TWO NOTES ON REGULARITY
IN GENERAL TOPOLOGY

by

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1. A characterization of regularity in topology.

2. On regular convergence spaces.

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A CHARACTER]ZATION OP REGULARITY IN TOPOLOGT

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We ehow in this note that the topological separation axiom T, (which we do not intend to imply T) can be regarded as a continuity condition for the convergence of filters. We express this condition in two quite distinct ways. This raises questions about connections between various continuity conditions for relations. We have only a partial answer to these questions.

Let (E,~C-) be a topological space. We denote by E* the set of all filters on $ which converge for ·r- to some point of E . For XC E , we put X* = i<p£E* : X £ <j>\ . Then 0* = 0 for the empty set, and

(xru)* = x*ni* , *ex* <%=^ x£x ,

for subsets X , Y of E , x £ E , and i = |l C E : x 6 XJ . We regard convergence of filters for TT as a relation q : E* —> E , writing a? q x if <p converges to x . This relation is a mapping if and only if (E,tr) is a Hausdorff space. For X CZ E , we have q(X*) = X̄ , the closure of X for ~cr .

It seems natural to impose a topology on E* by using the sets U* , with U open for X" , as a basis of open sets. The preceding paragraph shows that this works, and ve denote the topology of E* thus defined by ~c* . With this notation, wt; state the following theorem.

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(i) (E,T) is a $T_?$ space.

(ii) q : (E*,t*) —> (E.'tr) is left continuous.

(iii) (E,~n) is an R_n space and q is upper aepj.—continuous.

Proof. Assume first T_3 and consider f : A —>• E* and g : A —> E with f continuous and f(a) converging to g(a) for all a £A . If U is open in E and g(a) £ U , then g(a) *6* V CT V C U for some open V . For this V , we have V &f(a) , and V6f(x) implies g(x)£:V. Thus *a Crf'¹^*)* and f^{-1}(V*) *d g~* ^{1}(u) . This shows that g^{-1}(u) is open, and hence g continuous.

If q is left continuous, then q is upper semi-continuous by Theorem 2. If i q y , let A be the space with two points a , b , and with £b^ open, but not closed. Put f(a) = f(b) = i and g(a) = x , g(b) » y . Then f is continuous and f(u) q g(u) for u £ A . Thus g is continuous. If V is an open neighborhood of *x* , then g^{-1}(v) is open and a£ g^{-1}(v) . Thus g^{-1}(V) » A , and y ^.V . This shows that also y q x , and E is I_Q .

Assume now (iii), and let P be closed in E and x £ . E \ F . If i q y , then ŷ q x , and y£ F would imply x £F = F . Thus x^q^{-1}(F) . It follows that i £V* for an open set V with V*r\ q"¹(F) * *0* , But then x € V , and V n F = q(V*) *f*\ F = *0* . Thus E satisfies T_2 .

The following example shows that R_Q cannot be omitted from Theorem 3. The space with two points and three open sets (used in the proof of the Theorem) is TQ but not T_1 , and hence a fortiori not T_3 or R_Q . But one sees easily that q is bcth upper and lower semi-continuous for this space.

The significance of lower semi-continuity of q is not clear to this author. The following curious result does not help with this.

Theorem 4, $q : (E^*, \mathcal{C}^*) \to (E, \mathcal{C})$ is lower semi-continuous if $(E, \mathcal{C})$ is a finite topological space.

Proof. We show more: all sets $q^{-1}(X)$ are open in E^*. Since $q^{-1}(X)$ is the set union of all $q^{-1}(x)$, $x \in X$, it suffices to consider $q^{-1}(x)$ for $x \in E$. The set intersection of all neighborhoods of x is an open neighborhood V of x since E is finite, and obviously $q^{-1}(x) = V^*$ for this set.

One further example: for the space with three points and six open sets, q is lower semi-continuous (by Theorem 4), but not upper semi-continuous.

We conclude with the following remark. All results of this note remain valid if E^* is taken as the set of all convergent ultrafilters on E and filters are replaced by ultrafilters throughout.

Preferences

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ON REGULARITY FOR CONVERGENCE SPACES

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Cook and Fischer in [2] and independently Fleischer in [4] defined regularity for convergence spaces by a partial converse of the inverse limit property. We wish to show in this note that their regularity condition is in fact a continuity condition for filter convergence. Since convergence is in general a relation and not a mapping, this requires a definition of continuity for relations. We furnish such a definition which should also be useful elsewhere.

Cook and Fischer proved that regularity in their sense is implied by the topological axiom T_3, formulated in terms of filter convergence. Biesterfeldt [1] pointed out that the converse implication is also valid. However, his proof works only for T_0 spaces; we shall give a proof for all spaces.

We have to establish some notations. We write $\psi \geq \varphi$ for filters φ, ψ on a set E if ψ is finer than φ, i.e. if $\varphi \subset \psi$. We define a convergence space (E,q) as a pair consisting of a set E and a relation q from filters on E to E, subject only to the two Fréchet axioms, as follows.

L 1. If $x \in E$ and $\dot{x} = \{X \subset E : x \in X\}$, then $\dot{x}\, q\, x$.

L 2. If $\varphi\, q\, x$ and $\psi \geq \varphi$, then also $\psi\, q\, x$.

The third axiom used by Cook and Fischer [2] will not be needed.

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Let (E,q) be a convergence space. We denote by E* the set of all filters on E which converge for q to some point in E , and we consider q as a relation from E* to E . We put X* = £<p£E* : X<£<p} for *X <Z* E . Then

(1) (x m) * = X * m * and x £ X * <£«> x(£X

for subsets X , Y of E and x £ E . For a filter *W* on E* , we define a filter *(p*± on E , called the compression of *(p ,* by putting

(2) *£<£» 4=^ X*£ *(p*

for X C E , One verifies easily that 0* is indeed a filter. If *0′* is the filter on the set cf all filters on E with *(p* as basis, then *0* ~ 96 0*^{l}*,* the Kowalsky compression defined in [5]; see [6] for a proof of this.

The closure of a set X *C~* E with respect to q is the set X̄ = q(x*) . If *a* is a filter on E , then the sets X̄ with X£(T< form a filter basis; we denote jy &*T* the filter on E with this basis. This allows us to formulate T_3 in terms of filter convergence, as follows.

T_3 . (7 q x always implies {Tq x .

The following theorem is due to Cook-Fischer [2] and Biesterfeldt [l].

Theorem 1. A convergence space (E,q) satisfies T, if and only if the following axiom schema is valid for (E,q) .

(Reg) Let I be a set and f : I —-> E* and g : I •—»E mappings. If f(b) converges to g(b) for all b £ l and if *cr* is a filter on I such that (f(«r))# converges to a point x *(£E* , then g(«r) converges to x .

Proof. If the data of *(tieg)* are given, then b (~ I and X *£żt(b)* imply g(b)£ ^ , and thus f(B) *C* X* =^>. g(s) d X̄ for sets B C I and X *CZ.* E .

It follows immediately that g(<jr) 4-<p for a filter ff on I and cp = $(g(\sigma))_*$, and now T- and L 2 imply that g(<3") converges to x if $(f(or))_\#$ does.

For the converse, let I in (Reg) be the graph of q and f and g the projections, i.e. *t(<p_t x)-- Cp* and g(cP,x) <= *x if <jP q, x .* If I C E , let S_x « in (X*X X)¯ . Then clearly f(s_x) * X* and g(S_x) - *1* . If <p£ E* , then the sets S_X with *X£<X>* form a filter basis; we denote by S<p the filter on I with this basis. One sees easily that (f^S))$_\#$ - *(h* and g(S-) =5f . If we apply (Reg) withCT = *Sq>*, then T, results.

If f : (E,q) –*5* (F,r) is a continuous mapping of convergence spaces, then the induced filter mapping maps E* into P* . Thus f induces a mapping f* : E* – ^ P * , We note without proof that

$$(3) \qquad Y \in f^*(\varphi) \iff f^{-1}(Y) \in \varphi ,$$

for *m*£ E* and Y *CZ* P . This is a general filter property.

Let again (E,q) be a convergence space. We introduce a convergence structure q* on E* as follows. We put *(p* q* <p , for <p£E* and a filter *0* *on* E* , if *0_t 4q* ,* and we put in addition *<fi q** x for *x* £ E if <&. q *x* . Thus **q*(0) depends only** or *J*\ . We have the following result.

Theorem 2. (E*,q*) is a convergence space. *(£>* q* *&#* for every filter on E* with *0*+ 6E* . If 0q*p and *m*q x , then (4 q x . If f : (E,q) –V (P,r) is a continuous mapping of convergence spaces, then f* : (E*,q*) ⟶ (F*,r*) is also continuous.

Proof. X*e 4> ^=4> ^ 6- X* <£=> *X (≈f,* and thus $(^.)_\#$ • <T> , and φ q*<» , if <p£ E* . L 2 for q* is obvious since If* £ <^V if Cf.< ^6 . The middle parts of the Theorem are obvious from the definition of q* . The last part follows immediately from the definition of q^* , the implication $f(x)$ a y =^ $f^*(i) = y^*$, and the following bit of filter algebra.

Lemma 1. _If_ f : (E,q) •–> (P,r) _is a map of convergence spaces, then_

irH*))* - (f*)"1^*) sad., (f»(0))# - $f^*(\Phi_*)$,

£or YC P and a filter (f) on E* .

Prool' J We note for the first part that

$$\varphi \in (f^*)^{-1}(Y^*) \iff f^*(\varphi) \in Y^* \iff Y \in f^*(\varphi) \iff f^{-1}(Y) \in \varphi \ .$$

Now use (?) once more. The second part is established by

$$Y \in (f^*(\Phi))_* \iff Y^* \in f^*(\Phi) \iff (f^*)^{-1}(Y^*) = (f^{-1}(Y))^* \in \Phi$$
$$\iff f^{-1}(Y) \in \Phi_* \iff Y \in f^*(\Phi_*) \ .$$

This proves the Lemma.

We define mappings j : E —> E* and Pd : E** —^ E* by putting j(x) = i for x£ S and x(^) - (fa for 0£r E** « (E*)* .

Theorem ?. j : (E,q) –^ (E*,q*) _and_ «# : (E**,q**) –> (E,q) _are continuous maps of convergence spaces_.

Proof. We have j(x) C I* <==$ X <C Y for subsets X , T of E by the second part of (l), and it follows immediately that (j(o?))# * <£ for a filter e>£E* . Thus j(tf?) q* i if d? q x , and j is continuous.

For the second part, we need the following lemma.

Lemma 2- (^e(F))* * (*%)* _for a filter_ 5* fin E** .

Proof. If X CT E , then

$$x \in (\mathfrak{X}(\mathcal{F}))_* \Longleftrightarrow x^* \in \mathfrak{X}(\mathcal{F}) \Longleftrightarrow \mathfrak{X}^{-1}(x^*) \in \mathcal{F} \quad .$$

and Xfi(F4)t <#=> **<££-» «H> (*•)*££" .

Since (Pex'H**) <#=^ (p*£X* <£=*> $X \in \Phi_* \Longleftrightarrow x^* \in \Phi$

for (fie E** , we have x"1(x*) = (X*)* , and the Lemma is proved.

Now let .J" q** θ . If (^ =<f and ,jP# q ffi for some (7> £r E* , then ®=(fa. and J^ q* $* . If Jr* <&> then CFj*^.0* , and again ."F* q* </* • Stt* tnenala0 X (<3~) q* (^» by lamina 2 and the definition of q* . Thus ,5C is continuous.

The last part of Theorem 2 shows that T (E,q) « (E*,q*) and T f = f* define an endofunctor T of the category of convergence spaces. There is enough filter algebra scattered in this paper to prove that T and the mappings j and OC are the ingredients of a categorical triple (T, j,«£) . The existence of such a triple seems to be characteristic for topological regularity. We plan to take this up elsewhere in a more general context.

Let now (E,q) and (P,r) be convergence spaces. We call a relation u : E —> F <u>left continuous,</u> from (E,q) to (F,r) , if for every continuous mapping f : (A,p) .—^ (E,q) , where (A,p) is an arbitrary convergence space, and every mapping g : A —> P such that f(a) u g(a) for all u £A , the mapping g is continuous, from (A,p) to (P,r) . We shall not study this concept in depth here; we note only the following result.

<u>Theorem</u>^. <u>Let</u> (^,q) <u>and</u> (P,r) <u>be convergence spaces</u>. <u>A mapping</u> u : E .—s*'j <u>is a left continuous relation from</u> (E,q) <u>to</u> (P,r) <u>if and only if</u> u <u>is a continuous mapping</u>. <u>If</u> u : (E,q) —^ (P,r) <u>and</u> v : (P,r) —* (G,s) <u>are left continuous relations</u>, <u>then</u> v u : (E,q) —* (G,s) <u>is left continuous</u>.

Proof. A continuous mapping obviously is left continuous as a relation. Conversely, if a mapping u : E ••—:>F is a left continuous relation, then g = u for f = id E in the definition. Thus u is a continuous mapping.

If u : E —7 F and v : P —^ G are left continuous, and if f : A —^ E and h : A —? G satisfy f(a) (v u) h(a) for all a<ĉA , then there is g : A — > P such that f(a) u g(a) and g(a) v h(a) for all a £A . If f is continuous, then *g* is continuous, and then also h is continuous. Thus v u is left continuous.

Theorem 4 shows that convergence spaces and left continuous relations form a category into which the category of convergence spaces and continuous mappings is naturally embedded.

We have cleared the way for our main result.

Theorem 5. A convergence space (E,q) satisfies (Reg) if and only if convergence of filters defines a left continuous relation q : (E*,q*) ⟶ (E,q) .

Proof. Assume first (Reg) and consider f : (A,p) —^ (E*,q*) and g : A —* E such that f(a) converges to g(a) for all a(ĉA . If *er* p a , then f(b) q* f(a) and f(a) q g(a) . By the middle part of Theorem 2, (ffcr))# q g(a) in this situation. Thus *g((5)* q g(a) by (Reg), and g is continuous. This proves left continuity of q .

Let now q be left continuous, and consider the situation of (Reg). Adjoin a single point z to I to form A , and extend gr to a filter on A *t* also denoted by <*T* . Put & p z if *q*> 4<T or <*p*= *t* , and allow only à p a for a £ z in I . This defines a convergence space (A,p) . Extend f and g to A by putting f(z) = i and g(z) « x . This does not change f(<5") and g(5) .

If (f(er))» q x , then f(<y) q* f(z) , and thus f is continuous. Since f(a) q g(a) for a = z as well as a £ l , it follows that g is continuous. But then *g(&)* converges to x = g(z) , as required by (Reg).

Corollary. A separated convergence space (E,q) is regular if and only if q : (E*,q*) —*$* (2»q) is a continuous mapping.

Proof. If (E,q) is separated, then q is a mapping, and thus the Corollary just requires putting Theorems 4 and 5 together.

We conclude with some remarks. If (E,q) is a separated regular convergence space, then q : (E*,q*) —^ (E,q) is not only a continuous mapping but defines an algebra for the triple (T, j,i?e) mentioned earlier in this note. To prove this is an easy exercise in filter algebra.

The spaces considered in [<?] satisfy the following axiom in addition to L 1 and L 2.

L 3. If <*j*> q x and J^q y , then (^<_>$P») q x .

Here ^>cy.' = j l u Y : X^-qp and Y€=y^» T0686 spaces are the limit spaces of [5] and [3]. If (E,q) is * limit space, then so is (E*,q*) . This follows immediately from the fact, easily verified, that *((pulp)^* « *cf\ c* </» for filters *(ft* and ^ on E* . Thus all our results remain valid for the category of limit spaces. The only modification which is needed is in the proof of Theorem 5: we must put dp p z for <3> ^9"U ẑ .

If q is convergence of filters for a topology t of B , then one sees easily thj.t (E*,q*) is a neighborhood space, i.e. each <*p* € E* has a neighborhood filter N« such that <£ q* *f* <£=» 0 ^ U^ , for a filter (/) on E* . In general, q* is not the convergence of filters for a topology of E* ,, One sees without difficulty that the finest topology of B* with filter convergence coarser than q* has the sets U* , with U open for ~C t as a basis of open sets. The connections between this topology and T_3 for (&,*c) have been studied in [7].

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