LINEAR DIFFERENCE EQUATIONS: CLOSEDNESS OF COVARIANT SEQUENCES. APPENDIX.

by

Juan Jorge Schäffer

This report is an addendum to Report 69-34.

Report 70-7

University Libraries
Carnegie Mellon University
Pittsburgh PA 15213-3890

CARNEGIE-MELLON UNIVERSITY

5. Appendix: Other (b,d)-sequences

Suppose Y is a (b,d)-sequence for A, where (b,d) is an ^-pair such that supp (b) is infinite, but Y is not subcomplete. Is it still possible to infer the existence of a closed (b,d)-sequence "near" Y? An obvious candidate is cl Y, where (cl Y) (n) = cl(Y(n)), n e co; but even if we could show that cl Y is covariant (see Remark after Lemma 5.1), it might not be true that cl Y c X_{0d}. We therefore proceed indirectly, first constructing a subcomplete (b,d)-sequence with suitable properties.
The desirability of this investigation was suggested by results of G. Pecelli concerning dichotomies for functional-differential equations.

Let us recall [1; Lemma 6.3] for given deb/ : For each m € co the set $X_{d[m]}$ of d-solutions of $(I_{[m]})$ is a feubspace of $d_{[m]}(X)$ and the linear mapping II (m) : x -• x(m) : $X_{d[m]} \to X_{0d}(m)$ is jDounded and bijective.

Let Y be a covariant sequence, Y c X_{0d}. For each m G co we set $Y_a(m) = \Pi(m)(\mathrm{cl}_a(\Pi(m)^{-1}(Y(m)))) \subset \Pi(m)(X_{d[m]}) = X_{0d}(m)$, where cl_a indicates closure in $d_{[m]}(X)$ or, equivalently, in the subspace $X_{d[m]}$. Thus Y_a is subcomplete in X and, since II (m) is bounded and its range includes Y(ra)5

(5.1) Y(m) c $Y_d(m)$ a cl Y (m) m e ω.

We observe that the definition of Y_d may be rephrased as follows: u e $Y_d(m)$ if and only if, x being the solution of $(I_{[m]})$ with x(m) = u, there exists for every e > 0 a solution y of $(I_{[m]})$ in $Y_{[m]}$ such that x - y is a d-solution and | x - y |$_{\alpha}$ < 6.

5.1. Lemma, Let Y be a (b, d) - sequence for an /-pair (b^d) such that supp (b) is an infinite set (in particular, for a. ^-pair or a. £ - pair (£jd)) . Then Y_{d} is ^a subcomplete (b,d)-sequence.

Proof. It is sufficient to prove that Y_{d} is covariant: for then Y c Y, c X implies that Y-. is a (b^d) - sequence [1; Lemma 8.4^ (b)], and it is subcomplete by its definition. Let integers n n* n *>e given with n >= n O >= 0 . We are to prove that $(U(n,n_0))^{-1}(Y_d(n)) = Y_d(n_0)$.

Let $u \in Y_d(n_0)$ and $\epsilon > 0$ be given, and let x be the solution of $(I_{[n_0]})$ with $x(n_0) = u$; there exists a solution y of $(I_{[n_0]})$ in $Y_{[n_0]}$, such that $|x - y|_d < \epsilon$. But then $x_{[n]}, y_{[n]}$ are solutions of $(I_{[n]})$, $y_{[n]}$ is in $Y_{[n]}$, and $|x_{[n]} - y_{[n]}|_d \leq |x - y|_d < \epsilon$. Therefore $U(n,n_0)u = x(n) = x_{[n]}(n) \in Y_d(n)$, and hence $(U(n,n_0))^{-1}(Y_d(n)) \supset Y_d(n_0)$. It remains to prove the reverse inclusion.

We may choose $n_1 \in$ supp (b), $n_1 > n$. Since, by the preceding part of the proof applied to n, n_1, we have $(U(n,n_0))^{-1}(Y_d(n)) \subset (U(n,n_0))^{-1}(U(n_1,n))^{-1}(Y_d(n_1)) = (U(n_1,n_0))^{-1}(Y_d(n_1))$ it will be sufficient to prove that $(U(n_1,n_0))^{-1}(Y_d(n_1)) \subset Y_d(n_0)$.

Let $p > 1$ be given. Consider any $u \in (Ufn^n_0))^{-1}(Y_d(n_1))$ and any $\epsilon > 0$, and let x be the solution of $(I_{[n_0]})$ with $x(n_0) = u$. Thus $x(n_1) = UCn^^, n_0)x(n_0) \in Y_d(n_1)$. There exists, therefore, a solution y of $(I_{[n_1]})$ in $Y_{[n_1]}$ -- i.e., with $y(n_1) \in Y(n_1)$ --and such that $|x_{[n_1]} - y|_d < \epsilon$.

We define $f = (x(n_1) - y(n_1))\chi_{[1]} \ldots$ as we may, since $n_1 \in$ supp(b), and find

$$(5.2)\qquad \|f\|_{\underset{\sim}{b}} \leq \|x(n_1) - y(n_1)\| \, \|\chi^{n_1}\|_{\underset{\sim}{b}} \leq$$

$$\leq \|x_{[n_1]} - y\|_{\underset{\sim}{d}} \, \|\chi^{n_1}\|_{\underset{\sim}{d}}^{-1} \, \|\chi^{n_1}\|_{\underset{\sim}{b}} <$$

$$< \epsilon \, \|\chi^{n_1}\|_{\underset{\sim}{d}}^{-1} \, \|\chi^{n}\|_{\underset{\sim}{b}} .$$

Since Y is a $(\underset{\sim}{b},\underset{\sim}{d})$-sequence, there exists a solution z of (II) with this f such that z lies eventually in Y-- i.e., $z[n_1] \in Y[n_1]$ and

$$(5.3)\qquad \|z\|_{\underset{\sim}{d}} \leq \rho K_Y \|f\|_{\underset{\sim}{b}} .$$

Now $z_{[n_0]}$ is a solution of $(II_{[n_0]})$ with this f, and so is, obviously, $(x_{[n_1]} - y)_{[n_0]}$; also, x is a solution of $(I_{[n_0]})$. Therefore $v = z_{[n_0]} - (x_{[n_1]} - y)_{[n_0]} + x$ is a solution of $(I_{[n_0]})$; further, $v(n_1) = z(n_1) + y(n_1) \in Y(n_1)$, so that v lies in $Y_{[n_0]}$, and (5.2), (5.3) imply

$$\|x - v\|_{\underset{\sim}{d}} \leq \|x_{[n_1]} - y\|_{\underset{\sim}{d}} + \|z\|_{\underset{\sim}{d}} <$$

$$< \epsilon (1 + \rho K_Y \|\chi^{n_1}\|_{\underset{\sim}{d}}^{-1} \|\chi^{n_1}\|_{\underset{\sim}{b}}) .$$

Since $\epsilon > 0$ was arbitrary, it follows that $u = x(n_0) \in Y_d(n_0)$. Hence $(U(n_1,n_0))^{-1}(Y_d(n_1)) \subset Y_d(n_0)$, as was to be proved.

Remark. A slightly simplified version of the proof will show that the sequence cl Y is also covariant.

5.2. *Theorem*. *Let* Y *be a* (b,d)*-sequence for an* ℓ*-pair* (b,d) *such that* supp (b) *is an infinite set (in particular, for a* ℓ*-pair or a* t*-pair* (k,d)). *Then*:

(a); *if* $Y_d(m)$ *is closed (in particular, if* Y(m) *is closed) for some* $m \in \omega$, *then* cl Y = Y_d *is a closed* (b,d)*-sequence*;

(b); if $U(n_1,n_0)$ is compact for some $n_1, n_0 \in \omega$, $n_1 \geq n_0 \geq 0$, *then* cl Y = Y_d *is a closed* (b,d)*-sequence and its terms have constant finite co-dimension in* X.

Proof. Lemma 5.1, formula (5.1), and Theorem 4.1.