EXPONENTIAL STABILITY FOR A NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATION

by

R. C. MacCamy* and J. S. W. Wong

Report 70-11

April, 1970

* This author was supported by the National Science Foundation under Grant GP 11945.

1. Introduction.

This paper concerns the nonlinear functional differential equation,

$$\text{(E)} \qquad \dot{y}(t) = -\int_0^t a(t-\tau)g(y(\tau))d\tau + f(t).$$

This equation has been the subject of much study, mainly by Levin and Nohel (see for example [4]) and Hannsgen [2]. The object has been to obtain sufficient conditions for asymptotic stability. The best result in this direction appears in [2] and is contained in the following theorem.

Theorem I. *Suppose the following conditions are satisfied:*

(A_1) $\quad (-1)^k a^{(k)}(t) \geq 0$, $k = 0,1,2$, $a(t) \not\equiv$ *constant,*

(A_2) $\quad \xi g(\xi) > 0$ *for* $\xi \neq 0$; $\int_0^{\pm\infty} g(\xi)d\xi = \infty$.

(A_3) $\quad f \in L_1(0,\infty)$ *and either,*

(i) $|\dot{f}(t)| \leq M$ *for all* t

or (ii) $a \in L_1(0,\infty)$ *and* $|f(t)| \leq M$ *for all* t.

Then all solutions of (E) *tend to zero as* t *tends to infinity.*

Here we consider the rates of decay of solutions and in particular we give conditions for exponential decay. The basic result is as follows.

Theorem II. *Suppose* (A_1)-(A_2) *hold and in addition that:*

(H_1) $\quad a(t)-a(\infty) \leq Me^{-\alpha t}$, *for some positive constants* M *and* α, *where* $a(\infty)$ *denotes the limit of* $a(t)$ *at infinity,*

(H_2) $|f(t)| \leq Ne^{-\beta t}$, <u>for some positive constants</u> N <u>and</u> β,

(H_3) <u>If</u> $a(\infty) > 0$, <u>then</u> $|\dot{f}(t)| \leq B$ <u>for some positive constant</u> B,

<u>and</u>

(H_4) g <u>is differentiable in some neighborhood of origin and</u> $g'(0) > 0$.

<u>Then there exists a constant</u> $\gamma \leq \alpha$ <u>such that</u>:

(i) <u>If</u> $\beta > \gamma$, <u>every solution of</u> (1) <u>satisfies</u> $y(t) = O(e^{-\gamma' t})$ <u>as</u> $t \to \infty$, <u>for any</u> $\gamma' < \gamma$;

(ii) <u>If</u> $\beta < \gamma$, <u>every solution of</u> (1) <u>satisfies</u> $y(t) = O(e^{-\gamma' t})$ <u>as</u> $t \to \infty$, <u>for any</u> $\beta' < \beta$.

Note that the hypotheses of Theorem II imply (A_3).

<u>Remarks</u> (1) If $\beta = \gamma$ one must give more precise information about the behavior of f for large t and we omit the discussion of this case.

(2) As we indicate in section (2), the conditions (H_1) and (H_2) are essentially necessary in order to have exponential stability.

(3) Both the results and techniques here are similar to those contained in [5]. In [5] more general nonlinearities were admitted but the results were only local. Applied to the present situation they would assert that Theorem (II) holds if $|y(0)|$ is sufficiently small. We emphasize that Theorem (II) is, in contrast, a global result.

(4) The ideas of section (2) are related to those of Halanay [2]. The proofs in [2] are incorrect but have been recently clarified by the authors [6].

2. The Linearized Equation.

We use a kind of perturbation technique. We rewrite (E) in the form,

$$\dot{y}(t) = -\int_0^t a(t-\tau)g'(0)y(\tau)d\tau + F(t), \tag{2.1}$$

where,

$$F(t) = f(t) - \int_0^t a(t-\tau)(g(y(\tau)) - g'(0)y(\tau))d\tau. \tag{2.2}$$

From Theorem (I) we know that $y(t)$ tends to zero. Thus we can consider (E) as a perturbation on the linear equation (2.1). The present section is devoted to a study of (2.1) which is of some interest in itself. We prove the following result.

Lemma: Suppose (A_1), (H_1) and (H_4) hold. Then any solution of (2.1) can be written in the form:

$$y(t) = r(t)y(0) + \int_0^t r(t-\tau)F(\tau)d\tau \tag{2.3}$$

where the function $r(t)$* satisfies,

$$|r(t)| \leq \rho e^{-\gamma t} \text{ for some constants } \rho \text{ and } \gamma,\ 0 < \gamma < \alpha. \tag{2.4}$$

Moreover, if $a(\infty) \neq 0$ then $r(t)$ satisfies,

$$\left|\int_0^t r(\tau)d\tau\right| \leq \rho' e^{-\gamma t}, \tag{2.5}$$

where ρ' is some positive constant.

---

* The function r is referred to as the resolvent of equation (2.1).

Proof: We proceed formally by taking the Laplace transform of (2.1). This yields,

$$(2.6) \qquad s\hat{y}(s) - y(0) = -\hat{a}(s)g'(0)\hat{y}(s) + \hat{F}(s),$$

where $\hat{y}$, $\hat{a}$ and $\hat{F}$ denote the transforms of y, a and F respectively.

The estimate (H_1) yields,

$$(2.7) \qquad \hat{a}(s) = \frac{a(\infty)}{s} + \hat{b}(s),$$

where $\hat{b}$ is analytic in $\Re e\, s > -\alpha$. Thus we obtain, from (2.6),

$$(2.8) \qquad \hat{y}(s) = \hat{r}(s)\ (y(0) + \hat{F}(s)),$$

where

$$(2.9) \qquad \hat{r}(s) = \frac{1}{s + \hat{a}(s)g'(0)} = \frac{1}{s + g'(0)\left(\frac{a(\infty)}{s} + \hat{b}(s)\right)}$$

One obtains (2.3), formally, from (2.9) and the convolution theorem; r will be the inverse transform of $\hat{r}$. To prove estimates (2.4) and (2.5), we must study carefully the function $\hat{r}$. It is clearly analytic in $\Re e\, s > -\alpha$ except at the zeroes of the denominator which are poles. We claim, in fact, that $\hat{r}$ is analytic in $\Re e\, s \geq -\gamma$ for some $\gamma > 0$. This is proved in [5] and we review, briefly, the argument. It can be shown by a direct calculation that (A_1) implies the inequality,

$$(2.10) \qquad \Re e\, \hat{a}(s) > 0 \quad \text{on} \quad \Re e\, s = 0.$$

It follows that $s + g'(0)\hat{a}(s)$ cannot have zeroes <u>on</u> the imaginary axis. It can be verified, by integration by parts, that

$$(2.11) \qquad \hat{a}(s) = \frac{a(0)}{s} + \frac{a'(0)}{s^2} + o\left(\frac{1}{s^2}\right) \quad \text{as} \quad |s| \to \infty .$$

This equation shows that $s + g'(0)\hat{a}(s)$ cannot have zeroes of arbitrarily large absolute value. Thus the claim is verified.

Once the properties of $\hat{r}$ are known one can apply the inversion formula for Laplace transforms to obtain r. It is a straightforward matter to verify that (2.3) is indeed a solution of (2.1) and a standard argument shows that this solution is unique. The estimate (2.4) follows from the inversion formula and the fact that $\hat{r}$ is analytic in $\Re e\, s \geq -\gamma$.

The transform of the integral of $\hat{r}$ is given by $\hat{r}(s)/s$. When $a(\infty) \neq 0$ (2.9) shows that this function has a removable singularity at $s = 0$ and otherwise shares the analyticity properties of $\hat{r}$. Thus (2.5) holds when $a(\infty) \neq 0$. This completes the proof of the lemma.

<u>Remark</u>: The estimates (H_1) and (H_2) are in some sense necessary for exponential stability. Suppose we have a solution y of the linear equation (2.2) such that $y(t) = O(e^{-\beta t})$. Then $\hat{y}(s)$ is analytic in $\Re e\, s \geq -\beta$ and hence the right side of (2.8) must be also. But unless both a and F decay exponentially it cannot be that $\hat{r}$ and $\hat{F}$ are analytic in such a region.

3. Proof of Theorem II.

Theorem (I) and (H_4) imply that given any $\epsilon > 0$, there exists a $T > 0$ such that,

$$|g(y(\tau)) - g'(0)y(\tau)| \leq \epsilon \, |y(\tau)| \quad \text{in} \quad \tau \geq T. \tag{3.1}$$

We write, for $T > 0$,

$$F(t) = F_T(t) + G_T(t) \tag{3.2}$$

where,

$$\begin{aligned} F_T(t) &= \int_0^T a(t-\tau)\,(g(y(\tau)) - g'(0)y(\tau))\,d\tau \\ G_T(t) &= \int_T^\infty a(t-\tau)\,(g(y(\tau)) - g'(0)y(\tau))\,d\tau . \end{aligned} \tag{3.3}$$

We observe that (2.3) and (2.4) yield

$$|y(t)| \leq Y(t) + \rho e^{-\gamma t} \int_0^t e^{\gamma\tau} [F_T(\tau) + G_T(\tau)]\,d\tau, \tag{3.4}$$

where

$$Y(t) = \rho e^{-\gamma t}(|y(0)| + \int_0^t e^{\gamma\tau} |f(\tau)| d\tau) . \tag{3.5}$$

From (H_2) we have,

$$\begin{aligned} Y(t) &\leq K e^{-\gamma t} \quad \text{if} \quad \beta > \gamma \\ Y(t) &\leq K e^{-\beta t} \quad \text{if} \quad \beta < \gamma . \end{aligned} \tag{3.6}$$

Now we split the proof into two cases.

Case (1) $a(\infty) = 0$.

In this case (H_1) shows that $|F_T(\tau)| \leq K' e^{-\alpha\tau}$. Then (3.5) yields, if $\beta > \gamma$,

$$(3.7) \qquad |y(t)| \leq K'' e^{-\gamma t} + \rho e^{-\gamma t} \int_0^t e^{\gamma\tau} G_T(\tau)\, d\tau .$$

With $\epsilon > 0$ given, choose T so that (3.1) holds. Then for $t > T$ we obtain,

$$(3.8) \qquad |y(t)| \leq K'' e^{-\gamma t} + \rho\epsilon M e^{-\gamma t} \int_0^t e^{\gamma\tau} \left(\int_T^\tau e^{-\alpha(\tau-\mu)} |y(\mu)|\, d\mu\right) d\tau$$

$$\leq K'' e^{-\gamma t} + \rho\epsilon M e^{-\gamma t} \int_0^t |y(\mu)| \left(\int_\mu^t e^{\gamma\tau} e^{-\alpha(\tau-\mu)}\, d\tau\right) d\mu$$

$$\leq K'' e^{-\gamma t} + \frac{\rho\epsilon M}{\alpha-\gamma} e^{-\gamma t} \int_0^t |y(\mu)| e^{\gamma\mu}\, d\mu .$$

Set $z(t) = e^{\gamma t}|y(t)|$ so that (3.8) yields,

$$(3.9) \qquad z(t) \leq K'' + \nu \int_0^t z(\mu)\, d\mu \qquad \nu = \frac{\rho\epsilon M}{\alpha-\gamma} .$$

Since ϵ is arbitrary we deduce that

$$z(t) = O(e^{mt})$$

for arbitrary $m > 0$ and hence

$$y(t) = O(e^{-\gamma' t}) \quad \text{for any } \gamma' < \gamma .$$

If $\beta < \gamma$, similar calculations yield

$$y(t) = O(e^{-\beta' t}) \quad \text{for any } \beta' < \beta .$$

Case (2) $a(\infty) \neq 0$.

The proof proceeds in almost the same way. The only difference is that we no longer have the estimate $|F_T(\tau)| \leq K'e^{-\alpha\tau}$. Thus we need a preliminary step. We have, by (H_1),

$$F(\tau) = a(\infty)\int_0^{\tau}(g(y(\mu))-g'(0)\ y(\mu))d\mu + \tilde{F}(\tau)$$

where $\tilde{F}(\tau)$ is just like the term treated in case (1). For the first term we have,

$$a(\infty)\int_0^t r(t-\tau)\int_0^{\tau}(g(y(\mu))-g'(0))y(\mu))d\mu d\tau = a(\infty)\int_0^t r(\tau)\int_0^{t-\tau}(g(y(\mu))-g'(0)y(\mu))d\mu d\tau$$

$$= a(\infty)\Big\{\Big(\int_0^{\xi} r(\xi)d\xi\Big)\int_0^{t-\tau}(g(y(\mu))-g'(0)y(\mu))d\mu\Big|_0^T$$

$$+\int_0^t\Big(\int_0^{\tau} r(\xi)d\xi\Big)(g(y(t-\tau))-g'(0)y(t-\tau))d\tau\Big\}$$

$$= a(\infty)\int_0^t\Big(\int_0^{t-\tau} r(\xi)d\xi\Big)(g(y(\tau))-g'(0)y(\tau))d\tau\ .$$

By (2.5) of the lemma we have,

$$\Big|\int_0^{t-\tau}(r(\xi)d\xi\Big| \leq \rho' e^{-\gamma(t-\tau)}$$

and now we can proceed as in the preceding case.

Remark. The assumption (H_4) was dictated the method of proof. This linearization process is used in order to obtain a global result. Without (H_4), one can still obtain a local result, see [5]. On the other hand, without exponential decay of the resolvent $r(t)$, linearization still yields a local result for asymptotic stability, in fact for more general equations, see [7].

4. An Application to Control Theory.

Consider the autonomous system of $n + 1$ ordinary differential equations,

$$(4.1) \qquad \begin{cases} \dot{x} = Ax - \alpha g(y), \\ \dot{y} = \beta x - \gamma g(y), \end{cases}$$

where x, α are n-dimensional column vectors, A is an $n \times n$ matrix, β is an n-dimensional row vector, and y, γ are scalars. We solve the first n equations of (4.1) by the variation of constants formula and substitute into the last equation of (4.1). This yields,

$$(4.2) \qquad \dot{y}(t) = -\gamma g(y(t)) - \int_0^t \beta e^{A(t-\tau)} \alpha g(y(\tau))\, d\tau + \beta e^{At} x(0).$$

The problem of absolute stability in the theory of nonlinear controls [3] is to determine conditions on A, α, β, γ and g so that all solutions $y(t)$ of (4.2) tend to zero as t tends to infinity. In the special case of direct control, that is when $\gamma = 0$, equation (4.2) is of the form (E) and the present result concerning exponential stability becomes applicable. It is usually assumed that g satisfies (A_2) and A is a stable matrix. Suppose further that $\beta = \alpha^T$, the transpose of α. Finally assume that g satisfies (H_5) and that A is symmetric. Then one can show that $a(t-\tau) = \alpha^T e^{A(t-\tau)} \alpha$ satisfies (H_1) and (H_2). Hence it follows from our theorem that the control system (4.1) is not only absolutely stable but exponentially stable.

References

[1] Halanay, A., "On the Asymptotic Behavior of the Solutions of an Integro-Differential Equation", Journ. of Math. Anal. and Appl. Vol. 10, No. 2, 1965, pp.319-324.

[2] Hannsgen, K. B.,"On a Nonlinear Volterra Equation", Michigan Math. Journ. Vol. 12, 1965, pp.431-447.

[3] Lefschetz, S., Stability of Nonlinear Control Systems, Academic Press, New York, 1965.

[4] Levin, J. J. and J. A. Nohel, "Perturbations of a Nonlinear Volterra Equation", Mich. Math. Journ. Vol. 12, 1965, pp. 431-447

[5] MacCamy, R. C., "Exponential Stability for a Class of Functional Differential Equations", Tech. Rep. 69-17, Dept. of Math., Carnegie-Mellon Univ., March 1969.

[6] MacCamy, R. C. and J. S. W. Wong, "Stability Theorems for Some Functional Differential Equations", Tech. Rep. 70-10, Dept. of Mathematics, Carnegie-Mellon Univ., March 1970.

[7] Miller, R. K., J. A. Nohel and J. S. W. Wong, "Perturbations of Volterra Integral Equations", Journ. of Math. Anal. and Appl. Vol. 25, 1969, pp.676-691.

Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213