ORTHOGONALITY AND

NONLINEAR FUNCTIONALS

ON L_p-SPACES

by

K. Sundaresan

Report 70-21

Orthogonality and Nonlinear Functionals On L_p - spaces

K. Sundaresan

Let (X,Σ,μ) be a measure space. If x,y are two measurable real valued functions then x,y are said to be orthogonal in lattice theoretic sense, in short, $x \perp_L y$ if $\mu\{t \mid x(t)\, y(t) \neq o\} = o$. If F is a real valued function on the Banach space $L_p(\mu)$ $(1 \leq p \leq \infty)$ then F is said to be additive if $x,y \in L_p(\mu)$ and $x \perp_L y$ then $F(x + y) = F(x) + F(y)$. Integral representations of additive functionals on $L_p(\mu)$ has been the subject of extensive study in recent years. For these and related results we refer to Drewnowskii and Orlizc [1], Mizel and Sundaresan [2], Mizel [3] and Sundaresan [4].

The concept of orthogonality involved in the definition of additive functionals though very natural in these function spaces there are other concepts of orthogonality in a Banach space, in particular applicable to the function spaces $L_p(\mu)$, which are natural generalizations of the usual concept of orthogonality in Eucledian spaces and are of intrinsic geometric interest. For several such concepts of orthogonality we refer to James [5,6]. We are particularly interested in the definition of orthogonality

A M S Subject Classification. Primary 4610, 4635, 4780
Secondary 2825, 2816
Key words and phrases: Orthogonaltiy, additive functions, L_p - spaces, caratheodary function, integral representation, and non-atomic measure.

adopted in [6]. According to [6] if B is a real Banach space and $x,y \in B$ then x is orthogonal to y, in short, $x \perp_J y$ if $\| x + \lambda y \| \geq \| x \|$ for all real values of λ.

The purpose of the present paper is to study functionals F on spaces $L_p(\mu)$ where F is required to be additive in the sense if $x,y \in L_p(\mu)$ and $x \perp_J y$ then $F(x + y) = F(x) + F(y)$. For the motivation of study of such functionals it is enough to note that if $x,y \in L_p(\mu)$ and $x \perp_L y$ then $x \perp_J y$ while the following counter example shows that this implication cannot be reversed. Let μ be the Lebesque measure contracted to the unit interval I. Consider the functions x,y defined by $x = C_{[0,1/2]} - C_{[1/2,1]}$ and y is the constant function 1, where if M is a set C_M is the characterstic function of M. It is verified that $x,y \in L_p(\mu)$ and $x \perp_J y$ while x and y are not orthogonal in lattice theoretic sense.

Before proceeding to the main result we recall the necessary terminology, notations and few useful results.

In what follows p is arbitrary, $1 \leq p \leq \infty$, unless otherwise specified. (X,Σ,μ) is a fixed totally σ-finite nonatomic positive measure space. A function $\varphi : R \times X \to R$ is a Caratheodary function, in short a C-function, if

(1) $\varphi(\cdot,t) : R \to R$ is continuous for almost all $t \in X$ and

(2) $\varphi(r,\cdot) : X \to R$ is measurable for all $r \in R$. It is verified that if x is a measurable real valued function so is the function $\varphi \circ x$ defined by $\varphi \circ x(t) = \varphi(x(t),t)$. A C-function φ is said to be a C_p-function if $\varphi \circ x \in L_1(\mu)$ for all $x \in L_p(\mu)$.

It is known, Krasnoselśkii [7], that if $p < \infty$ and $\mu(X) < \infty$ then a C - function φ is a C_p - function if and only if there exists a function $\alpha \in L_1(\mu)$ and a **positive number** b such that

$$|\varphi(r,t)| \leq \alpha(t) + b\,|r|^p$$

for all $r \in R$.

A function $F: L_p(\mu) \to R$ is an A_L-function if it satisfies the following condition. (1) If $x,y \in L_p(\mu)$ and $x \perp_L y$ then $F(x + y) = F(x) + F(y)$. (2) If $p = \infty$ then F is uniformly continuous relative to L_∞ norm on each bounded subset $M \subset L_\infty(\mu)$ and if $p < \infty$ then F has the same property provided M is supported by a set of finite measure. (3) F is continuous on the Banach space $L_p(\mu)$ if $p < \infty$ and is continuous with respect to bounded a.e convergence if $p = \infty$. F is said to be an A_J - function if F satisfies the conditions (2) and (3) stated above in addition to (1′) if $x,y \in L_p(\mu)$ and $x \perp_J y$ then $F(x + y) = F(x) + F(y)$. Since $x \perp_L y \Rightarrow x \perp_J y$ every A_J - function is an A_L - function.

We recall the following theorem stated in [2] in a form suitable for our purpose.

Theorem 1. Let F be a real valued function on $L_p(\mu)$ $(1 \leq p \leq \infty)$. Then F is an A_L - function if and only if there exists a C_p - function $\varphi : R \times X \to R$ such that for all $x \in L_p(\mu)$

$$F(x) = \int_X \varphi \circ x \, d\mu$$

where $\varphi(o,t) = o$ for t a.e in X.

The main purpose of the paper is to characterise the class of all A_J - functions on $L_p(\mu)$. Before proceeding to the characterization we need a useful lemma from functional equations. In the rest of the paper if $M \subset X$, the characterstic function of M is denoted by I_M.

Lemma 1. Let φ be a continuous function on $R \to R$ satisfying the functional equation

$\varphi(p + q) + \varphi(p - q) = \varphi(q) + \varphi(-q) + 2\,\varphi(p)$, $\varphi(o) = o$,

then there exists real constants C_1 and C_2 such that

$\varphi(t) = C_1 t^2 + C_2 t$ for all $t \in R$.

Proof. From equations (1) and (2) it is seen that $\varphi(2t) = 3\,\varphi(t) + \varphi(-t)$ for all $t \in R$. Hence if $G(t) = \varphi(t) + \varphi(-t)$ then $G(2t) = 2^2 G(t)$. Assuming inductively that $G(mt) = m^2 G(t)$ for all natural numbers $m \leq n$ and substituting $p = nt, q = t$ in (1) it is verified that $G((m + 1) t) = (m + 1)^2 G(t)$.

Thus $G\left(\frac{m}{n}\right) = \frac{m^2}{n^2} G(1)$ for all rationals $\frac{m}{n}$. Since G is continuous it follows that $G(t) = t^2 G(1)$ for all $t \in R$. Let ψ be the function $\psi(t) = \varphi(t) - \frac{1}{2} ct^2$ where $G(1) = C$. Then ψ is a continuous function and ψ satisfies the functional equation $\psi(r + s) + \psi(r - s) = 2\,\psi(r)$, $\psi(o) = o$. Hence $\psi(t) = a\,t$ for all $t \in R$ where a is a constant. Hence $\varphi(t) = at + \frac{1}{2} ct^2$ completing the proof of the lemma.

Theorem 2. Let F be a function on $L_p(\mu) \to R$.

(a) If $p \neq 2$ then F is an A_J - function if and only if there exists a real valued measurable function β on X such that if

$x \in L_p(\mu)$ then the function $\beta(t)$ is in $L_1(\mu)$ and

$F(x) = \int_X \beta(t)\, x(t)\, d\,\mu(t).$

(b) If $p = 2$ then F is an A_J-function on $L_p(\mu)$ if and only if there is a real number c and a function β as in (a) such that for all $x \in L_2(\mu)$

$$F(x) = \int (cx^2(t) + \beta(t)\, x(t)\, d\,\mu(t)$$

Proof. Since the functions $\varphi_i : R \times X \to R$, $i = 1,2$ defined by $\varphi_1(r,t) = \beta(t)\, r$ and $\varphi_2(r,t) = cr^2 + \beta(t)\, r$ are verified to be C_p and C_2 functions respectively it follows from Theorem 1 that the function F defined by the equations in (a) and (b) is an A_L - function on the corresponding space $L_p(\mu)$. If $p \neq 2$ F is linear on $L_p(\mu)$ and certainly it is an A_J - function by noting that in a Hilbert space $x \perp_J y$ if and only if the inner product $(x,y) = o$.

Conversely let F be an A_J - function on the space $L_p(\mu)$. Since an A_J - function is also an A_L - function it follows from Theorem 1 that there exists a C_p - function φ such that for all $x \in L_p(\mu)$

$$F(x) = \int_X \varphi(x(t),t\, d\,\mu(t).$$

Since $\mu(X) > o$ and μ is nonatomic there exists a measurable set M, $o < \mu(M) < \infty$, a pair of disjoint measurable subsets M_1, M_2 of M such that $\mu(M_i) > o$, $i = 1,3$ and $M_1 \cup M_2 = M$. Let r,s be any two real numbers and x,y be the functions $x = r\, I_{M_1} \dot{-} I_{M_2}$ and $y = s\, I_M$. It is verified that $x,y \in L_p(\mu)$ and $x \perp_J y$. Hence $F(x + y)$ $= F(x) + D(y)$. Thus from the integral representation of F it is verified that $\int_X \varphi(s,t)\, d\,\mu(t) + \int_{M_1} \varphi(r,t)\, d\,\mu(t) + \int_{M_2} \varphi(-r,t)\, d\,\varphi(t)$

$= \int_M \varphi(r + s,t)\, d\,\mu(t) + \int_M \varphi(s - r.t)\, d\,\mu(t)$

Similarly considering the functions $r I_{M_1}$ and $s I_M$ we obtain an equation same as the preceeding one except that M_1 and M_2 are to be interchanged. From these equations it is verified that

$$\int_M 2\,\varphi(s,t)\,d\mu(t) + \int \varphi(r,t)\,d\mu(t) + \int \varphi(-r,t)\,d\mu(t)$$

$$= \int_M [\varphi(r+s,t) + \varphi(s-r,t)]\,d\mu(t).$$

Since this equation is true for every measurable set M, $\mu(M) < \infty$, and μ is nonatomic it follows from the uniqueness assertion in Radon - Nikodym theorem that for a given pair of real numbers r,s and for t a e that

$$(*) \quad \varphi(r+s,t) + \varphi(s-r,t) = \varphi(r,t) + \varphi(-r,t) + 2\,\varphi(s,t).$$

Let now Q be the set of rationals. Since $Q \times Q$ is a countable set it is verified that there exists a μ - null set N such that for all $t \notin N$ the equation $(*)$ continues to be true for all $(r,s) \in Q \times Q$. Since $\varphi(\cdot,t)$ is a continuous function for t a.e. the equation $(*)$ is verified for all $(r,s) \in R \times R$ and for t outside a nullset. Since $\varphi(o,t) = o$ a.e. it follows from Lemma 1 that there exist functions α,β on $X \to R$ such that for t a.e.

$$\varphi(r,t) = \alpha(t)\,r^2 + \beta(t)\,r.$$

Since φ is a C - function $\varphi(1,\cdot)$ and $\varphi(-1,\cdot)$ are measurable functions. Thus α,β could be assumed to be measurable functions.

We proceed to show that α is a constant a e and further if $p \neq 2$ then $\alpha = o$

We assert (A) if s is any real number then either $\mu\{t \mid \alpha(t) > s\} = o$ or $\mu\{t \mid \alpha(t) < s\} = o$ and (B) if for some real number c, $\mu(\alpha^{-1}(c)) > o$ then $\alpha(t) = c$ for t a.e. i.e. $\mu(\alpha^{-1}(c)) = \mu(X)$.

If the assertion in (A) is false there exist measurable sets M_1, M_2, $o < \mu(M_1) = \mu(M_2) < \infty$ such that $M_1 \cap M_2 = \emptyset$ and $\alpha(t) > s$ on M_1 and $\alpha(t) < s$ on M_2. Let $x_1 = I_{M_1} - I_{M_2}$ and $y_1 = I_M$ where $M = M_1 \cup M_2$. It follows that $x_1 \perp_J y_1$. Hence $F(x_1 + y_1) = F(x_1) + F(y_1)$. Now appealing to the integral representation of F and noting $\varphi(r,t) = \alpha(t)\, r^2 + \beta(t)\, r$ it is verified that $\int_{M_1} (\alpha(t) - s)\, d\,\mu(t) - \int_{M_2} (\alpha(t) - s)\, d\,\mu(t) = o$

since $\mu(M_1) = \mu(M_2)$. However this equation contradicts the fact that $\mu(M_i) > o$ for $i = 1,2$ and $\alpha(t) - s > o$ on M_1 and $\alpha(t) - s < o$ on M_2. This completes the proof of (A). We proceed to verify (B). Let c be as in the hypothesis of (B). If $\alpha(t) = c$ for t a e is not true then either $\mu\{t \mid \alpha(t) > c\}$ or $\alpha\{t \mid \alpha(t) < c\}$ is positive. Let for definiteness $\mu(\alpha^{-1}]\, c, \infty\, [) > o$. Since $\lim_{n\to\infty} \mu(\alpha^{-1}]\, c + 1/n, \infty\, [)$ $= \mu(\alpha^{-1}]\, c, \infty\, [)$ there is an integer n_1 such that $\mu(\alpha^{-1}]\, c + 1/n_1, \infty\, [) > o$. Thus if $c < s < c + 1/n_1$ then $\mu(\alpha^{-1}] - \infty, s\, [$ and $\mu(\alpha^{-1}]\, s, \infty\, [$ are both positive contradicting (A). Thus (B) is verified.

Next we proceed to show (C) that there is a real number c such that $\mu(\alpha^{-1}(c)) > o$. If this is not true then (*) for every real number r, $\mu(\alpha^{-1}(r)) = o$. Thus from (A) it is verified that either $\mu(\alpha^{-1}]\, o, \infty\, [) = \mu(X)$ or $\mu(\alpha^{-1}] - \infty, o\, [) = \mu(X)$. Let us assume the first alternative is true. Then recalling (*)

it is verified that $o < \mu(\alpha^{-1}]\ o,\infty[) = \sum_{n\geq o} \mu(\alpha^{-1}]\ n,n+1[)$.

Thus from (A) and (*) it is verified that there exists an interval $]m,m+1[$ such that $\mu(\alpha^{-1}]\ m,m+1[) = \mu(X)$. Once again from (*) it follows that if $F_1 = [m,m+1]$ then $\mu(\alpha^{-1}(F_1)) = \mu(X)$. Noting that $\mu(\alpha^{-1}]\ m,m+1[= \mu(\alpha^{-1}]\ m,m+1/2[) + \mu(\alpha^{-1}]\ m+1/2,m+1[)$ and repeating the above argument it is verified that there is a closed interval $F_2 \subsetneq F_1$, length $F_2 = \frac{1}{2}$ such that $\mu(\alpha^{-1}(F_2)) = \mu(X)$. Repeating this procedure there is a decreasing sequence of closed intervals $\{F_i\}$ such that length $F_i \to o$ and $\mu(\alpha^{-1}(F_i)) = \mu(X)$. Thus if c is is the real number $\{c\} = \cap F_i$ then $\mu(\alpha^{-1}(c)) = \mu(X)$ contradicting (*). Hence (C) is verified.

From (B) and (C) it follows that α is a constant function. This completes the proof of the theorem for $p = 2$.

We complete the proof of the theorem by showing that if $p = 2$, $\alpha(t) = o$ a.e.

If $1 \leq p < 2$ since $\varphi \circ x \in L_p(\mu)$ whenever $x \in L_p(\mu)$ as noted earlier restricting our attention to a measurable set M, $o < \mu(M) < \infty$, we find that there is a function $a \in L_p(\mu)$ and a positive number b such that

$$|\ \alpha(t)\ r^2 + \beta(t)\ r\ | \leq a(t) + b\ |\ r\ |^p, \text{ a.e. in } M.$$

From this inequality since $p < 2$ it follows that $\alpha(t) = o$ a.e. in M. Thus $\alpha(t) = o$ a.e.

Next we proceed to the case $p > 2$ but $p \neq \infty$. Let $\alpha(t) = c$ a.e. Then from the implication $x \perp_J y \Rightarrow F(x) + F(y) = F(x + y)$ it is verified that

(s) $x \perp_J y \Rightarrow c \int_X x(t)\, y(t)\, d\,\mu(t) = o$.

Let $c \neq o$. Let M_1, M_2 be two disjoint measurable subsets of X such that $o < \mu(M_1) = \mu(M_2) < \infty$. Consider the functions $x = 2^{\frac{1}{p-1}} I_{M_1} + 8^{\frac{1}{p-1}} I_{M_2}$ and $y = 4\, I_{M_1} - I_{M_2}$. Then it is verified that $x \perp_J y$. Thus from (s) since $c \neq o$ it follows that $\int_X x(t)\, y(t)\, d\,\mu(t) = o$. By direct evaluation of the integral it is verified that the integral is not equal to o, obtaining a contradiction. Hence $\alpha(t) = o$ a.e.

The case when $p = \infty$ is similarly dealt except instead of x, y of the preceeding case we consider the functions defined as follows. Let M_1, M_2, M_3 be three pair wise disjoint measurable sets such that $o < \mu(M_1) = \mu(M_2) = \mu(M_3) < \infty$. Let $x = I_M$ and $y = -\, I_{M_2}$ where $M = M_1 \cup M_2 \cup M_3$. It is verified that $x, y \in L_\infty(\mu)$ and $x \perp_J y$. However $\int_X x(t)\, y(t)\, d\,\mu(t) \neq o$. Thus as in the preceeding case $\alpha(t) = o$ a.e.

Remark: We note that since F is a continuous function on $L_p(\mu)$ that the function $x \to \int \beta(t)\, x(t)\, d\,\mu$ is a continuous linear functional. Thus if $1 \leq p < \infty$ then $\beta \in L_q(\mu)$ where $q = \frac{p}{p-1}$ if $p > 1$ and $q = \infty$ if $p = 1$.

Thus we note that every A_J - function on $L_2(\mu)$ is of the form $F(x) = c \|x\|^2 + f(x)$ where c is some real number and f is a continuous linear functional on $L_2(\mu)$.

In conclusion we note that if B is a Banach space and if there is a function $F : B \to R$ such that (1) $x \perp_J y \Rightarrow F(x + y) = F(x) + F(y)$ and (2) $F(x) = c \| x \|^2 + f(x)$ for some real number $c \neq o$ and for some $f \in B^*$ (the dual of B) and dim $B \geq 3$ then B is a Hilbert space. For if there is such a function F then it is verified that

$$x \perp_J y \Rightarrow \| x + y \|^2 = \| x \|^2 + \| y \|^2 .$$

Thus if $x \perp_J y$ then since $\lambda x \perp_J y$ for all $\lambda \in R$ it follows that $\| y + \lambda x \|^2 = \lambda^2 \| x \|^2 + \| y \|^2 \geq \| y \|^2$. Thus $y \perp_J x$. Hence orthogonality (J) is symmetric. Thus from the footnote on p. 283 in [6] it follows that B is a Hilbert space. This observatiin provides an alternate proof of the part asserting $\alpha(t) = o$ a.c. if $p \neq 2$ in the proof of Theorem 2 in this paper.

As already noted in the introduction there are other concepts of orthogonality in normed linear spaces, [5]. However the one considered in this paper is the most interesting since it is closely related to hyperplanes and linear functionals as one notices by referring to [6].

References

1. L. Drewnowskii and W. Orlicz, On Orthogonally Additive Fuctionals, Bull. Acad. Polon. Sci., Sér. Sci. math. astronom. et. Phys. XVI(1968), 883-888.

2. V. J. Mizel and K. Sundaresan, Representation of Additive and Biadditive Functionals, Arch. Rational Mech. Anal. 30(1968), 102-126.

3. V. J. Mizel, Representation of Nonlinear Transformations on L^p spaces, Bull. Amer. Math. Soc. 75(1969), 164-168.

4. K. Sundaresan, Additive Functionals on Orlicz spaces, Studia Math. XXXII(1969), 269-276.

5. R. C. James, Orthogonality In Normed Linear spaces, Duke Math J. 12(1945), 291-302.

6. R. C. James, Orthogonality And Linear Functionals In Normed Linear Spaces, Trans. Amer. Math. Soc. 61(1947), 265-292.

7. M. A. Krasnoselskii, Topological Methods in the theory of nonlinear integral equations, translated by J. Burlak, Macmillan, New York, 1964.

Carnegie-Mellon University
Pittsburgh, PA

Errata

Page	Line	Read	Instead
1	9	Orlicz	Orlizc
5	1	$\beta(t)\ x(t)$	$\beta(t)$
5	24	$\int_M$	$\int_X$
6	1	$r\ I_{M_2} - r\ I_{M_1}$	$r\ I_{M_1}$