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CHARACTERIZATION  
OF THE REAL LINE

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# ON THE TOPOLOGICAL CHARACTERIZATION OF THE REAL LINE

by S. P. Franklin and G. V, Krishnarao.

## Introduction.

In 1936 A\* J. Ward ([Wa]) characterized the real numbers topologically among the metric spaces. (In fact his proof seems valid for regular spaces.) He showed that every metric space that is separable, connected and locally connected, and in which each point is a strong cut point<sup>1</sup> is homeomorphic to  $3R$ . Recently V. B. Buch was led by work on continuously ordered spaces to ask if the same characterization holds for Hausdorff spaces. Here we answer that question negatively and go on to give a new (and very short) proof of Ward's Theorem (for regular spaces) based on recent theorems ([Wh]) of the late G. T. Whyburn, to whom this note is respectfully dedicated.

For the convenience of the reader we shall quote some special cases of the results of Whyburn that we use. Let  $X$  be a connected and locally connected Hausdorff space. A nonempty closed subset  $A$  of  $X$  is called an A-set if each component of its complement has precisely one boundary point. Each A-set is itself connected and locally connected and any nonempty intersection of A-sets is again an A-set. For any two points  $a, b \in X$ ,  $E(a, b) \cup \{a, b\}$  is compact, where  $E(a, b)$  is the set of points of  $X$  which separate  $a$  and  $b$ .

Addendum to the Introduction

(to be the last sentence)

This theorem is then combined with even more recent results of Rok and of Wattel to give a slightly different topological characterization of  $3R$  among the Hausdorff spaces.

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1. Suppose every point of  $X$  is a strong cut point. For every pair of distinct points  $a, b$  in  $X$  let  $C_a$  and  $C^a$  be the components of  $X - a$  and  $X - b$  containing  $b$  and  $a$  respectively, and  $B(a, b) = (C_a \cup C^a) \cup \{a, b\}$ .

LEMMA. Let  $X$  be a connected and locally connected separable regular space, in which every point is a strong cut point, every  $B(a, b)$  is an arc.

PROOF.  $B(a, b)$  is an  $A$ -set and hence is connected and locally connected. It is also a separable regular space with just two noncut points\*

Claim: Every point of  $C_a \cup C^a$  separates  $a$  and  $b$ . If not, let  $x \in C_a \cup C^a$  and let  $B(a, b) - x = A_x \cup C_{xx}$  be a separation with  $a$  and  $b$  in  $A_x$ . Then no point of  $C$  separates  $a$  and  $b$ .

Take  $A$  to be the intersection of all the  $A$ -sets,  $A_x \cup \{x\}$  with  $x$  in  $B(a, b)$  and not separating  $a$  and  $b$ .

$A$  is itself an  $A$ -set in  $B(a, b)$  and hence is connected and locally connected. Since  $B(a, b)$  is separable,  $B(a, b) - A$  has only a countable number of components each having a single boundary point in  $A$ . If  $M$  is the set of boundary points,  $A - (M \cup \{a, b\})$  is precisely the set of points of  $B(a, b)$  which separate  $a$  and  $b$ . Indeed, if  $x \in A - (a, b)$  does not separate  $a$  and  $b$ ,  $x$  would belong to  $M$ . But  $A - M = E(a, b) \cup \{a, b\}$  is a compact and therefore closed subspace of  $B(a, b)$  and therefore of  $A$ . Now  $M$  is an open countable locally connected regular subspace of  $A$  and hence  $([U])$  a discrete subspace

which contradicts the connectedness of  $A$ . Therefore  $B(a,b)$  consists precisely of  $a,b$  and the points which separate  $a$  and  $b$ .

Thus  $B(a,b)$  is a compact separable connected and regular space with just two non-cut points and hence is an arc (Theorem 11.15• in [Wi]).

THEOREM. A connected and locally connected separable and regular space, in which every point is a strong cut point is homeomorphic to the real line.

PROOF. The previous lemma and the separability of the space imply that it is the union of a countable number of arcs; we can find a subcollection of these arcs,  $\{I_m : m \text{ an integer}\}$  so that  $I_m$  and  $I_{m+1}$  have just one end point in common,

Remarks. The Lemma and the Theorem both hold (with the same proof) for functionally Hausdorff<sup>3</sup> spaces instead of regular spaces, since it is easy to see that a connected functionally Hausdorff space with more than one point must have at least the cardinality of the continuum. One may also derive the second countability of  $X$  from the proof of the Lemma and so deduce the Theorem from the metric case. However this does not work in the functionally Hausdorff case. As a third alternative, one can easily conclude from the Lemma that  $X$  is a 1-manifold. Since it is separable and is composed of only cut points, it is not compact and is therefore homeomorphic to  $2R$  ([P] Theorem 2). This also holds in the functionally Hausdorff case.

Addendum to Section 1

(to be the last paragraph)

Combining recent results of Kok [Ko] and Wattel [Wat], local connectedness can be replaced by either local compactness or rim compactness in this theorem. Indeed, Kok shows [Ko, Theorem 1] in a connected Hausdorff space, each point being a strong cut point is equivalent to  $(S^*)$  given three distinct points, some one separates the other two. Wattel [Wat, Theorem] shows that for a connected  $T_1$ -space satisfying  $(S^*)$ , local connectedness\* local compactness and rim-compactness are equivalent\*. Hence it follows from the theorem that a separable connected locally compact Hausdorff space in which each point is a strong cut point is homeomorphic to the real line. This characterises the line topologically among the Hausdorff spaces\*.

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2. The counterexample in the Hausdorff case is constructed from a countable, connected and locally connected Hausdorff space due to F. B. Jones which we describe briefly for the convenience of the reader. (Another such example was recently published by A. M. Kirch [K].) Let  $P$  be a countable collection of disjoint pairs of irrational numbers such that given any pair  $(a,b) \in \mathbb{Q}$  of rational numbers and any neighborhoods  $N_a$  and  $N_b$  in of  $a$  and  $b$  respectively, there is a pair  $(x,y) \in P$  with  $x \in N_a$  and  $y \in N_b$ . Let  $C = \mathbb{Q} \cup P$  with basic open sets of the form

$$(T \cap \mathbb{Q}) \cup \{p \in P \mid p \subset T\}$$

where  $T$  is either an open interval in  $\mathbb{R}$  or the union of two such.  $C$  then is a countable connected and locally connected Hausdorff space in which no point is a cut point.

For each  $x \in C$ , let  $I_x$  be a copy of the half-open interval  $[0,1)$ , with its zero denoted by  $0_x$ . Let  $W$  be the disjoint topological sum (coproduct) of these  $I_x$  and let

$A = \{C^x \mid x \in C\}$ . Since  $A$  is a discrete subset of  $W$ , the function

$f: A \rightarrow C$ , defined by  $f(0_x) = x$ , is continuous. The ad-

junction space  $W \cup A$  is the desired separable, connected and

~~locally connected Hausdorff space in which every point is a~~

~~strong cut point that is not homeomorphic to  $\mathbb{R}$  since it is~~

~~not regular.~~

Three further, and somewhat technical, remarks must be made. (1) A countable, connected and locally connected Hausdorff space in which each point is a strong cut point can be constructed by attaching to each point of  $C$  a copy of  $C$  (instead of  $I$ ) and repeating this process  $\omega$  times, taking the inductive limit topology on the union. (2) In fact, any countable, connected and locally connected Hausdorff space will serve to produce a space such as that in (1). First one takes its Shimrat homogeneous extension ([Sh]) which preserves all these properties. If no point is a cut point one proceeds as in (1). If each point is a cut point, unwanted components can be judiciously pruned to make each point a strong cut point. (3) There is a countable, connected and locally connected Urysohn space<sup>4</sup> due to F. B. Jones and A. H. Stone. Hence there are such spaces in which each point is a strong cut point, and counterexamples which are Urysohn spaces can be constructed.

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- 1# A strong cut point is one whose complement has precisely two components.
2. Including  $T_1$
3. A functionally Hausdorff space is one in which each pair of distinct points can be separated by a continuous real-valued function.
4. A Urysohn space is one in which each pair of distinct points have disjoint closed neighborhoods. Every regular space is a Urysohn space.

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Addendum to the References

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