SOME EXAMPLES IN TOPOLOGY

by

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Section one is concerned with variations on the theme of an ordinal compactification of the integers. Several applications are found, yielding for instance, an example previously known only modulo the continuum hypothesis, and a counterexample to a published assertion.

Section two is concerned with zero-one sequences and section three with spaces built from sequential fans. Of two old problems of Čech, one is solved and one partly solved.

Since the sections are more or less independent, each will have its own introduction. Sequential spaces form the connecting thread, although not all the examples are concerned with them.

§1. Spaces such as $\beta\mathbb{N}$ and $I^{\aleph_1}$ provide ready examples of separable compact Hausdorff spaces which are not sequential[(1)]. But these are of "large" cardinality, i.e. $2^{\mathfrak{c}}$. The space $\omega_1 + 1$ with the order topology is a non-sequential, compact Hausdorff space of "small" cardinality, i.e. $\aleph_1$, but, unfortunately, it isn't separable. This leads one naturally to ask if there is a non-sequential, but separable, compact Hausdorff space of small cardinality. Such a space can be produced simply by conjoining known theorems as follows.

Magill ([M] Theorem 2.1) showed that if any Hausdorff space K is the continuous image of $\beta X \setminus X$, with X locally compact Hausdorff, then there is a compactification γX of X with $\gamma X \setminus X$ homeomorphic to K. Parovicenko ([P] Theorem 1) proved that every compact Hausdorff space of weight $\leq \aleph_1$ is the continuous image of $\beta\mathbb{N} \setminus N$. From these results one obtains

EXAMPLE 1.1. <u>There is a compactification $\gamma\mathbb{N}$ of $\mathbb{N}$ with $\gamma\mathbb{N} \setminus \mathbb{N}$ homeomorphic to $\omega_1 + 1$, and hence there is a non-sequential, but separable, compact Hausdorff space of cardinality $\aleph_1$.</u>

By providing a specific construction of the space $\gamma\mathbb{N}$, which is done below, we can assure (modulo the continuum hypothesis (CH)) that no sequence in $\mathbb{N}$ converges to $\omega_1 \in \gamma\mathbb{N}$. Then by modifying the topology of $\gamma\mathbb{N}$ at the point ω_1, we get

EXAMPLE 1.2. (CH) <u>There is a sequentially compact, Hausdorff c-space</u>[(2)] <u>which isn't sequential</u>. This corrects a mistake of the first author, showing that the Proposition in $[F_4]$ is false. This space, however, is not regular. The existence of such an example which is regular is still an open question.

Several fairly complicated examples have been given of separable and normal but non-paracompact spaces (see [R_1], [McA]). Having $\gamma\mathbb{N}$ in hand, the simple expedient of omitting the point ω_1 from $\gamma\mathbb{N}$ yields

EXAMPLE 1.3. $\gamma\mathbb{N}\setminus\{\omega_1\}$ is a first countable, locally compact space of cardinality $\aleph_1$ which is normal and separable, but not metacompact (and hence not paracompact). CH isn't needed here. Example 1.3 is similar to that of [R_1].

In response to a question of E. Michael, Mrs. Rudin ([R_2]1) constructed, modulo CH, a normal, sequentially compact, but non-compact space with a separable, metric, locally compact, dense subset. Again modulo CH, we can construct such a space with even nicer properties (i.e. first countability, local compactness) as well as a simpler proof.

EXAMPLE 1.4. (CH) $\gamma\mathbb{N}$ can be constructed so that $\gamma\mathbb{N}\setminus\{\omega_1\}$ is a first countable, zero-dimensional, locally compact, normal, sequentially compact, but not compact space with a countable, discrete, dense subspace.

W. W. Comfort asked, in a private communication, whether or not a separable, sequentially compact, but non-compact space can be constructed without appeal to the continuum hypothesis. We can now answer this affirmatively.

EXAMPLE 1.5. There is a separable, sequentially compact, locally compact normal, space which isn't compact. The construction does not depend on CH. However, first countability is lost. This raises the new

QUESTION: Can a first countable separable, sequentially compact, but not compact space be produced without appeal to CH?

For any space X, let sX be its sequential coreflection, i.e. the same set with the sequentially open sets as topology (see [F_3] proof of 5.2). This leads us to

EXAMPLE 1.6. $s\gamma\mathbb{N}$ is a separable, sequentially compact Hausdorff space which isn't compact.

The continuum hypothesis is not required. However, without it we cannot be sure the space is regular, i.e. that the point ω_1 can be separated from the now closed set ω_1. Hence we feel that Example 1.6 is not a satisfactory answer to Comfort's question. With CH, the point ω_1 can be made isolated in $s\gamma\mathbb{N}$ so that $s\gamma\mathbb{N}$ is another space with all the properties of Example 1.4.

With a little care, it can be insured that $s\gamma\mathbb{N}$ isn't regular yielding

EXAMPLE 1.7. There is a compact Hausdorff space whose sequential coreflection isn't even regular. This shows that in general one can expect little preservation of properties under topological coreflections.

Zenor [Z] has introduced a property between countable paracompactness and paracompactness, and has shown that together with

Hausdorffness it implies regularity. He gives an example to show that a countably paracompact Hausdorff space need not be regular. Example 1.7 shows that this may also be the case even if the space is separable and sequentially compact.

CONSTRUCTIONS.

We first define recursively a family $\{U_\alpha\}_{\alpha < \omega_1}$ of non-empty clopen subsets of $\beta\mathbb{N} \setminus \mathbb{N}$ such that $U_\alpha \subsetneqq U_\beta$ whenever $\alpha < \beta$. Let U_0 be any proper non-empty clopen set, and having found suitable U_α for each $\alpha < \gamma$, let $F_\alpha = (\beta\mathbb{N} \setminus \mathbb{N}) \setminus U_\alpha$. Since any G_δ formed from non-empty open sets in $\beta\mathbb{N} \setminus \mathbb{N}$ has a non-empty interior ([G-J] 65.8 p.99), let A be a non-empty clopen subset of $\cap \{F_\alpha | \alpha < \gamma\}$ and write $A = B \cup C$ with each of B and C non-empty and clopen. Let $U_\gamma = (\beta\mathbb{N} \setminus \mathbb{N}) \setminus B$. Clearly $U_\alpha \subsetneqq U_\gamma$ for all $\alpha < \gamma$ and the existence of B insures that the process can continue, i.e. that $F_\gamma \neq \emptyset$.

Adopting the same method employed in the usual proof of Urysohn's lemma (see, for example, [V] or [K]), we construct a continuous function from $\beta\mathbb{N} \setminus \mathbb{N}$ onto $\omega_1 + 1$. The theorem of Magill, previously quoted, now assures the existence of the desired compactification $\gamma\mathbb{N}$, with $\gamma\mathbb{N} \setminus \mathbb{N}$ homeomorphic to $\omega_1 + 1$.

Since $\omega_1 + 1$ can be thought of as a closed non-sequential subspace of $\gamma\mathbb{N}$ ($\mathbb{N}$ being locally compact), $\gamma\mathbb{N}$ cannot be sequential, and so Example 1.1 is complete.

Let $Y = \gamma\mathbb{N} \setminus \{\omega_1\}$. Then $\beta Y = \gamma\mathbb{N}$ is totally disconnected and hence zero-dimensional, being compact ([G-J] 16.17, p.247). Thus Y is also zero-dimensional ([G-J] 16.11, p.245). We will use this fact to show that Y is normal.

If A and B are disjoint closed subsets of Y and A_1 and B_1 are their respective intersections with ω_1 thought of as a subset of Y, one of them, say A_1 is compact ([G-J] 5.12(b), p.74). Then there is a clopen subset U of Y containing A_1 and missing B. Then $U \cup A$ is a clopen subset of Y containing A and missing B. Indeed, points of $A \setminus U$ are isolated, whence $U \cup A$ is open; they can accumulate only in $A_1 \subseteq U$, whence $U \cup A$ is closed. Thus Y is normal.

If C is any compact open subset of $\omega_1 \subseteq Y$, then for some compact open $U \subseteq Y$ $C = U \cap \omega_1$. By removing the countably many points of $U \cap \mathbb{N}$ one at a time, we see that C is a G_δ. But each point of $\omega_1 \subseteq Y$ is the intersection of countably many such C. Hence each point of Y is a G_δ. Since Y is locally compact (being an open subset of $\gamma\mathbb{N}$), it is first countable.

Noting that Y has cardinality $\aleph_1$, and that, having ω_1 as a closed subset, it isn't metacompact completes Example 1.3.

We now turn our attention to $s\gamma\mathbb{N}$. Since the cardinality of $\gamma\mathbb{N}$ is $\aleph_1$ it is sequentially compact ([F_4] Corollary p.598). But $s\gamma\mathbb{N}$, having the same convergent sequences, must then also be sequentially compact. Since $\gamma\mathbb{N}$ is not sequential, $s\gamma\mathbb{N}$ carries a strictly finer topology and is therefore not compact. Thus Example 1.6 is complete.

In the original construction of $\gamma\mathbb{N}$ a transfinite sequence $\{U_\alpha\}_{\alpha < \omega_1}$ of clopen subsets of $\beta\mathbb{N} \setminus \mathbb{N}$ was employed. By confining our construction to the complement of some given proper clopen subset U, we can assure that $\cup\{U_\alpha | \alpha < \omega_1\}$ isn't dense in $\beta\mathbb{N} \setminus \mathbb{N}$. Every such U is of the form $(cl_{\beta\mathbb{N}} A) \setminus \mathbb{N}$ for some subset A of $\mathbb{N}$ ([G-J] 65.4, p.99). Any sequence in $\gamma\mathbb{N}$ which is an enumeration of A must converge to ω_1. Indeed ω_1 is its only cluster point by the continuity of the function $\beta\mathbb{N} \longrightarrow \gamma\mathbb{N}$, and hence its limit point since we are in a compact Hausdorff space.

Now suppose we have separated the point ω_1 from the set ω_1 by disjoint open subsets V and W of $s\gamma\mathbb{N}$. V then must eventually contain any sequence in $\mathbb{N}$ converging to the point ω_1. Hence $(cl_{\beta\mathbb{N}} V \setminus \{\omega_1\}) \setminus \mathbb{N}$ is in fact the complement in $\beta\mathbb{N} \setminus \mathbb{N}$ of the closure of $\cup\{U_\alpha | \alpha < \omega_1\}$. This leaves the closure of that union open, a contradiction ([G-J] 6W.3, p.100). Thus if some sequence in $\mathbb{N}$ converges to the point ω_1 in $\gamma\mathbb{N}$, then $s\gamma\mathbb{N}$ isn't regular. This completes Example 1.7.

The continuum hypothesis can be used to assure that no sequence from $\mathbb{N}$ converges to ω_1. The transfinite recursion used to construct the $\{U_\alpha\}_{\alpha < \omega_1}$ can be continued so long as $\cup\{U_\alpha\}$ isn't dense in $\beta\mathbb{N} \setminus \mathbb{N}$. That G_δ's in $\beta\mathbb{N} \setminus \mathbb{N}$ have non-empty interiors insures that the process won't terminate for some $\delta < \omega_1$. This was the crucial fact of the first construction. However, by cardinality, the process must terminate for some $\delta \geq \omega_1$, i.e. $\cup\{U_\alpha\}_{\alpha < \delta}$ is dense in $\beta\mathbb{N} \setminus \mathbb{N}$. Since there are exactly $\mathfrak{c}$

clopen sets in $\beta\mathbb{N} \setminus \mathbb{N}$ (again [G-J] 6S.4, p.99), and a new one is created for each α, the cardinality of δ is less or equal to $\mathfrak{t}$. Assuming CH, the cardinality of δ is $\aleph_1$ and hence there is a cofinal subset of δ of type ω_1 (otherwise $\cup\{U_\alpha\}_{\alpha < \delta}$ couldn't be dense). The U_α's indexed by this subset form a strictly ascending chain of clopen sets indexed by ω_1 whose union is dense in $\beta\mathbb{N} \setminus \mathbb{N}$. If $\gamma\mathbb{N}$ is constructed from this chain, then no sequence in $\mathbb{N}$ will converge to the point ω_1.

In this case, removing ω_1 leaves $\gamma\mathbb{N} \setminus \{\omega_1\}$ sequentially compact (recall that $\gamma\mathbb{N}$ is always sequentially compact, regardless of its construction). Since the other properties are independent of the choice of the U_α's, this completes Example 1.4. A similar example occurs by taking $s\gamma\mathbb{N}$ in this case, since ω_1 then becomes isolated (being sequentially open) and hence $s\gamma\mathbb{N}$ has the same properties as $\gamma\mathbb{N} \setminus \{\omega_1\}$.

Having carefully constructed $\gamma\mathbb{N}$ so that no sequence in $\mathbb{N}$ converges to the point ω_1, let $\mathfrak{F}$ be the trace on $\mathbb{N}$ of the neighborhood filter of ω_1 in $\gamma\mathbb{N}$. Let $\mathfrak{U}$ be any ultrafilter containing $\mathfrak{F}$. Let X be a space whose underlying set is $\gamma\mathbb{N}$, and in which the neighborhoods of points are as in $\gamma\mathbb{N}$, except that a basic neighborhood of the point ω_1 is of the form $\{\omega_1\} \cup U$, where $U \in \mathfrak{U}$. The space X is clearly sequentially compact due to the careful construction of $\gamma\mathbb{N}$.

The space X is not sequential since $\{\omega_1\}$ is sequentially open but not open. It is a c-space since each point other than ω_1 has a countable neighborhood base, and ω_1 has a base of countable neighborhoods. X is clearly Hausdorff, completing Example 1.2, but fails to be regular since the point ω_1 doesn't have a basis of closed neighborhoods. This leaves the question: Is a sequentially compact regular c-space always sequential? An affirmative answer would be enough to restore faith in the now doubtful Theorem B of [F_4].

For Example 1.5, let δ be as on page 1.6. Without CH we can only conclude that $\omega_1 \leq \delta \leq \mathfrak{t}$ and that δ has no countable cofinal subset. Defining a function $\beta\mathbb{N} \setminus \mathbb{N} \to \delta + 1$ as before we obtain a compactification X' of $\mathbb{N}$ with remainder $X' \setminus \mathbb{N} = \delta + 1$ in the order topology. Since X' is totally disconnected and therefore zero-dimensional, $X = X' \setminus \{\delta\}$ is also zero-dimensional. Since δ has no countable cofinal subset, of two disjoint closed subsets of δ, one must be compact. These two facts are all that is required to prove that X is normal (see page 1.5).

It remains to show that X is sequentially compact. If we assume that $c < 2^{\aleph_1}$ this follows as before. However this is only replacing one independent set theoretic assumption (CH) by another, albeit weaker, one ($c < 2^{\aleph_1}$). This can be avoided as follows. Suppose $\{x_n\}$ is any sequence of distinct points in X. If $\{x_n\} \cap \delta$ is infinite, $\{x_n\}$ has a convergent subsequence since δ is sequentially compact. If not, we may assume that

$\{x_n\} \subseteq \mathbb{N}$. Then $\{x_n\}^* = cl_{\beta\mathbb{N}}\{x_n\} \setminus \{x_n\}$ must intersect some U_α since their union is dense. Now $U_\alpha = A_\alpha^*$ for some $A_\alpha \subseteq \mathbb{N}$ and $A_\alpha^* \cap \{x_n\}^* \neq \emptyset$ implies that $A_\alpha \cap \{x_n\}$ is infinite. This intersection, thought of as a subsequence of $\{x_n\}$ has cluster points, relative to $\beta\mathbb{N}$, only in U_α, and hence has α as its only cluster point relative to X'. Thus the subsequence converges to α in X and the proof is complete.

We wish to thank W. W. Comfort for several valuable comments concerning this section.

§2. An important subclass of the sequential spaces are the Fréchet spaces, i.e. those in which the closure of any set is simply the set of all its sequential limits. Clearly every first countable space (and hence every metric space) is Fréchet. Examples of Fréchet spaces which are not first-countable fairly abound: the reals with the integers identified, the plane with the X-axis shrunk to a point, any CW complex which isn't locally finite, etc. An example of a sequential space which isn't Fréchet can be found in [F_2] (Example 2.2).

In 1937 E. Čech asked if there was a Fréchet space (in the convergence space, not the topological, sense, (see [N], p.3) in which no point had a countable basis of neighborhoods. J. Novak produced such a convergence space ([N] §6, p.16) which wasn't a topological convergence, and remarked that he didn't know a topological example ([N], p.17).

We now have such an example; it will be presented in Section 3. On hearing of our example, Professor Novak informed us that he also had such an example (quite different from ours) which he described in terms of convergence groups of sets, convergence being order convergence relative to inclusion.

This section is devoted to an account of what we believe to be an example which is essentially the same as Novak's, in a different, and more accessible guise, that of zero-one sequences.

EXAMPLE 2.1. <u>There is a zero-dimensional topological group which is a Fréchet space but is not first countable</u>.

Let 2^{ω_1} be the topological product of two point discrete spaces $\{0,1\}_\alpha$ indexed by the countable ordinals ω_1. Let X be the subspace of 2^{ω_1} consisting of all functions taking the value 1 at most countably many times.

X as a subspace of 2^{ω_1} is a topological group under pointwise addition and hence is homogeneous and completely regular. It is also zero-dimensional, since 2^{ω_1} is. It fails to be locally compact since it is a non-open dense subset of 2^{ω_1}.

No point of X is a G_δ; thus X isn't first countable. Indeed, countably many basic open sets can restrict only countably many coordinates and hence cannot intersect in a point, even in X.

If $F \subseteq X$ and $g \in \text{cl } F$, then there is a countable subset F' of F with $g \in \text{cl } F'$, i.e. X is a c-space. Indeed, let $\alpha_o = \sup g^{-1}(1)$ and choose a countable subset F_o of F containing a function belonging to each of those basic neighborhoods of g which restrict only coordinates $\leq \alpha_o$. Let $\alpha_1 \geq \sup \{\alpha < \omega_1 \,|\, f(\alpha) = 1$ for some $f \in F_o\}$. Choose a countable subset F_1 of F meeting every basic neighborhood of g restricting only coordinates $\leq \alpha_1$. Having chosen F_n similarly for α_n, let $\alpha_{n+1} \geq \sup \{\alpha < \omega_1 \,|\, f(\alpha) = 1$ for some $f \in F_n\}$. Thus we construct recursively a sequence of ordinals $\alpha_o, \alpha_1, \ldots$ (without loss of generality we may take them strictly increasing -- we are indebted to F. G. Slaughter, Jr. for this simplification) and a sequence of countable subsets of F having the property that each basic neighborhood of g which restricts no coordinate α strictly between α_n and α_{n+1} (i.e. $\alpha_n < \alpha \leq \alpha_{n+1}$) meets F_n.

Since each basic neighborhood of g restricts only finitely many coordinates, it leaves some $(\alpha_n, \alpha_{n+1}]$ unrestricted and so meets F_n. Hence $g \in \mathrm{cl} \bigcup F_n$, which is countable.

Finally X is a Fréchet space. Indeed, if $g \in \mathrm{cl}\, F$ choose a countable subset F' of F with $g \in \mathrm{cl}\, F'$. Let $\{\alpha_1, \alpha_2, \ldots\}$ by some enumeration of the set of coordinates mapped to 1 either by g or by some $f \in F'$. For each $n < \omega_0$, let $F_n = \{f \in F' \mid f(\alpha_i) = g(\alpha_i),\ i \leq n\}$. Then for each n, $g \in \mathrm{cl}\, F_n$ (hence $F_n \neq \emptyset$) and if we choose f_n arbitrarily from F_n, $\{f_n\}$ will converge (pointwise) to f. This completes the proof.

The authors are indebted to M. Venkataraman for suggesting that Novak's example might be recast in this simple way, and to T. Soundararajan for a key idea in the original proof.

It has recently come to our attention that an as yet unpublished result of Noble [N_1] considerably generalizes Example 2.1. A Σ-subspace of a product space is one consisting of all those functions agreeing with a given fixed function except at countably many indexes. Noble proves ([N_1] Theorem 2.1) that any Σ-subspace of a product of first countable spaces is Fréchet. Example 2.1 is an immediate specialization.

§3. Since every first-countable space with unique sequential limits is Hausdorff, and every first-countable space is Fréchet, it is natural to ask whether every Fréchet space with unique sequential limits is Hausdorff. Several examples have been given to show this isn't the case (see for example [F_1]).

Again it was E. Čech who asked whether there was a Fréchet space with unique sequential limits in which no pair of points have disjoint neighborhoods.

In this section, we propose to give a totally different (and <u>countable</u>) solution to the problem of Section 2, and then to give a partial answer to the problem of Čech mentioned in the preceding paragraph.

The method employed in both cases is that of attaching of spaces, which can be traced from Urysohn [U] to Hewitt [H] to Shimrat's homogeneous extension [S] to the more recent applications in [A-F] and [F_5]. The construction of S_ω in [A-F] provides a good warm-up for the examples of this section.

EXAMPLE 3.1. <u>There is a countable, zero-dimensional, Hausdorff, homogeneous, Fréchet space which isn't first countable</u>.

Note that this space, being σ-compact and regular, has many nice topological properties, i.e. paracompactness etc. It fails to be locally compact, as did Example 2.1. It would be interesting to know if a homogeneous Fréchet space could also be locally compact without being first countable.

EXAMPLE 3.2. <u>There is a countable, homogeneous, sequential space with unique sequential limits in which no pair of distinct points have disjoint neighborhoods</u>.

Since every Fréchet space is sequential, this may be considered as partial solution to Čech's problem.

CONSTRUCTION OF 3.1

We begin by constructing a sequential fan F. Take denumerably many copies of a convergent sequence together with its limit point (i.e. copies of $\{^1/_n | n \in \mathbb{N}\} \cup \{0\}$ as subsets of the real line) and identify the limit points, calling the new point 0 and the new set F. Provide F with the quotient topology after having taken a disjoint topological sum of the convergent sequences with limits. The resulting sequential fan F has each of its countably many points isolated except for 0, which fails to have a countable basis of neighborhoods. The space F, however, is certainly a zero-dimensional, Hausdorff, Fréchet space. (Some of these assertions can be most quickly verified by recognizing F as a closed subset of the real line with the integers identified).

We now begin the attaching process in earnest. Let $F_1 = F$ and construct F_2 by attaching to each isolated point x of F_1 a copy F^x of F (all these various copies being kept scrupulously disjoint before attaching) identifying x with the zero 0^x of F^x. For a topology, each isolated point of each F^x will remain isolated in F_2; a basic F_2-neighborhood of 0^x will be simply a basic F^x-neighborhood considered as a subset of F_2; for a basic F_2-neighborhood of $0(\in F_1)$ in F_2, choose an

F_1-neighborhood U of 0 and take $U \cup \bigcup\{F^x | x \in U\}$ with the appropriate identifications. Clearly F_1 is a closed subspace of F_2, F_2 is a Fréchet space, and most importantly, no point in F_1 has a countable basis of neighborhoods in F_2.

We proceed by recursion. Having constructed F_{n-1}, a copy of F is attached at each isolated point of F_{n-1} to arrive at F_n. If F^x is attached at x we will say that $y \leq x$ for each $y \in F^x$. This relation is defined at each stage of the construction; its transitive closure is a partial order on F_n. Define the rank of a point p in F_n as 0 for $p = 0$ in F_1, and as the least i such that $p \in F_i$ otherwise. For $p \in F_n$ of rank n, let p be isolated. Otherwise, let U be a neighborhood of p in F^p (take $F^0 = F_1$). Then the sets $U^* = \{y \in F_n | y \leq x$ for some $x \in U\}$ will form a neighborhood basis for p. Again each F_i for $i < n$ is a closed subspace of F_n, F_n is a Fréchet space (this requires a little thought), and no point of rank $< n$ has a countable basis of neighborhoods in F_n.

Now let $F_\omega = \bigcup F_n$. We may either take the inductive limit topology (the F_n's together with their inclusion maps form an inductive system; F_ω is the limit) or, preferably, we may extend the partial order to F_ω and use the U^* as basic neighborhoods as before. It amounts to the same thing. Each F_n is a closed subspace of F_ω, F_ω is Fréchet and no point of F_ω has a countable basis of neighborhoods.

It is obvious that F_ω is Hausdorff. Each U^* is clopen (recall that U is a neighborhood of p in some F^p); whence F_ω is zero-dimensional. For each $p \in F_\omega$ let $I(p)$ be the principal ideal generated by p, i.e. $I(p) = \{y \in F_\omega | y \leq p\}$. Then each $I(p)$ and each $F_\omega \setminus I(p)$ is homeomorphic to F_ω. Homogeneity follows easily by finite induction. This completes Example 3.1.

It would be interesting to know if F_ω can support a group structure as does Example 2.1.

One should note that having specified any infinite cardinal m, an example similar to F_ω can be constructed having all the same properties (except countability) with the character of each point $\geq$ m. One simply must put more sequences in the fan.

If one wanted only a sequential space instead of a Fréchet space in Example 3.1, it could be had simply by taking Shimrat's homogeneous extension of F. The Fréchet property is lost through quotients generally, where sequentialness is not. The space in this case would apparently be much more complicated as a set than F_ω.

CONSTRUCTION OF 3.2

Example 3.2 is significantly more complex than is Example 3.1. This is so on two counts: first the basic building block is more complicated; secondly, the attaching is performed at two points each time instead of at one, and this is done for "almost all" pairs of points.

The basic building block D is sort of a "sequential fan with two pivots". Precisely, D consists of an infinite sequence $B_1, B_2, \ldots$ of pairwise disjoint countably infinite sets of isolated points (B_n is called the <u>n th blade</u>) together with two additional distinct points O and O' whose neighborhoods are described as follows: to form a basic neighborhood of O, one may discard finitely many points (including possibly zero) from each even numbered blade, as well as finitely many odd numbered blades in their entirety; the basic neighborhoods of O' are formed similarly with odd and even interchanged. The resulting countable space is D.

It is clear from the definition that O and O' have no disjoint pair of neighborhoods. If O is in the closure of some subset A of D, then either $A \cap B_n$ is infinite for some even n, in which case any enumeration of $A \cap B_n$ is a sequence in A converging to O, or A has a non-empty intersection with infinitely many blades of odd index, in which case a point chosen arbitrarily from each of these intersections gives rise again to a sequence in A converging to O. Using the dual (in the sense of odd and even) we conclude that D is a Fréchet space. Any sequence in $D \setminus \{O\}$ converging to O must be either infinitely many times in some even numbered blade or else only finitely many times in each of infinitely many odd numbered blades. In any event, it cannot also converge to O'. Hence sequential limits are unique and we have another example such as was mentioned in the first paragraph of this section, i.e. a non-Hausdorff, Fréchet space with unique sequential limits.

The non-Hausdorffness occurs only at the points 0 and $0'$. We will use the attaching process to construct a space D_ω in which every pair of distinct points looks like a complicated version of the pair $0,0'$.

Let D_o be the two point discrete space $\{a,b\}$. To get D_1, simply attach a copy of D to D_o identifying 0 with a and $0'$ with b. We will say that the pair of distinct points are <u>joined</u> if they have been identified with the points 0 and $0'$ of some copy of D. Thus a and b are joined in D_1. To get D_2, to each pair of distinct non-joined points of D_1 attach a copy of D. In general, having constructed D_n, for each pair $\{x,y\}$ of non-joined points of D_n choose a copy $D^{\{x,y\}}$ of D and attach it at x and y, thus arriving at D_{n+1}. Each D_n is a proper subset of D_{n+1}. Let D_ω be the union of all the D_n.

We topologize D_ω a bit at a time. Having given D_o the discrete topology and each copy $D^{\{x,y\}}$ of D the topology described above for D, each D_{n+1} can be regarded as a quotient of the disjoint topological sum of D_n and countably many copies of D. Give D_n the quotient topology. As before, D_ω can be regarded as the inductive limit of an inductive system composed of the D_n and compositions of their inclusion maps into each other. Give D_ω the inductive limit topology.

Clearly D_ω is sequential (since we began with sequential spaces and essentially performed only sums and quotients (see $[F_2]$ 1.2, 1.6, 1.7). It is also clear that no pair of distinct points of D_ω have disjoint neighborhoods. It remains only to show that sequential limits are unique. This becomes clear once one realizes that a sequence can converge to a point x only if A is eventually in some finite number of $D^{\{x,y\}}$.

Unfortunately D_ω isn't a Fréchet space, so that the problem of Čech remains open. One might be tempted to redefine the topology of D_ω in a manner more analogous to that of F_ω in order to make D_ω Fréchet. This can surely be done, but the uniqueness of sequential limits is lost in the process.

FOOTNOTES

(1) A set is _sequentially open_ if no sequence outside converges to a point inside. A _sequential space_ is one in which every sequentially open set is open.

(2) A c-_space_ is one in which the closure of each set is the union of the closures of its countable subsets.

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