# NORMS AND DETERMINANTS OF LINEAR MAPPINGS

by

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## 1. Introduction and Preliminaries.

Some years ago, B. L. van der Waerden called the author's attention to the inequality

$$\|T^{-1}\| \ |\det T| \ \leqq \ \|T\|^{n-1} \tag{1.1}$$

valid for an invertible linear mapping $T$ of an $n$-dimensional real euclidean vector space into itself, with respect to the operator norm. Following a discussion on possible applications of the Cayley-Hamilton Theorem to problems in analysis, W. A. Coppel remarked that that theorem could be used to show that for any norm, not necessarily euclidean, one still has

$$\|T^{-1}\| \ |\det T| \ \leqq \ (2^n - 1)\|T\|^{n-1} . \tag{1.2}$$

It is our purpose in this paper to give some more precise estimates of this type and to illustrate some methods that are useful in exploring the geometrical properties of normed spaces.

$X$ shall generally denote a finite-dimensional real normed vector space, $n = \dim X$ its dimension, $\| \ \|$ its norm, $\Sigma$ its unit ball (in full, $\| \ \|_X$, $\Sigma(X)$). $X$ is <u>euclidean</u> if its norm is determined by an inner product; this is the case if and only if $\Sigma$ is an ellipsoid. (The definition of an ellipsoid depends only on the underlying linear structure of $X$.)

$\tilde{X}$ shall denote the normed ($n^2$-dimensional real) algebra of linear operators on $X$, with the operator norm. The following discussion of determinants and adjugates depends on the linear structure of $X$ and $\tilde{X}$ only, together with their natural topologies, not on the specific norm.

The determinant function $\det : \tilde{X} \to R$ is well defined, and $T \in \tilde{X}$ is invertible if and only if $\det T \neq 0$. We are interested in the continuous function $T \mapsto (\det T)T^{-1}$, defined for invertible $T \in \tilde{X}$. It is useful to know that this function can be extended to a continuous function $\operatorname{adj} : \tilde{X} \to \tilde{X}$;

since the set of invertible elements is dense in $\tilde{E}$, this extension is unique; thus adj is characterized by being continuous and satisfying

$$(1.3) \qquad T \operatorname{adj} T = (\det T) I, \qquad T \in \tilde{X}.$$

The function adj can be defined by the identity

$$\det(T+A) = \det T + \operatorname{tr}(A \operatorname{adj} T) + o(A), \qquad T, A \in \tilde{X}, \quad A \to 0;$$

further, if $\det(T - \lambda I) = \sum_{0}^{n} p_i(T)\lambda^i$ is the characteristic polynomial of $T$, $\operatorname{adj} T$ is given by

$$(1.4) \qquad \operatorname{adj} T = -\sum_{1}^{n} p_i(T) T^{i-1}$$

Finally, if a basis of $X$ is specified, the matrix of $\operatorname{adj} T$ is the 'adjugate transpose' of the matrix of $T$, i.e., its $(j,i)$-entry is the cofactor of the $(i,j)$-entry of the matrix of $T$. The verification, in terms of (1.3) and continuity, of these well-known statements is omitted. Formula (1.4) follows from the Cayley-Hamilton Theorem and leads almost immediately to the estimate (1.2).

We return to the consideration of the normed space $X$. We are interested in an inequality of the form

$$(1.5) \qquad \|T^{-1}\| \, |\det T| \leqq c\|T\|^{n-1}$$

valid for all invertible $T \in \tilde{X}$. Since (1.5) is homogeneous (i.e., preserved under multiplication of $T$ by a scalar), we are led to define

$$(1.6) \qquad \begin{aligned} k(X) &= \sup\{\|T^{-1}\| \, |\det T| : T \in \Sigma(\tilde{X}) \text{ invertible}\} \\ &= \max\{\|\operatorname{adj} T\| : T \in \Sigma(\tilde{X})\}. \end{aligned}$$

The maximum of the continuous function $T \mapsto \|\operatorname{adj} T\|$ on the compact set $\Sigma(\tilde{X})$ is attained. If $X$ and $Y$ are congruent (i.e., isometric) normed spaces, we obviously have $k(X) = k(Y)$.

We also define

$$(1.7) \qquad k_n = \sup\{k(X) : X \text{ a real normed space, } \dim X = n\}, \qquad n = 1,2,\ldots .$$

The set-theoretical difficulties of this definition are illusory: since congruent spaces have the same $k$, the supremuum in (1.7) need be taken only over the <u>set</u> of normed spaces defined on one and the same $n$-dimensional real linear space. By (1.7), $k_n$ is defined as the least $c$ for which (1.5) holds for every invertible operator on every $n$-dimensional normed space.

The main result of this paper is that $k_2 = 2$ and $2 \leqq k_n \leqq (n^n/(n-1)^{n-1})^{1/2} < (ne)^{1/2}$, $n = 3,4,\ldots$. It is conjectured that, in fact, $k_n = 2$, $n = 3,4,\ldots$, also. Of course $k_1 = 1$, and the trivial one-dimensional case will be excluded without further mention in what follows.

## 2. $\ell^1$ - spaces and a lower bound.

In this section we shall compute $k(\ell_n^1)$, where $\ell_n^1$ is the linear space $R^n$ of column vectors $x = (x^j)$, $j = 1,\ldots,n$, with the norm $\|x\|_1 = \sum_1^n |x^j|$ and the unit ball $\Sigma_1 = \Sigma(\ell_n^1) = \{x \in R^n : \sum_1^n |x^j| \leq 1\}$.

The elements of $(\ell_n^1)^\sim$ are square matrices $T = (T_j^i)$; $T_j$ denotes the $j^{th}$ column of $T$. Thus $I_1,\ldots,I_n$, the columns of the identity $I$, constitute the natural basis of $\ell_n^1$, and $T_j = TI_j$, $j = 1,\ldots,n$. $\Sigma_1$ is the balanced convex hull of $\{I_1,\ldots,I_n\}$. The use of matrix language could be avoided, but it is quite natural for a space with a built-in basis. For $T \in (\ell_n^1)^\sim$ we have

$$\|T\|_1 = \max_j \|T_j\| = \max_j \sum_{i=1}^n |T_j^i| . \tag{2.1}$$

As mentioned in Section 1, $(\text{adj } T)_i^j$ is the cofactor of $T_j^i$ in $T$.

2.1. Lemma. <u>$\|\text{adj } T\|_1$, considered as a function of $T_1,\ldots,T_n$, is a convex function of each argument separately.</u>

Proof. Let $h$, $1 \leqq h \leqq n$, be specified and $T_j$ be fixed for all $j \neq h$. In its dependence on $T_h$, $(\operatorname{adj} T)^j_i$ is constant if $j = h$, and linear if $j \neq h$, for all $i,j$. Thus $T_h \mapsto \operatorname{adj} T : \ell^1_n \to (\ell^1_n)^\sim$ **is an affine mapping.** But $A \mapsto \|A\|_1 : (\ell^1_n)^\sim \to R$ is a convex function; the conclusion follows.

2.2. Theorem. $k(\ell^1_n) = 2$, $n = 2,3,\ldots$.

Proof. 1. According to (1.6) and (2.1),

$$k(\ell^1_n) = \max\{\|\operatorname{adj} T\|_1 : T_j \in \Sigma_1,\ j = 1,\ldots,n\}. \tag{2.2}$$

By Lemma 2.1, $\|\operatorname{adj} T\|_1$ is, for each $h, 1 \leqq h \leqq n$, a convex function of $T_h$ (the other $T_j$ being fixed). Now a convex real-valued function attains its maximum on a compact convex set at least at one extreme point. If we start from a point at which the maximum in (2.2) is attained, we may replace the $T_j$, one at a time, by an extreme point of $\Sigma_1$ without changing the value of $\|\operatorname{adj} T\|_1$. Thus the maximum in (2.2) is attained at least at one point at which each $T_j$ is an extreme point of $\Sigma_1$.

The extreme points of $\Sigma_1$ are exactly $\pm I_1,\ldots,\pm I_n$. The preceding discussion implies that the maximum in (2.2) need be sought only in the finite set $P$ of those $T$ that map $\{\pm I_1,\ldots,\pm I_n\}$ into itself and thus have, in each column, exactly one non-zero entry, namely a $1$ or a $-1$. Thus

$$k(\ell^1_n) = \max\{\|\operatorname{adj} T\|_1 : T \in P\}. \tag{2.3}$$

2. We claim that, for $T \in P$, $\|\text{adj } T\|_1 = 0, 1,$ or $2$ according as the number of rows of $T$ consisting only of zeros ("zero-rows") is greater than one, zero, or one, respectively.

If $T$ has two zero-rows or more, every cofactor is $0$, and therefore $\text{adj } T = 0$. If $T$ has no zero-rows, it has exactly one non-zero entry (a $1$ or a $-1$) in each column and in each row. If $T^T$ denotes the transpose of $T$, we have $|\det T| = 1$ and $T^T = T^{-1}$ by direct verification; so $\text{adj } T = T^{-1} \det T = \pm T^T$ also has exactly one non-zero entry (a $1$ or a $-1$) in each column, and (2.1) yields $\|\text{adj } T\|_1 = 1$.

Assume, finally, that the $h$th row is the only zero-row of $T$. Then there must be exactly two non-zero entries in one other row, say $T^k_p, T^k_q$ (each $1$ or $-1$), and each row other than the $h$th and the $k$th contains exactly one such entry. Now the cofactor of $T^i_j$ in $T$ is $0$ unless $i = h$ and $j = p$ or $q$ : indeed, it is the determinant (up to sign) of a minor matrix that has a zero-row if $i \neq h$, and has two columns that are equal or opposite if $j \neq p, q$. The cofactors of $T^h_p$ and $T^h_q$ are the determinants (up to sign) of minor matrices that have exactly one non-zero entry (a $1$ or a $-1$) in each column and in each row; these co-factors are the only non-zero entries of $\text{adj}\, T$, namely $(\text{adj } T)^p_h$, $(\text{adj } T)^q_h$, and each is $1$ or $-1$. By (2.1),

$\|\mathrm{adj}\ T\|_1 = 2$. This completes the proof of our claim.

3. Since there does exist $T \in P$ with exactly one zero-row (e.g., set $T_j = I_j$, $j = 1,\ldots,n-1$, and $T_n = I_{n-1}$), we conclude from (2.3) that $k(\ell_n^1) = 2$.

2.3. <u>Corollary</u>. $k_n \geqq 2$, $n = 2,3,\ldots$.

## 3. Loewner ellipsoids and upper bounds.

Let $E$ be an n-dimensional real linear space. A *central convex body* in $E$ is a compact convex set with non-empty interior and $0$ as a centre of symmetry. We want to discuss the *volume* of convex bodies in $E$; now volume is (a restriction of) a Haar measure on $E$; since any two Haar measures are proportional, there will be no loss in choosing one arbitrary but fixed determination of the volume; vol $U$ will denote the volume of $U$ in this determination. If $T : E \to E$ is an invertible linear mapping and $U$ is a central convex body in $E$, then $TU$ is another, and vol $TU = |\det T|$ vol $U$.

We shall require the following interesting property of ellipsoids.

3.1. *Lemma*. (Loewner). *If $U$ is a central convex body in $E$, then among the central ellipsoids containing $U$ there is exactly one of least volume.*

*Proof*. See, e.g., [4; 16.10]. (The sketchiness of the existence proof given there is easily remedied.)

The ellipsoid singled out by Lemma 3.1 shall be termed the *Loewner ellipsoid* of $U$, and denoted by $L(U)$.

*Remark*. Lemma 3.1 is the special case for central convex bodies of a much more general result; Busemann, in the book quoted above, has a version of intermediate generality and attributes the result to C. Loewner. Behrend [2],[3] discussed the two-dimensional case in detail. A deep application of the existence and uniqueness of Loewner ellipsoids was made by Gromov [6].

3.2. Lemma. *If* $U$ *is a central convex body in* $E$ *and* $T : E \to E$ *is an invertible linear mapping, then* $TL(U) = L(TU)$.

Proof. $TL(U)$ is a central ellipsoid containing $TU$, and $T^{-1}L(TU)$ is a central ellipsoid containing $T^{-1}TU = U$. By the minimality property of the Loewner ellipsoids,

$$1 \leqq \frac{\text{vol } TL(U)}{\text{vol } L(TU)} = \frac{|\det T|}{|\det T|} \cdot \frac{\text{vol } L(U)}{\text{vol } T^{-1}L(TU)} \leqq 1.$$

Thus equality holds at both ends; by the uniqueness of the Loewner ellipsoid we must have $TL(U) = L(TU)$.

We now return to our normed space $X$. We first perform a geometric computation for euclidean $X$; this will lead, incidentally, to a proof of (1.1).

3.3. Lemma. *Assume that* $X$ *is euclidean and that* $T\in\tilde{X}$ *is invertible. Let* $a_1,\ldots,a_n$ *be the half-axes of the ellipsoid* $T\Sigma$, *where* $0 < a_1 \leqq \cdots \leqq a_n$. *Then* $\|T\| = a_n$, $\|T^{-1}\| = a_1^{-1}$, $|\det T| = \prod_1^n a_i$.

*Proof*. Obvious from euclidean geometry and the fact $|\det T| \text{vol } \Sigma = \text{vol } T\Sigma = (\prod_1^n a_i) \text{vol } \Sigma$.

3.4. Theorem. *If* $X$ *is euclidean, then* $k(X) = 1$.

*Proof*. If $T\in\tilde{X}$ is invertible and $\|T\| \leqq 1$, Lemma 3.3 implies $\|T^{-1}\| |\det T| = \prod_2^n a_i \leqq a_n^{n-1} \leqq 1$. On the other hand, $k(X) \geqq \|I^{-1}\| |\det I| = 1$.

To deal with a normed space $X$ that is not necessarily euclidean, we consider the auxiliary euclidean space $X_L$ on the same underlying linear space, and whose unit ball is the Loewner ellipsoid of the unit ball of $X$ : $\Sigma(X_L) = L(\Sigma)$.

The operator algebra $(X_L)^\sim$ is algebraically identical to $X$. Norms of elements of $X_L$ and of $(X_L)^\sim$ will be distinguished by a subscript $L$.

3.5. <u>Lemma</u>. <u>If</u> $T\epsilon X$ <u>is invertible and</u> $\|T\| \leqq 1$, <u>then</u>

$$\|T^{-1}\|_L |\det T| < (n/(n-1))^{(n-1)/2} \tag{3.1}$$

<u>Proof</u>. $T\Sigma(X_L) = TL(\Sigma) = L(T\Sigma)$ by Lemma 3.2. Let $a_1,\ldots,a_n$ be the half-axes of this ellipsoid in the euclidean space $X_L$, where $0 < a_1 \leqq \ldots \leqq a_n$. By Lemma 3.3 applied to $X_L$,

$$\|T^{-1}\|_L |\det T| = \prod_2^n a_i . \tag{3.2}$$

For an appropriate orthonormal basis in $X_L$ we have
$L(\Sigma) = \{x\epsilon X_L : \sum_1^n x_i^2 \leqq 1\}$ $L(T\Sigma) = \{x\epsilon X_L : \sum_1^n a_i^{-2} x_i^2 \leqq 1\}$,

where $x_1,\ldots,x_n$ are the coordinates of $x$ with respect to this basis. For $\lambda, 0 \leqq \lambda \leqq 1$, consider the ellipsoid

$$V(\lambda) = \{x\epsilon X_L : \sum_1^n (\lambda + (1-\lambda)a_i^{-2}) x_i^2 \leqq 1\}, \quad 0 \leqq \lambda \leqq 1.$$

Then $\|T\| \leqq 1$ implies $T\Sigma = \Sigma \cap T\Sigma \subset L(\Sigma) \cap L(T\Sigma) \subset V(\lambda)$ for each $\lambda$. By Lemma 3.1,

$$\text{vol } V(0) = \text{vol } L(T\Sigma) \leqq \text{vol } V(\lambda) = \left(\prod_1^n (\lambda+(1-\lambda)a_i^{-2})\right)^{-1/2} \text{vol } L(\Sigma),$$
$$0 \leqq \lambda \leqq 1,$$

and therefore

$$0 \leqq (d/d\lambda)\log \text{vol } V(\lambda)\big|_{\lambda=0} = \tfrac{1}{2}\sum_1^n (1-a_i^2),$$

so that

$$(3.3) \qquad \sum_1^n a_i^2 = n.$$

From the inequality between arithmetic and geometric means and from (3.3) we obtain

$$(3.4) \quad \left(\prod_2^n a_i^2\right)^{1/(n-1)} \leqq \left(\sum_2^n a_i^2\right)/(n-1) < \left(\sum_1^n a_i^2\right)/(n-1) \leqq n/(n-1).$$

Combination of (3.2) and (3.4) yields (3.1).

In order to obtain an upper estimate for $k(X)$ from this lemma, we must be able to bound $\|T^{-1}\|$ in terms of $\|T^{-1}\|_L$. For this purpose we introduce a parameter that measures the closeness of $\Sigma$ and $L(\Sigma)$, namely $\rho(X) = \min\{r \in R : L(\Sigma) \subset r\Sigma\} \geqq 1$. This parameter is 1 if and only if $X$ is euclidean; we shall show presently that it never exceeds $n^{1/2}$.

3.6. <u>Theorem</u>. $1 \leqq k(X) \leqq (n/n-1))^{(n-1)/2} \rho(X)$.

<u>Proof</u>. $k(X) \geqq \|I^{-1}\| |\det I| = 1$. On the other hand, let $J : X \to X_L$ be the identity mapping considered as a mapping from the normed space $X$ to the normed space $X_L$. Then $\|J\| = 1$, $\|J^{-1}\| = \rho(X)$. If $T \in \tilde{X}$ is invertible,

$$\|T^{-1}\| = \|J^{-1} T^{-1} J\| \leqq \|J\| \|J^{-1}\| \|T^{-1}\|_L = \rho(X) \|T^{-1}\|_L .$$

Combination of this inequality with Lemma 3.5 and (1.6) yields the conclusion.

For an upper bound of $k_n$ we need a bound for $\rho(X)$; in finding one, we recover a precise form of a result of John [7].

3.7. <u>Lemma</u>. $\rho(X) \leqq n^{1/2}$, i.e., $L(\Sigma) \subset n^{1/2}\Sigma$.

<u>Proof</u>. Let $\partial\Sigma$ denote the boundary of $\Sigma$. Choose $u_0 \in \partial\Sigma$ so that $d = \|u_0\|_L = \min\{\|u\|_L : u \in \partial\Sigma\} > 0$. We are to prove that $n^{1/2} d \geqq 1$. Assume that this is not the case, i.e., that $nd^2 < 1$.

Choose an orthonormal basis of the euclidean space $X_L$ so that its first element is $d^{-1}u_0$, and let $x_1,\ldots,x_n$ be the coordinates of the point $x$ with respect to this basis. Now $x\in\Sigma$ implies $x \in L(\Sigma)$ and $|x_1| \leqq d$, so that

$$\frac{1}{d^2} x_1^2 + \frac{n-1}{1-d^2} \sum_2^n x_i^2 \leqq \frac{1}{d^2} x_1^2 + \frac{n-1}{1-d^2}(1-x_1^2) = \frac{n-1}{1-d^2} + \frac{1-nd^2}{d^2(1-d^2)} x_1^2 \leqq n, x\in U;$$

thus $\Sigma$ is contained in the ellipsoid

$$V = \{x \in X_L : \frac{1}{nd^2} x_1^2 + \frac{n-1}{n(1-d^2)} \sum_2^n x_i^2 \leqq 1\}.$$

By Lemma 3.1, vol $L(\Sigma) <$ vol $V$, i.e.,

$$(3.5) \qquad 1 < nd^2 \left(\frac{n(1-d^2)}{n-1}\right)^{n-1} = \frac{n^n}{(n-1)^{n-1}} d^2(1-d^2)^{n-1} .$$

But the function $t \to t(1-t)^{n-1}$ is increasing in the interval $[0,n^{-1}]$; therefore $nd^2 < 1$ implies

$$d^2(1-d^2)^{n-1} < n^{-1}(1-n^{-1})^{n-1} = \frac{(n-1)^{n-1}}{n^n} ,$$

and this contradicts (3.5).

<u>Remark</u>. $\rho(X)$ does attain the bound $n^{1/2}$, e.g. for $X = \ell_n^1$, $X = \ell_n^\infty$.

3.8. <u>Theorem</u>. $2 \leqq k_n \leqq (n^n/(n-1)^{n-1})^{1/2} < (ne)^{1/2}$ <u>for</u> $n = 2,3,\ldots$; <u>in particular</u>, $k_2 = 2$.

<u>Proof</u>. Theorem 3.6, Lemma 3.7, Corollary 2.3.

## 4. Comments and conjectures

We collect in this section some additional comments and some open questions.

For given $X$, let $X^*$ be the normed dual space. The mapping $T \mapsto T^* : \tilde{X} \to (X^*)^\sim$ is bijective and preserves invertibility, the inverse, the norm, and the determinant. This allows us to compute $k(X^*)$.

4.1. Theorem. For each $X$, $k(X^*) = k(X)$.

4.2. Corollary. *If the unit ball of* $X$ *is a parallelotope, then* $k(X) = 2$.

*Proof.* $X$ is then congruent to $\ell_n^\infty$, which is itself congruent to $(\ell_n^1)^*$. The conclusion follows from Theorems 2.2 and 4.1.

Banach and Mazur defined a 'distance' between (complete) normed spaces [1; pp. 242-243]; see also [5; p. 156], [9; pp. 72-74]. For our purposes--restricted to finite-dimensional spaces--this may be formulated as follows. If $X,Y$ are real normed spaces of the same dimension,

$$(4.1) \qquad \Delta(X,Y) = \min\{\log \|S\|\|S^{-1}\| : S : X \to Y \text{ an invertible linear mapping}\}.$$

The minimum is attained; $\Delta$ does not change if either space is replaced by one congruent to it, and $\Delta$ induces, on the set of congruence classes of n-dimensional spaces, a *metric* with respect to which this set is a compact space. This makes it again possible to overcome the set-theoretical difficulties of the definition. The preceding remarks show that a real-valued function defined for n-dimensional real normed spaces and continuous with respect to the pseudometric $\Delta$ (hence congruence-invariant) attains its maximum (cf. [9; Corollary 6.4]).

4.3. <u>Theorem</u>. <u>If</u> $X, Y$ <u>are</u> n-<u>dimensional real normed spaces,</u> $|\log k(Y) - \log k(X)| \leq n\Delta(X,Y)$. <u>The supremum in</u> (1.7) <u>is attained</u>.

<u>Proof</u>. Let $S : X \to Y$ be linear and invertible. If $T \in \tilde{Y}$ is invertible, so is $S^{-1}TS \in \tilde{X}$; and $\det S^{-1}TS = \det T$. Then

$$\|T^{-1}\| |\det T| \leqq \|S\|\|S^{-1}\|\|(S^{-1}TS)^{-1}\||\det S^{-1}TS| \leqq$$

$$\leqq k(X)\|S\|\|S^{-1}\|\|S^{-1}TS\|^{n-1} \leqq k(X)(\|S\|\|S^{-1}\|)^n\|T\|^{n-1} .$$

Therefore $k(Y) \leqq (\|S\|\|S^{-1}\|)^n k(X)$. Interchanging $X$ and $Y$ (and $S$ and $S^{-1}$) in this computation and using (4.1) we arrive at the conclusion. The paragraph preceding this theorem then shows that the supremum in (1.7) is attained.

Any two n-dimensional euclidean normed spaces are congruent. It is therefore meaningful to set, for every n-dimensional normed space $X$, $\Delta_e(X) = \Delta(X,Y)$, where $Y$ is any n-dimensional euclidean space.

4.4. <u>Corollary</u>. $k(X) \leqq \exp(n\Delta_e(X))$.

<u>Proof</u>. Theorems 3.4 and 4.3.

This estimate for $k(X)$ is a generalization of Theorem 3.4--while Theorem 3.6 is not--but it is good only for small $\Delta_e(X)$. Observe that, by Lemma 3.7, (4.2) $$\Delta_e(X) = \Delta(X,X_L) \leqq \log \rho(X) \leqq \frac{1}{2} \log n,$$

and that equality holds throughout in (4.2) if $X = \ell_n^1$ or $X = \ell_n^\infty$. Corollary 4.4 and (4.2) yield the estimate $k_n \leqq n^{n/2}$ ; this is exact for $n = 2$ (Theorem 3.8), but much cruder than the bound given by Theorem 3.8 for greater $n$.

We have seen that, in a way relevant to the computation of $k(X)$ (cf. Theorem 2.6 and Lemma 2.7, Theorem 4.3 and Corollary 4.4), the n-dimensional spaces 'farthest' from the euclidean spaces include $\ell_n^1$ and $\ell_n^\infty$. This motivates the following conjecture.

4.5. Conjecture. $k(X) \leqq 2$ for all $X$; thus $k_n = 2$ for $n = 3,4,\ldots.$

In this **connection**, it may also be asked what spaces $X$, if any, share with the euclidean ones (Theorem 3.4) the value $k(X) = 1$. A complete answer is available for dimension 2. Assume $\dim X = 2$ and let $\partial\Sigma$ be the boundary of $\Sigma$. $X$ shall be said to be a Radon plane if for any $u,v \in \partial\Sigma$ such that $v$ has the direction of a supporting line of $\Sigma$ at $u$, $u$ has the direction of a supporting line of $\Sigma$ at $v$. Radon [8] first pointed out that $X$ with these properties need not be euclidean, even if $\Sigma$ is strictly convex (when the condition asserts that every diameter has a 'conjugate diameter'). The description in [4; p. 104] is easily extended to all Radon planes, even those whose $\Sigma$ is not strictly convex or smooth. A careful use of this description, as well as the study of special linear mappings, yield the following characterization; the proof cannot be given in detail here.

4.6. Theorem. *If* $\dim X = 2$, *then* $k(X) = 1$ *if and only if* $X$ *is a Radon plane.*

Inasmuch as Radon planes are characterized among normed planes by 'symmetry of orthogonality', and this property is characteristic of euclidean spaces if the dimension is greater than two, the following conjecture appears plausible.

4.7. Conjecture. *If* $\dim X > 2$, *then* $k(X) = 1$ (*if and*) *only if* $X$ *is euclidean.*

As long as our information on $k_n$ is not complete, it is interesting to compare values of $k_n$ for different $n$. For this purpose we define, for each n-dimensional real normed space $X$, the (n+1)-dimensional normed space $X \oplus R$ : algebraically, this is the outer direct sum of $X$ and $R$; the norm is $\|x\oplus r\| = \max\{\|x\|, |r|\}$, so that $\Sigma(X\oplus R) = \Sigma(X) \oplus [-1,1]$. If $T\in\tilde{X}$, $s\in R$, we define $T\oplus sI\in(X\oplus R)^{\sim}$ by $(T\oplus sI)(x\oplus r) = Tx\oplus sr$; we find $\|T\oplus sI\| = \max\{\|T\|, |s|\}$, $\det(T\oplus sI) = s \det T$.

4.8. Lemma. $k(X\oplus R) \geqq k(X)$.

Proof. For every invertible $T\in\Sigma(\tilde{X})$ we have $\|T\oplus I\| \leqq 1$, $\det(T\oplus I) = \det T$, $(T\oplus I)^{-1} = T^{-1}\oplus I$. Now $\|T^{-1}\| \geqq \|T\|^{-1} \geqq 1$. Thus $\|T^{-1}\||\det T| = \|(T\oplus I)^{-1}\||\det(T\oplus I)| \leqq k(X\oplus R)$. Since this holds for all invertible $T\in\Sigma(\tilde{X})$, the conclusion follows.

4.9. Theorem. *The sequence* $(k_n)$ *is non-decreasing.*

Proof. For every n-dimensional $X$, $k(X) \leqq k(X\oplus R) \leqq k_{n+1}$ (Lemma 4.8). Therefore $k_n \leqq k_{n+1}$.

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