $H$ containing $k$. Since the simple subcoalgebras of $H$ are one-dimensional they are a fortiori separable, and by the uniqueness of the previously defined $\rho : H \to H_s$, $\rho = \rho'$. Moreover, $H_s = k(G(H))$ and our results give the same decomposition up to isomorphism as Sweedler's. In particular in the case that $k$ is an algebraically closed field the above conditions are satisfied for a Hopf algebra $H$ over $k$ and we recover Konstant's result on the decomposition of a Hopf algebra over an algebraically closed field.

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