ON EPI-REFLECTIVE HULLS

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Let G be a class of Tychonoff spaces[1] which is inversely preserved under perfect maps[2]. Let T2(G) be the epi-reflective hull[3] of G in the category T2 of Hausdorff spaces ([H_2],p.121), i.e., the full subcategory of T2 whose objects are the G-compact spaces*[4] ([E-M], [Bl], fJYU], [H_1]) .

THEOREM. X <u>is an object of</u> T2(G) <u>iff there is a family of subsets of</u> gX, <u>each belonging to</u> G, <u>whose intersection is</u> X.

This characterizes the epi-reflective hull of very many classes of Tychonoff spaces, for example, those defined by any of the following properties: countable compactness, local compactness, g-compactness, paracompactness, countable paracompactness, metacompactness, strong-paracompactness (fH-IJ), isocompactness (fBaJ). The Lindelöf spaces ([H-I]), the topologically complete spaces ([F_2]), and the M-spaces' and M*-spaces of Morita (fM_1J) also form suitable classes.

The theorem also yields an analogous characterization of the reflection ax of a Tychonoff space X in T2(G) as follows.[5]

COROLLARY. αX *is the intersection of all subsets of* βX *containing* X *and belonging to* G.

(The requirement that X be a Tychonoff space is simplifying but not necessary. For any Hausdorff space X substitute $\beta(\alpha X)$, where αX is the reflection of X in the category of Tychonoff spaces ([H]), for βX and the Corollary remains true).

The proof of the Theorem is a modification of an argument due to Frolik ([F_1] Theorem 2.1).

To say that

(1) G *is inversely preserved under perfect maps*

means that whenever $f: X \to Y$ is perfect and $Y \in G$, then $X \in G$ also. (1) implies that

(2) G *is closed hereditary.*

Indeed, the inclusion map on a closed subspace is perfect.

(1) also implies that

(3) *The product of a space in* G *and a compact space is again in* G.

This follows since the projection $X \times K \to X$ is perfect whenever K is compact. If G contains a non-empty space, (3) implies that it must contain every compact space, as do all the classes mentioned above.

(2) and (3) also imply (1) if G is replete[5]. Indeed, the map $\theta: X \to \beta X \times Y$ induced by $e_\beta: X \to \beta X$ and f is an embedding. Since f is perfect, $\theta(X) = G(\beta f) \cap (\beta X \times Y)$, where $G(\beta f)$, the graph of βf, is closed in $\beta X \times \beta Y$. Hence X is homeomorphic to a closed subspace of a member of G, and hence is in G.

If $X = \bigcap A_\alpha$ where each $A_\alpha \subseteq \beta X$ and each $A_\alpha \in G$, then X is homeomorphic to the diagonal (a closed subspace, by Hausdorffness) of $\prod A_\alpha$. Thus the sufficiency of the condition is easy.

Conversely, suppose that X is a closed subspace of $\prod A_\alpha$ $(\alpha < \lambda)$, with each $A_\alpha \in G$. For each $\alpha < \lambda$, let $B_\alpha = A_\alpha \times \prod\{\beta A_\gamma \mid \gamma \neq \alpha\}$. By (3), each $B_\alpha \in G$. But $X \subset \prod A_\alpha \subset B_\alpha$. Hence let $C_\alpha = \mathrm{cl}_{B_\alpha} X$ for each $\alpha < \lambda$. By (2) each $C_\alpha \in G$.

Applying β to the inclusion map $i_\alpha: X \to C_\alpha$ yields a perfect map $\beta i_\alpha: \beta X \to \beta C_\alpha$. Let $D_\alpha = (\beta i_\alpha)^{-1}(C_\alpha)$. Since the restriction of a perfect map to a full inverse image is again perfect, it follows from (1) that each $D_\alpha \in G$. Thus we have the desired family $\{D_\alpha\}_{\alpha < \lambda}$ of G-subspaces of βX, each containing X. It remains to show that their intersection is precisely X.

Let Δ and Δ' be the diagonals of $(\prod\{\beta A_\alpha \mid \alpha < \lambda\})$ and X^λ respectively. Since X is closed in $\prod A_\alpha$, $\Delta' = \Delta \cap \prod\{C_\alpha \mid \alpha < \lambda\}$. (It is clear that $\Delta' \subset \Delta \cap \prod C_\alpha$. If $p \in \Delta \cap \prod C_\alpha = \Delta \cap \prod \mathrm{cl}_{B_\alpha} X$,

then for each a < A, v_{α}(p) e B_{α}. Since p is a constant function, it follows that for each a, *ir* (p) e IIA <= B . But also
' a ^ a — a
7r (p) € C . Therefore 7T (p) e IIA D cl X. Since X is closed
CX CC OC & B
a λ
in *UA* , *ir* (p) e X and p e A' •) Since A is closed in (IlgA) ,
CX xJL CX
A^r is closed in lie . This is the crucial fact needed to complete the proof.
CX

Let D = (1{D (a < A}. For each *a* < A, let <*p* : D—» C be
QC CC CC
the restriction of the map pi : p x —> gC , and let <*p:D—>* IIC
CX CC CC
be their product. <*p\x* is a homeomorphism of X onto A^1 £ nc .
CX

Now A' is dense in *<p(D)*, since X is dense in D. But A^1 is closed, and hence A^1 = <*P(D)*. But <p(D \ X) c IIC \ *(p(X)* (fG- J], *p.* 92).
CC

Therefore D\ X = *0* and the proof of the Theorem is complete.

For the proof of the Corollary, note that 3X = 3(aX), since maps from X to a compact space (necessarily in G) can be extended in two steps to 3(aX). Thus by the Theorem, aX is the intersection of some such family. If A is any *G*-subset of pX containing X, the inclusion map X —^ A factors through the inclusion map X——> aX, via another inclusion map, since A and aX are both subsets of |3X. This proves the Corollary.

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FOOTNOTES

(1) i.e., completely regular T_1 spaces. All classes of spaces will be assumed to be Tychonoff spaces.

(2) A perfect map is a closed continuous function such that the inverse image of each point is compact. See (1) in the text for inverse preservation.

(3) The epi-reflective subcategories of $T2$ form a complete lattice. The epi-reflective hull of a class of Hausdorff spaces is the smallest epi-reflective subcategory containing it.

(4) If G is a class of spaces, the G-compact spaces are just those homeomorphic to a closed subspace of some product of spaces from G.

(5) The author is endebted to H. Herrlich for this observation.

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