ON OPEN EXTENSIONS OF MAPS

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§0. INTRODUCTION

In recent years there has been some interest in trying
to improve the behavior of maps by extending their domains.
For example, in 1953 Whyburn showed that every map is the
restriction of a compact map [7]. Similarly Krovelec
proved in 1967 that each locally perfect map can be extended
to a perfect map [12] and in an as yet unpublished paper,
Dick man obtained the same result for arbitrary maps [4].
Here we show that every map can be extended to an open map
so that certain properties of the domain and range are pre-
served in the new domain. These results are then used to
obtain analogues and improvements of recent theorems of
Arhangelskii, Coban, Hodel, and Proizvolov.

§1. OPEN EXTENSIONS

Let $f : X \text{ "•} Y$ be a function, not necessarily con-
tinuous, from a topological space X into a topological
space Y. We shall call a point $X \in X$ and its image

$f(x) \in Y$ singular points of X and Y respectively, if there is an opsn set U of X containing x whose image $f(U)$ is not a neighbourhood of $f(x)$. The function f is open just in case there are no singular points of X (or equivalent of Y)•

For each singular point x of X, let Y_x be a copy of Y. Let $W = X \odot (\odot Y^x)$, where the second disjoint topological sum is taken over all singular points of *X.* By identifying each singular point $X \in X$ (with X thought of as a subset of W) with its image $f(x)$ (as a point of $Y_x \subset W$) we arrive at a quotient space X* of W, The inclusion map $i : X \to W$ composes with the quotient map $q : W \to X^*$ to give an imbedding of X into X*. Hence we may think of X* as an ^ctension of X.

Let $f, : W \to Y$ be the function whose restriction to X is f and whose restriction to each Y_x is the identity map $1 : Y \to Y$. We leave the reader to verify that the unique function $f^* : X^* \to Y$ satisfying $f^* \circ q = f-$ is an open extension of f. Recapitulating, we have that

1.1 There is an overspace X* of X and an open function f* : X* → Y whose restriction to X is. f; f* is^ continuous just in case f is.

Since only sums and quotients were used in the construction of X* it follows at once that

1.2 <u>Any</u> <u>coreflexive</u> <u>property</u>^**1**'<u>of</u> X <u>and</u> Y <u>is</u> <u>preserved</u> <u>in</u> X*.

In particular, if X and Y are countably, sequentially or
cpmpactly generated^(.2)so is X*, If X and Y are locally
connected, or are P-spaces, or are chain net spaces*^**3**'so is X*.

It is routine to verify that

1.3 X* satisfies the separation <u>axioms</u> T , T,, T$_o$, T$_o$, and
‾ ‾ ‾ ‾ ‾ ‾ ‾ * ‾ _ _   -   O . JL   z   J -
T$_3$ **₁**/$_2$ w̶h̶e̶n̶e̶v̶e̶r̶ X a̶n̶d̶ Y d̶o̶.

We shall prove only the last case, that of complete regularity.
Suppose F is a closed subset of X* and p^F. If q"^{1}(p) H X = 0
(where q : W -• X* is the quotient map), then q"^{1} (p) €Y for some x.
Then there is a real valued function * on Y__ which is zero
at q^{-1}(p) and one at x•* f (x) eY and on q"^{1} (F) H Y . Extend *
continuously to all of W by taking it constantly one on X and
on each Y$_{x}$,, x^1 / x. This extended $ defines a real valued
function on X* which separates p and F. In the other case,
if $^x{_o}$eq"^{1}(p) n x, let *$_Q$: X - R be zero at X$_Q$ and one on
q·"^{1} (F) fl X. For each singular x, choose *$_x$: Y -• R which is
one on q"^{1} (F) 0 Y and such that *$_x$ (f($_x$)) = *$_o$ (x) . These
functions combine to form one $: W -* R which in turn induces
a real valued function on X* separating p and F.

X* can also be realized as an adjunction space. Let F be
the closed discrete subset of © Y whose intersection with each Y$_x$
is its singular point f (x) . The map ^{x}g :^{g}P -• X which sends each f (x)
to x, yields the adjunction space ® Y U X which is homeomorphic
to X*. Using this representation we see that

1.4 X* Is. normal, hereditarily normal, perfectly normal,
collectionwise normal, or fully normal (i.e. paracompact)
whenever X and Y are.

The first three of these properties are preserved under sums
aid adjunctions. For the other three, the assertion follows from
a theorem of Tsuda [16].

If x is a singular point of X, then f(x) ∈Y. and
q(x) = q(f(x)). If X and Y are (pathwise) connected, so are
their continuous images q(X) and q(Y.). Since each q(X) 0 q(Y.) / 0,
                                                          x   x
U(q(x)Uq(Yᵣ)) is (pathwise) connected. Thus we have that

1.5 X* is^ (pathwise) connected whenever X and Y are.

Let X be the plane set consisting of the union of the
closed intervals [-1, 1] on the two axes. Let Y = [-1, 1] on one
axis and let f be the restriction to X of the projection onto
this same axis. Each point on one axis, except the origin is a
singular point of X. This example shows that

1.6 X* need not preserve metrizability, either axiom of counta-
bility, weight or local weight, separability or density, the
Lindelöf property, or (countable, sequential, pseudo) compactness.

(Another open extension of a map f can be given whose domain
will preserve many of these properties. Since the family $\{1_v, f\}$
separates points and also separates points from closed sets, the
evaluation map e : X - X x Y (given by e(x) = (x, f(x))) is an

embedding. Hence X is homeomorphic to e(X), the graph of £,
and the projection ir_y : X x Y "* Y restricted to e(X) is
essentially f. Thus each map is the restriction of a projection
map. If X is compact, f is the restriction of ja clopen map.
This is^ also the case if X is^ countably compact and Y a subspace
of a^ sequential space [5J. Clearly any finitely productive property
of X and Y is preserved in the domain of the projection. Hence
most of the properties mentioned in 1.6 are preserved.)

By imposing restrictions on the set S of singular points
of X, X* may be induced to preserve many other properties.
For example

1.7 If the singular points of X do not accumulate, metrizability,
local compactness and local weight are preserved in X*. Further
if the cardinality of S i^ not more than the larger of the
weights of X and Y, then neither is the weight of X*.

(Here we assume that X and Y are T_1 spaces.)
We first show that the quotient map q : W-* X* is a closed
mapping under the given hypothesis. Suppose F is a closed
subset of W and {pg} is a net in q(F) converging to a point p
in X*. Now q(F) = q(F fi X) U (U q(P fl Y)) and the restriction
                                    x
of q to X and to each Y is an embedding. Since q(X) and
each q(Y) is closed in X* we conclude that q(F 0 x) and
each q(F H Y_v) is closed in X* also. Thus if (p*,} is fre-
quently in q(F H x) or_1 in some q(F H Y), p must belong to q(F)
                                              x
and we are done. If q" (p) = {y} with yeY , let U = Y
                                              x   x.

If $q^{-1}(p) = \{x\}$ where x is a non-singular point of X, let U

be a neighborhood of x in X which is free of singular points.

If $q^{-1}(p) = (x, f(x))$ let U be the union of Y_x and a neigh-

borhood of x in X which is free of singular points of X other

than x. In any case, q(U) is a neighborhood of p which has a

non-empty intersection with at most one q(Y fl F) . But $\{p^\wedge_a\}$ is

eventually in q(U) and hence is frequently in either q(U) fl q(F fl X)

or q(U) fl q(F D Y_x) . Hence q is a closed map. Since each $q^{-1}(p)$

is at most a doubleton, q is a perfect map. Since metrizability

and local compactness are preserved under sums and perfect maps,

the first two assertions of 1.7 are proved.

Suppose m is the larger of the local weights of X and Y.

If $q^{-1}(p) = fy)$ where yeY_x $\{f(x)\}$, the image of a base at y

under q is a base at p. If $q^{-1}(p) = \{x\}$ where x is a non-

singular point of X, a base can be chosen at x whose members

contain no singular points. The image of this

base under q is a base at p. If $q^{-1}(p) = \{x, f(x)\}$, choose a

base IB at x whose members contain only one singular point, and

choose a base If at $f(x)eY$, with the cardinality of U and IB

no larger than m. The images under q of sets of the form B U V

with BelB and Vel/, form a base at p of cardinality no larger

than m.

For what remains we need only note that weight w
= weight X + card S. weight Y $<^\wedge$ m + m^2 = m and that perfect maps
do not increase weight.

Example 1.6 shows that the restrictions imposed on the set S of singular points in 1.7 are not superfluous. By replacing one of the intervals [-1, 1] in 1.6by a sequence converging to zero, one can easily see that these cannot be weakened even to S being a countable discrete subset.

§2. FINITE-TO-ONE MAPPINGS

In 1966, Proizvolov [15] showed that weight and metrizability are inversely preserved in locally compact spaces under open finite-to-one maps. Later that year, Arhangelskii [1] [2] showed they were always inversely preserved under clopen finite-to-one maps. In 1967, Coban [3] proved that hereditary paracompactness (metacompactness, Lindelof) are inversely preserved under open finite-to-one maps. (Some separation axioms are required for all these results.)

If f is a finite-to-one mapping and the set S of singular points of X is finite (i.e. f is open except at finitely many points) then f* : X* -» Y is an open, finite-to-one map and hence X* will inherit properties from Y by the results quoted in the paragraph above. (1.3 shows that the needed separation axioms (see below) also lift properly.) Since these properties are all hereditary, X must also enjoy them. Thus we see that it is sufficient to require that f be open except at finitely many points to arrive at the desired conclusion. For the, convenience of the reader ye list precise statements of the improved theorems. (Assume all spaces to be Hausdorff, and that f is continuous and onto.)

2.1 (Proizvolov) If X and Y are locally compact, and f is finite-to-one and open except at finitely many points, then weight X £ weight Y. £f_ Y Is metrizable, so is X. In the proof, 1.7 as well as Proizvolov's original theorem must be used.

2.2 (Arhangelskiĭ) £f. X and Y are completely regular and f is a, finite-to-one closed map which is open except at finitely many points, then weight X < 'weight Y. Ij: Y is. metrizable, so is X.

Here, in addition to 1.3 we need to only note that with S finite, f* is closed iff f is closed.

2.3 (Coban) Ij: f jj3 finite-to-one and open except at finitely many points, then X ^s hereditarily paracompact (metacompact. Lindelöf) whenever Y JLS. (For paracompactness X is required to be regular.)

Simple examples can be given to show that the conditions on the singular points of f cannot simply be omitted in 2.1, 2.2, and 2.3. First let X = fcty Y = {0} U {1/n|ne»} <_ B and f : X -* Y arise from n - 1/n. Y is locally compact, second countable and metrizable. X is locally compact, has weight c and is not metrizable, even though f is a perfect map. This covers 2.1 and 2.2.

For 2.3 let Y be as before and let X^f be the ordinal com-pactification of U recently constructed by Franklin and Rajagopalan [7], i.e. X' = IT U u^ + 1 with UN embedded as an open dense subspace, u^ + 1 embedded as a closed subspace, IT fl a₁ + 1 = 0

and X^1 compact Hausdorff. Let $X = X^! \setminus [^\wedge\}$ and define $f : X \sim> Y$ by $f(n) = 1/n$ and $f(x) = 0$ otherwise. Y is hereditarily Lindelöf (and much more) while X fails to be metacompact.

§3. DIMENSION

(in this section all spaces are assumed to be metric, and f is continuous and onto. In 1963 Hodel [11] showed that dimension cannot be lowered by open maps f such that each $f"^1(y)$ is discrete. The technique of the last section can be used to improve this result also.

3.1 (Hodel) iE the singular points of X do not accumulate and if each $f"^1(y)$ JLS. discrete, then dim $X \leq^\wedge$ dim Y.

For the proof we use 1.7, Hodel's original theorem and that X is a closed subspace of $X*$.
Hodel's theorem (in both the original and the improved version) holds true for not necessarily continuous f if Y is taken to be locally compact and separable.

To show that some hypothesis is needed on f, one need only look at Paeno's map of the interval onto the square.

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<u>Footnotes</u>

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(1)   See Herrlich [10] for the definition and properties of coreflective subcategories.

(2)   A countably generated space is one ^{Tf} determined by countable subsets[11] in the sense of Moore and Mrowka [14]. Sequentially generated spaces are the sequential spaces [6] and compactly generated spaces the k-spaces.

(3)   For the coreflexivity of local connectedness, see Gleason [9], A P-space is one in which each G_δ is open (see Gilman and Jerison [8]). A chain net is one whose underlying directed set is a chain. Chain nets are said to suffice if any set containing the limits of all its convergent chain nets is closed (see Misra [13]).