SPACES OP CONTINUOUS FUNCTIONS

INTO A BANACH SPACE I

by

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SPACES OF CONTINUOUS FUNCTIONS INTO A BANACH SPACE

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1. Introduction

Let X be a compact Hausdorff space and E be a Banach space. Let C(X,E) denote the Banach space of E-valued continuous functions equipped with the usual supremum norm. The Banach-Stone theorem. Day [4], asserts that if *X,Y* are compact Hausdorff spaces then X is homeomorphic with Y if and only if there is a linear isometry on C(X,R) onto C(Y,R) where R is the real line. Subsequently Jerison [8] investigated the problem of extending Banach-Stone theorem with R replaced by an arbitrary Banach space E. In [8] it is proved that the theorem remains true if (*) any two T-sets in E are discrepant and in particular if E is a strictly convex space; however it is shown that the theorem is false in general. We aim here to investigate the same problem in the case when the space E does not satisfy the condition (*) • Among others we consider in this paper similar problems that arise when E is a smooth Banach space or the unit cell of E is quasi-cylindrical.

Apart from discussing the above problems we obtain some auxilliary results concerning the spaces $C(X,E)$ which are also of intrinsic interest. Among others we provide a useful characterization of extreme points of the unit cell of the dual space of $C(X,E)$ and determine the functions f in $C(X,E)$ such that the norm is G-differentiable at f.

2. Preliminaries

Before proceeding to the main results of the paper we recall the necessary terminology and notation and few useful results.

Throughout the paper E is a fixed real Banach space of dimension ≥ 2. If B is a Banach space we denote the dual space of B by B^*. The norms of the various Banach spaces that enter our discussion are all denoted by the same symbol $\|\ \|$ as there is no occasion for confusion. The unit cell of $B\,(B^*)$ is denoted by $U_B(U^*_B)$

and $S_n(S^*_D)$ is the boundary of $U_Q(U^*_{Ho})$. If X is a compact Hausdorff space the unit cells of C(X,E) and its dual are denoted by U_A and U^*_X respectively.

If K is a convex set Ext K is the set of extreme points of K.

In the sequel we make use of the following functions e and II. The function e on E* x X into (C(X,E))* is defined by setting *e(l,p)* (f) = <t(f(p)). We have |e(*,,p)(f)| = |*(f(p))| £ \\l\\ llf(p)|! ≤ \\l\\ ||f||. Thus for a fixed p |e*(l,p)*) ≤ \}l\\ . On the other hand if $f^x{}_n$) *i*-s a sequence in S_{hi} such that *I* (x_{3D}) -» Il-tl then considering the functions f^n e $C(X_j)E)$ defined by $f^n(q) = x^n$ for all q e X it is verified that e(^,p)(f_n) -• |Kj| and ||f^n I] = 1 . Thus ||e(<t,p)|j = ||-t|| and for a fixed p e X, e(,p) is a linear isometry on E* into *(C(X,E))*.* The function II on E into C(X,E) is defined by setting II(x) (p) = x for all x e E and p e X . It is verified that II is a linear isometry on E into C(X,E) .

We recall few geometric properties of a Banach space. If B is a Banach space and x *e* S_B then a functional U B * is said to support U_ at x if H|| = 1 = *I* (x) . The cell U_, is said to be smooth at x if there exists one and only one hyperplane of support at x. A Banach space B is said to be smooth if U_B is smooth at all points $x \in S_B$. It is known, Mazur [10], that U_B is smooth at x if and only if (*) $\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} = G(x;y)$ exists for all $y \in B$. Further, if the limit exists then $G(x; \;)$ is a linear functional supporting U_B at x. For each $x \in S_B$ let $v(x)$ be the set of all linear functionals supporting U_B at x. Then v is a set valued mapping on S_B into $2^{S_B^*}$. If follows from Hahn Banach theorem that $v(x) \neq \emptyset$ for all $x \in S_B$. Further if C is the set of smooth points in S_B, it follows from the preceding observations that $v|C$ might be considered as a function on $C \to S_B^*$. The set valued mapping v is called the spherical image map of S_B, Cudia [3].

We proceed to define certain distinguished subsets of a Banach space which are useful later in the paper. An M-set in a Banach space B is a maximal convex subset of S_B. A T-set in B is the half cone of nonnegative multiples of vectors in a M-set. For a discussion of these sets we refer to [8]. Two T-sets T_1, T_2 are said to be discrepant if either $T_1 \cap T_2 = \{0\}$ or if there exists a T-set T_3 such that $T_1 \cap T_3 = \{0\} = T_2 \cap T_3$.

It is verified by applying Zorn's lemma that if $x \in S_B$ then there is a M - set M_x containing x. Further if $x \in S_B$ and $\{x\}$ is a M- set then any two T-sets are discrepant. For if T_0 is the T-set $\{Ax \mid A \geq 0\}$ and T_1 is another T-set then it is verified using the maximality of M- sets that $T_0 \cap T_1 = \{0\}$. In particular it follows that the norm in B is strictly convex then any two T-sets are discrepant for then there exists only one point belonging to a M - set.

We summarize some properties of M- sets which are required in the last section of the paper in the following remark.

Remark 1. If M is a M- set in the Banach space B then it follows from the separation theorem [4], that there exists a linear functional $f \in B^*$ such that $||f|| = 1$ and $\inf_{x \in M} f(x) \geq \sup_{x \in U_B^\circ} f(x)$ where U_B° is the interior of the cell U_B. Hence if H is the hyperplane $f^{-1}(1)$ then $M \subset H$. Further since $H \cap S_B$ is a convex set and M is a maximal convex set we conclude that $H \cap S_B = M$. We refer to H as a hyperplane supporting U_B along M. Further we note that if ther exists a point $x \in M$ and if U_B is smooth at x then from the definition of smooth point it follows the hyperplane supporting U_B along M is unique.

We conclude the preliminary remarks from geometry of Banach spaces by stating the definition of quasi-cylinders and a known result, theorem 5.2 [8] for convenience of reference. Before stating the definition we recall that if C is a convex set in a Banach space B the relative interior of C is the interior of C relative to the affine subspace of B spanned by C. The relative boundary of C is $C \sim$ relative interior C. The relative interior and boundary are respectively denoted by rel-int and rel-bd .

Definition 1 . Let B be a Banach space and D be the set of smooth points in S_D. Then the unit cell $U_\_$. is said to be a quasi-cylinder if it satisfies the following conditions.

(1) Ext U_{J3} is a closed subset of S_{oB} and D U Ext $U_B = S_B$.

(2) There are two antipodal M - sets M^1 and M^2 in S_{oB} such that Ext U_B c M^1 U M^2 and if H is the hyperplane supporting U_B along M^1 then M^1 has a nonempty interior relative to H.

(3) x e D implies x is in the rel-int of a M - set of B .

(4) There exists a point p e M^1 and a closed subspace L of B such that $p + (U_Q$ fl $L) = M^1$.

We proceed to provide an example of a quasi-cylinder. Let $(E, \| \|)$ be a Hilbert space and L be a subspace of E of deficiency 1. Let $x \in L$ and $\|x\| = 1$. Let C be the complete cylinder erected on $L \cap U_{H}$ with generators parallel to x. Let U be the portion of C cut out by the hyperplanes $x + L$ and $-x + L$. Then U is the unit cell of a Banach space $(E, \| \|_0)$ topologically equivalent to $(E, \| \|)$ and U is a quasi-cylinder. From this example of a quasi-cylinder we conclude that if the unit-cell of a Banach space is a quasi-cylinder then the T-sets need not necessarily be discrepant. In this connection we refer to example 4.5 in [8].

For convenience of reference we state the theorem 5.2 in [8] below.

Theorem 1. [Jerison]. Let X, Y be two compact Hausdorff spaces and E be a Banach space such that any two T-sets in E are discrepant. Then X is homeomorphic with Y if there is a linear isometry on $C(X,E)$ onto $C(Y,E)$.

We need some concepts from the theory of vector valued measures. For an account of vector valued measures we refer to Edwards [6] and Dinculeanu [5]. An E-valued Borel measure on a topological space X is a measure defined on the Borel sets of X with values in E. If μ is a Borel measure on X with values in E then the variation of ji is defined by

$$V(ju) = \sup \left\{ \sum_{i=1}^{n} HMV l_x ! \quad I \quad (B_1, B_2, \ldots B_n) \in P \right\}$$

where P is the set of finite Borel partitions of X . When X is a compact Hausdorff space an E-valued Borel measure pt is said to be regular if V *(fj.)* is a regular set function, see in this connection proposition 21 on page 318 in [5]. If $V_V(E)$ is the set of all E-valued regular Borel maesures *i* of finite variation defined on the compact Hausdorff space X then with the usual definitions of addition and Scalar multiplication $V_V(E)$ is a linear space. Further equipped with the norm $\|\mu\| = V(\mu)$, $V_Y(E)$ is a Banach space.

Let $\mu \in V_X(E^*)$ and $S: X \to E$ be a step function i.e. S is of the form $\sum_{i=1}^{n} \chi(B_i) x_i$ where $\{B_i\}_{i=1}^{n}$ is a finite family of pairwise disjoint Borel sets in X and $x_i \in E$ for $1 \leq i \leq n$ and $\chi(B_i)$ is the characteristic function of B_i. Then the Stieltje's integral $\int S d\mu$ is defined by $\int S d\mu = \sum_{i=1}^{n} \mu(B_i)(x_i)$. Since X is a compact Hausdorff space corresponding to any function $f \in C(X,E)$ there exists a sequence of step functions S_n uniformly converging to f. The Stieltje's integral $\int f d\mu$ is defined to be the $\lim_{n \to \infty} \int S_n d\mu$. Since μ is of finite variation the integral exists for all $f \in C(X,E)$. For a detailed account of this theory of integration we refer to Bochner and Taylor [2] and to sections 7 and 8 in Chapter II in [5] and the representation theorem stated below could be deduced from the corollary 2 on page 387 in [5]. In this connection we refer also to Bochner and Taylor [2], Gowurin [7] and Singer [11]. The papers [2], [7] are concerned with the representation problem when X is the unit interval while the more general case when X is an arbitrary compact Hausdorff space has been dealt in [11].

Theorem 2. If X is a compact Hausdorff space there exists a linear isometry σ on $(C(X,E))^*$ onto $V_{\omega}(E^*)$ such that

$$L(f) = \int_X f \, d\sigma(L)$$

for all $L \in (C(X,E))^*$ where the integral is the Stieltje's integral defined in the preceding paragraph.

3. Smooth Points in $C(X,E)$.

We proceed next to characterize the functions $f \in C(X,E)$ such that the norm in $C(X,E)$ is G-differentiable at f. We first establish a lemma useful in the subsequent discussion.

Lemma 1. Let X be a compact Hausdorff space and E be a Banach space. Then

$$\text{Ext}\,(U^*_{\_}) = e(\text{Ext}\, U^*_{\_} \times X)$$

where e is the map defined earlier.

Proof. Let $\ell \in \text{Ext}\, U^*_E$ and $p \in X$. Since $e(\cdot,p)$ is an isometry and $\|\ell\| = 1$, $\|e(\ell,p)\| = 1$. Let $L_1, L_2 \in U_X$ be such that $\ell(l,p) = \frac{L_1+L_2}{2}$. We claim that $L_1 = L_2 = \ell(l,p)$. Let $\mu_1, \mu_2 \in V_X(E^*)$ be such that $\mu_i = \alpha(L_i)$ where α is the isometry described in theorem 2. We claim that if M is a Borel set in X and $p \notin M$ then the variation $\left\| \frac{\mu_1 + \mu_2}{2} \Big| M \right\| = 0$. For if this is not true by the regularity of the measure $\frac{\mu_1 + \mu_2}{2}$ there exists a compact set $C \subset M$ such that if ν is the contraction of $\frac{\mu_1+\mu_2}{2}$ to C then the variation of ν is a positive number δ. Since $\left\|\frac{L_1+L_2}{2}\right\| = 1$, $\left\|\frac{\mu_1+\mu_2}{2}\right\| = 1$. Hence if ν' is the contraction of $\frac{\mu_1+\mu_2}{2}$ to $X \sim C$ then from the definition of variation it follows that $\|\nu\| + \|\nu'\| = \left\|\frac{\mu_1+\mu_2}{2}\right\| = 1$. Thus $\|\nu'\| = 1 - \delta$. Let $\{x_n\}_{n \geq 1}$ be a sequence of vectors in E, $\|x_n\| = 1$ such that $\ell(x_n) \to 1$. Since $p \notin C$ and C is compact there exists a sequence $\{f_n\}_{n \geq 1}$ in $C(X,E)$ such that $\|f_n\| = 1$, $f_n(p) = x_n$ and $f_n(q) = 0$ if $q \in X \sim \{p\}$ for all $n \geq 1$. Now

$$|\ell(f_n(p))| = |\ell(l,p)(f_n)| = \left|\frac{L_1+L_2}{2}(f_n)\right|$$

$$= \left|\int_{X \sim C} f_n \, d\frac{\mu_1+\mu_2}{2}\right| \leq \|\nu'\| \, \|f_n\| \leq 1 - \delta .$$

Hence $\frac{\mu_1 + \mu_2}{2}(M) = 0$ if M is a Borel set and $p \notin M$. Thus it is verified that

$$1 = \left\| \frac{\mu_1 + \mu_2}{2} \right\| = \left\| \frac{\mu_1\{p\} + \mu_2\{p\}}{2} \right\| \leq 1 .$$

Since $\|\mu_1\| = \|\mu_2\| = 1$, $\|\mu_1(p)\| \leq 1$ and $\|\mu_2(p)\| \leq 1$. Hence the preceding equations imply $\|\mu_1(p)\| = \|\mu_2(p)\| = 1$. Thus if $x \in E$ $L_1(\Pi(x)) = \int_X \Pi(x)\,d\mu_1 = \mu_1\{p\}(x)$. It follows similarly that $L_2(\Pi(x)) = \mu_2\{p\}(x)$. Further $\alpha(\ell,p)(\Pi(x)) = \ell(x)$. Hence by our choice of L_1 and L_2 it follows that $\ell = \frac{\mu_1\{p\} + \mu_2\{p\}}{2}$. Since $\ell \in \text{Ext } U_E^*$, $\mu_1\{p\} = \mu_2\{p\}$. Thus $L_1 = L_2$ and $\alpha(\ell,p) \in \text{Ext } U_X^*$.

Conversely let $L \in \text{Ext } U_X^*$. We verify that there exist $\ell \in \text{Ext } U_E^*$ and $p \in X$ such that $L = \alpha(\ell,p)$. Let α be the map assured by theorem 2 and let $\phi(L) = \mu$. Since $\|L\| = 1$, $\|\mu\| = 1$. We claim that there is a point $p \in X$ such that if M is a Borel set and $p \notin M$ then $\mu(M) = 0$. For convenience the contraction of the Borel measure μ to a Borel set $N \subset X$ will be denoted by $\mu|N$.

As a first step we verify that there exists a point $p \in X$ such that $\mu\{p\} \neq 0$. For if $\mu\{p\} = 0$ for all

$p \in X$ then there exists a point $p \in X$ such that whenever N is a neighborhood of p then $\mu(N) \neq 0$. This statement is verified as follows. If for each point $p \in X$ there exists a neighborhood N_p such that $\mu(N_p) = 0$ then since X is a compact space there exists a finite set $\{p_1, p_2, \ldots p_n\}$ $\subset X$ such that $X = \bigcup_{1 \leq i \leq n} N_{p_i}$. Thus $1 = \|\mu\| \leq \sum_{i=1}^{n} \|\mu | N_{p_i}\| = 0.$

Hence we obtain a contradiction. Thus there exists a point $p \in X$ such that for all neighborhoods N of p, $\mu(N) \neq 0$. Since $\mu(p) = 0$ and the measure μ is regular there exist two neighborhoods N_1 and N_2 of p such that $N_1 \subset N_2$ and $0 < \|\mu|N_1\| < \|\mu|N_2\| < 1$. We verify in such a case $\mu \notin \text{Ext } U_X^*$, thus obtaining a contradiction. Let $N_2 \sim N_1 = N_3$ and μ' be the measure in $V^X(E^*)$ defined by $\mu' = \|\mu|N_3\| (\mu|N_1) - \|\mu|N_1\| (\mu|N_3)$. Thus $\mu + \mu' = (\mu|X \sim N_2) + (1 + \|\mu|N_3\|)(\mu|N_1) + (1 - \|\mu|N_1\|)(\mu|N_3)$ and $\mu - \mu' = (\mu|X \sim N_2) + (1 - \|\mu|N_3\|)(\mu|N_1) + (1 + \|\mu|N_1\|)(\mu|N_3)$. From the above equations and the definition of variation it follows that $\|\mu + \mu'\| \leq \|\mu|X \sim N_2\| + \|\mu|N_1\| + \|\mu|N_3\| = \|\mu\| = 1$. Similarly it is verified that $\|\mu - \mu'\| \leq 1$. Since $\mu = \frac{\mu + \mu'}{2} + \frac{\mu - \mu'}{2}$ and $\mu' \neq 0$ it follows that μ is not an extreme point of the unit cell of

$V_V(E^*)$. Since the map a is a linear isometry it follows that $L \notin \text{Ext } U^*_Y$. Thus we conclude that there exists a point $p_0 \in X$ such that $M(P(J \wedge ^\circ$ ' Suppose now that $||\mu\{p_0\}|| < 1$. Then if $Y = X \sim \{p\}$, $0 < ||\mu|Y|| < 1$ and the above argument applied to $\{P_n\}$ and Y in place of N_1 and N_3 will again contradict the assumption that $L \in \text{Ext } U^*_X$. Thus $||\mu\{p_0\}|| = 1$ and for all Borel sets M such that $p_0 \notin M$ it follows that $\mu(M) = 0$. Let $M(P_0) = I$ • If $f \in C(X,E)$ then $L(f) = \int_X f \, d\mu = \mu\{p_0\}f(p_0) = e(\ell, p_0)(f)$.

Hence $L = e(\ell, p_0)$. If $I = \frac{\ell_1 + \ell_2}{2} > ||\ell_1|| = 1 = ||\ell_2||$ then it is verified that $L = \frac{e(\ell_1, p_0) + e(\ell_2, p_0)}{2}$. Since $L \in \text{Ext } U^*_X$ we conclude that $e(\ell_1, p_0) = e(\ell_2, p_0)$. Hence $e(\ell_1, p_0)(n(x)) = e(\ell_2, p_0)(n(x))$ for all $x \in E$. Thus $\ell_1(x) = \ell_2(x)$ for all $x \in E$ i.e. $\ell_1 = \ell_2$. Thus $\ell \in \text{Ext } U^*_E$ and this completes the proof of the lemma.

In the next theorem we provide a characterization of those functions $f \in C(X,E)$ such that the $||\ \ ||$ is G-differentiable at f. The theorem thus generalizes the known result for the case when $E = R$, discussed in Banach [1]. Before proceeding to the theorem we wish to state a couple of remarks.

The Remark 2 is an immediate consequence of the definition of the function e.

Remark 2. Let $x \in E$ with $\|x\| = 1$ and let $\ell \in E^*$ be such that $\|\ell\| = \ell(x) = 1 = \|x\|$. Let $f \in C(X,E)$, $\|f\| = 1$ be such that for some point $q \in X$, $f(q) = x$. Then the linear functional $e(\ell,q)$ is of unit norm and the hyperplane $e(\ell,q)^{-1}(1)$ supports the unit cell of $C(X,E)$ at f.

Remark 3. Let F be an arbitrary Banach space and $x \in F$ with $\|x\| = 1$. Let $B(x) = \{f \mid f \in E^*, \|f\| = 1 = f(x)\}$. Then $B(x)$ is a nonempty w^*-compact convex subset of U_F^* and $\operatorname{Ext} B(x) \subseteq \operatorname{Ext} U_F^*$.

Proof. From the Hahn-Banach theorem it follows that $B(x) \neq \emptyset$. Further it is verified that $B(x)$ is a w^*-closed convex subset of U_F^* and since by Alaoglu's theorem U_F^* is w^*-compact it follows that $B(x)$ is a w^*-compact set. Let $f \in \operatorname{Ext} B(x)$ and let $f = \frac{g_1 + g_2}{2}$, where $g_i \in U_F^*$ for $i = 1,2$. $f \in B(x)$ implies that $\|g_1\| = \|g_2\| = 1 = g_1(x) = g_2(x)$. Thus $g_1, g_2 \in B(x)$. Since $f \in \operatorname{Ext} B(x)$ it is inferred that $g_1 = g_2 = f$. Hence $f \in \operatorname{Ext} U_F^*$.

Theorem 3. If X is a compact Hausdorff space and f e C(X,E),, ||flj = 1 then the unit cell of C(XJ)E) is smooth at f if and only if there exists a point q e X such that 1 = ||f(q)|| > ||f(q')|| for all q' £ q and U_E is smooth at f(q).

Proof. We prove first that the condition in the theorem is necessary. Let the unit cell in C(X,E) be smooth at f . If possible let q_1, q_2 be two distinct points in X such that 1 = HffqAU = $||f(q_2)||$. Let $l_{\pm}, l_2$ e E* be such that

$$||\ell_1|| = ||\ell_2|| = \ell_1(f(q_1)) = \ell_2(f(q_2)) = 1$$

Thus ||e(*1,q1)|| = ||e(t2,q2)|| = e f ^ q ^ f) = e(*2,q2)(f) = 1 where e is the map defined in section 2. Since the unit cell in C(X,E) is smooth at f it follows from the above equations that e(<t1,q1) = e(<t2Jq2) • Hence for g e C(X,,E) $\ell_1(g(q_1)) = \ell_2(g(q_2$ ^ * Let now X}Y e E **be such that** I_1, (x) ^ ^2^^ ' Since X is a compact Hausdorff space there exists a continuous function g_0: X -• E such that $g_0(q_1)$ = x and S Q ^ 0 ^ = v * For such a function g_0 we have ^l^0^cj-l^ ^ ^2^g0^q2^' TJ^1VLS a contradiction is obtained and there exists only one point q e X such that 1 = ||f(q)|| = ||f|| .

We proceed to show that the norm in E is smooth at f(q) . For if the norm is not smooth at f(g) let £', £" be two distinct linear functionals in E* supporting LL, at f (q) . As noted in Remark 2 the hyperplanes $e(l_1, q)^{-1}(1)$ and e^-j^)"^{1}(1) support the unit cell of C(X,E) at f . Since the norm in C(X,E) is smooth at f it follows that e(£,,q) = eflj^q) • Now evaluating these functionals over the E-valued constant functions in C(X,E) it follows that $I_1 = l_2$ contradicting the assumption £, ^ £ • Thus the norm in E is smooth at f and this completes the proof of the necessity of the condition.

Conversely suppose f e C(X,E) and ||f]| = 1 . Let q be the only point in X such that 1 = ||f(q)|| and further let the norm in E be smooth at f(q) . If *I* e E* , with \\l\\ = 1 , supports U_E at f(q) then ||e(*,q)|| = 1 and e(-t^q)$^{-1}$ (1) is a hyperplane supporting the unit cell in C(X,E) at f . If the norm in CCX^E) is not smooth at f and if

B(f) = { L | L e (C(X,E))* , |JL|! = 1 = L(f))

then B(f) is a w*-compact convex subset of the unit cell U_X^* and card B(f) ^_ 2 . By the Krein-MiI'm an theorem. Day [4], B(f) = w*-closure of the convex hull of Ext B(f). Since card B(f) ^> 2 it follows that there are at least two distinct extreme points L_i, i = 1,2 in B(f). From remark 2 L_i are extreme points of U^*_A. We complete the proof by showing that $L_1 = L_2$ = e(£,q) thus obtaining a contradiction. From Lenun[a] 1 it follows that there is a point v e X and a linear functional I_O e E*, $1K_n$ll = 1 such that $L_1 = e(l_Q, r)$. Thus $eM,_Q, r)(f) = -t_Q(f(r)) = 1$.

Since ||f(v)|| 1 |]f|| = 1 = \\l_Q\\ it is verified that ||f(r)|| = 1, From the choice of f we conclude that q = r. Since the cell TX-$_{\hbar i}$ is smooth at f(q), £ = <t_u. Thus $L_x = e(l,q)$.

From a similar argument we conclude that L«$_2$ = ef-C^q). Thus $L_1 = L_2$ and card B(f) = 1. Thus the norm is smooth at f and the condition in the theorem is sufficient.

4. Spaces of maps into Banach spaces

We next proceed to the main theorem of the paper. In the course of the proof of the theorem we make use of the following lemmas.

Lemma 2. If X is a compact Hausdorff space and f is an extreme point of the unit cell of *C(X,E)* then ||f(t)|| = 1 for all t e X.

Proof. Since $\|f\| = 1$ if $\|f(t)\| \neq 1$ for $t \in X$ then there exists a point $t_0 \in X$ such that $\|f(t_0)\| = \delta < 1$. Since X is a compact space and f is a continuous function there exists a compact neighborhood N of t_0 and a number δ', $0 \leq \delta' < 1$ such that $\|f(t)\| \leq \delta'$ for all $t \in N$. Since X is a compact Hausdorff space there exists a continuous function $c : X \to [0, 1 - \delta']$ such that $c(t_0) = 1 - \delta'$ and $c(t) = 0$ if $t \notin N$. Let a be a vector in E such that $\|a\| = 1$. Let g be the function on $X \to E$ defined by $g(t) = c(t)a$. Then $g \in C(X,E)$, $\|f \pm g\| \leq 1$ and $g \neq 0$. Thus $f \notin \text{Ext}\, U_X$. This completes the proof of the Lemma.

Lemma 3. Let M be a convex subset of S_E and H be a hyperplane such that $M = H \cap S_E$. If the interior of M relative to H is nonempty and f is an extreme point of the unit cell U_X of $C(X,E)$ then $f(p) \notin$ relative interior of M for all $p \in X$.

Proof. We note first that the rel-int M is a subset of the interior of M relative to S_E. For let, $\ell \in E^*$, $\|\ell\| = 1$ be such that $H = \ell^{-1}(1)$. Let x be in the rel-int M. Since the interior of M relative to H is non-empty the core of M relative to H is nonempty. Thus the smallest affine space containing M is H . Thus the interior of M relative to H is also the rel-int M. Hence there exists a 6 > 0 such that if ||hj| < 6 and I(h) = 0 then x + h e M . We note that corresponding to each y e E there exists a unique scalar A_y and a unique vector h_y e $l^{-1}(0)$ such that $y = A_Y x + h_Y$. Since y -• A_y is a continuous linear functional, y -• h_y is also a continuous function on E -• E . Let G = {y [$||h_y||$ < 4 and A_y > 1/2 } . By the preceding observations G is a open subset of E. Now if y e G.d S_E then 1 = ||y|| = $||A_y x + h_y||$ = $|A_y|$ | | x + ^/y |l. Since ^/y || < 6 and $I(h_y)$ =0 from the choice of 6 it follows that $||x + \frac{h_y}{y}|| = 1$. Thus A_Y =1 noting that y e G . Hence I(y) =1 i.e. G O S_E c M and x is in the interior of M relative to S_E . To complete the proof of the lemma let f e Ext U . From Lemma 2 it follows that range f c S^ . Now if possible let there be a point p e X such that f(p) e rel int M . Thus f (p) e interior of M relative to S,, as seen from the observation in the preceding paragraph. Since f is a continuous function there exists an open set G , p € G and f (G) c S_E . Let C be a compact neighborhood of p with C c G. Thus f(c) is a compact subset of the interior of M relative to S_E . Thus there exists a

$\delta > 0$ such that if $y \in S_E$ and $\|y - z\| < \delta$ for some $z \in f(x)$ then $y \in$ interior of M relative to S_E. With H and l chosen as in the preceding paragraph it is verified that if $l(h) = 0$ and $\|h\| < \delta$ then $z + h \in M$ for all $z \in f(c)$. Let h be such a fixed vector. Since X is a compact Hausdorff space there exists a continuous function $\varphi : X \to [0,1]$ such that $\varphi(C) \subset \{1\}$ and $\varphi(X \sim G) \subseteq \{0\}$. Let g_1, g_2 be two functions on $X \to E$ such that $g_1(q) = f(q) + \varphi(q)\,h$ and $g_2(q) = f(q) - \varphi(q)\,h$. It is verified that $g_1, g_2 \in C(X,E)$, $\|g_1\| = \|g_2\| = 1$, $f = \frac{g_1+g_2}{2}$, and $g_1 \neq f \neq g_2$. Hence $f \notin \operatorname{Ext} U_X$ and the proof is complete.

Lemma 4. If M is a M- set in a Banach space E then $\operatorname{Ext} M \subset \operatorname{Ext} U_E$.

Proof. Let $e \in \operatorname{Ext} M$ and let H be a hyperplane supporting U_E along M. Let $H = l^{-1}(1)$ for some $l \in E^*$, $\|l\| = 1$. If $e = \frac{y+z}{2}$, $y, z \in U_E$ then it is verified that $l(y) = l(z) = 1$. Thus $y, z \in H$ and $\|y\| = \|z\| = 1$. Hence $y, z \in M$. Since $e \in \operatorname{Ext} M$, $e = y = z$. Thus $e \in \operatorname{Ext} U_E$ and $\operatorname{Ext} M \subset \operatorname{Ext} U_E$.

Lemma 5. Let E be a Banach space with a quasi-cylindrical unit cell. If M is a M- set rel—bd M = Ext M .

Proof. Let x e rel-bd M . If x 4 Ext M then x 4 Ext U_E . Thus from the condition (1) in the definition of a quasi-cylinder it follows that x is a smooth point of S^, . Thus there is only one hyperplane H supporting $U^\wedge_{.hi}$ at x and H fl $S_{w\,hi}$ = M. Since distinct M— sets are in distinct hyperplanes supporting U_ it follows that M is the only M- set such that x e M . Since x is a smooth point from the condition (3) in the definition of a quasi-cylinder it follows that x e rel-intM thus obtaining a contradiction. Hence rel-bd M c Ext M. Since every extreme point M is in the rel-bd M it follows that rel-bd M = Ext M .

Remark 4. From the preceding lemma it follows that if U_E is quasi-cylindrical then for a M- set M in E, M = Conv (Ext M) .

Lemma 6. Let E be as in the preceding lemma. if P_1, P_2 are two distinct M — sets then Card (P_1 n PO <_ 1 .

Proof. If possible let x,y e Pi fl P, and x ≠ Y • Since P_i , i = 1,2 are convex $\frac{x+y}{2}$ e P_1 (1 P_2 . Since

$\frac{x+y}{2}$ & Ext U_{H_i} it is a smooth point in S_E. Thus if H is the hyperplane supporting U_E at $\frac{x+y}{2}$ it follows that $H = H_1 = H_2$ where H_i, $i = 1,2$ are the hyperplanes supporting U_E along P_i, $i = 1,2$. Since distinct M- sets are in distinct hyperplanes of support a contradiction is obtained and the proof of the lemma is complete.

Remark 5. If the Banach space E in the preceding lemma contains two non-discrepant T-sets then as observed in § 1 if M is any M - set then Card $M \geq 2$. Thus from the remark it follows that Card Ext $M \geq 2$. If now M_1, M_2 are the two antipodal M - sets such that Ext $U_E \subset M_1 \cup M_2$ then Ext $M \subset M_1 \cup M_2$. From lemmas 4 and 6 it follows that if $M \neq M_i$, $i = 1,2$ then M has precisely two extreme points one belonging to Ext M_1 and the other belonging to Ext M_2.

Lemma 7. Let E be a Banach with a quasi-cylindrical unit cell. Let M_1, M_2, p and L be as in (2) and (4) of Definition 1. If $e_i \in$ Ext M_i $i = 1,2$ and if $[e_1, e_2] \subset S_E$ then $\frac{e_1 - e_2}{2} = p$.

Proof. Let $V = L \cap U_E$. Hence from (4) of Definition 1 it is seen that $p + V = M_1$. We proceed to verify that

(*) $p + \frac{e_1 + e_2}{2} = e_1$ and $-p + \frac{e_1 + e_2}{2} = e_2$. We note that since $M_1 = p + V$ ($M_2 = -p + V$), $e_1 - p$, $e_2 + p$ are in V.

Hence $\frac{e_1 + e_2}{2} \in V$ since V is convex. Further since $\| \frac{e_1 + e_2}{2} \| = 1$ and $e_1 - p$, $e_2 + p$ are in U_E it is verified that $\| e_1 - p \| = \| e_2 + p \| = 1$ and $[e_1 - p, e_2 + p] \subseteq S_E$. Since $\frac{e_1 + e_2}{2} \in S_0 \sim \text{Ext}\, U_E$ and U_E is a quasi-cylider $\frac{e_1 + e_2}{2}$ is a smooth point of S_E.

Hence there is exactly one hyperplane H supporting U_E at $\frac{e_1 + e_2}{2}$. Since $\frac{e_1 + e_2}{2}$ is a point in $[e_1, e_2]$ and $[e_1 - p, e_2 + p]$ and these line segments are subsets of S_E it is verified that $H \cap S_E \supseteq [e_1, e_2] \cup [e_1 - p, e_2 + p]$. Since H is the only hyperplane supporting U_E at $\frac{e_1 + e_2}{2}$, it follows from Remark 1 that if M is an M - set containing $\frac{e_1 + e_2}{2}$ then $M = H \cap S_E$. It is inferred from Remark 5 that $M = [e_1, e_2]$. Hence $[e_1 - p, e_2 + p] \subset [e_1, e_2]$. If the equations in (*) are false then $e_1 - p \neq e_2 + p$. Since $e_1 - p$, $e_2 + p$ are in the subspace L it is verified that $[e_1, e_2] \subset L$. Hence $e_1 \in V$. Thus $p + e_1 \in M_1$. However, since $p, e_1 \in M_1$ and M_1 is convex, $\frac{p + e_1}{2} \in M_1$. Since $M_1 \subseteq S_E$

it follows that $\|p + e_{\pm}\| = \|\frac{p+e}{2}\| = 1$ which is a contradiction. Since either of the equations in (*) implies $\frac{e_1 - e_2}{2} = p$ the proof is completed.

Lemma 8. Let E be a Banach space containing two non-discrepant T-sets and let U_E be a quasi-cylinder. Then if $f \in C(X,E)$ is an extreme point of U_A then $f(p)$ is in Ext U_E for all $p \in X$.

Proof. Let M_1, M_2 be the pair of antipodal M- sets of U_E such that Ext $U_E \subset M_1 \cup M_2$ with interior of

(Continued)

$M_1(M_2)$ relative to the hyperplane supporting U_E along M_1 (along M_2) non-empty. Let f be as in the hypothesis of the lemma. Then from Lemma 3 it follows that $f(p)$ is not in rel int M_i, $i = 1,2$ for all $p \in X$. Thus if for $p \in X$, $f(p) \notin \operatorname{Ext} U_E$ then $f(p) \notin M_1 \cup M_2$. Since $M_1 \cup M_2$ is closed there is a compact neighborhood C of p such that for all $q \in C$, $f(q) \in U_E \sim (M_1 \cup M_2)$. From Lemma 2 it follows that $f \in e \subset S_E \sim (M_1 \cup M_2)$. From Remarks 4 and 5 it follows that for each $q \in C$ there is a uniquely determined pair of extreme points e_q^i with $e_q^i \in M_i$ $i = 1,2$ such that $f(q) \in [e_q^1, e_q^2]$. Since $f(q) \notin M_i$, $i = 1,2$ there exists a function $\lambda : C \to]0,1[$ such that $f(q) = \lambda(q) e_q^1 + (1 - \lambda(q)) e_q^2$. We verify that λ is a continuous function. For if $\{q_\alpha\}$ is a set in C converging to a point q then since f is a continuous function $\|f(q_\alpha) - f(q)\| \to 0$. Thus if ℓ is the continuous linear functional on E such that $\|\ell\| = 1$ and $M_1 = \ell^{-1}(1) \cap S_E$ then $\ell(f(q_\alpha)) \to \ell(f(q))$. Hence $\lambda(q_\alpha) \to \lambda(q)$ since $\ell(e_{q_\alpha}^1) = 1$ and $\ell(e_{q_\alpha}^2) = -1$. Thus λ is a continuous function. Since C is a compact set there exists an $a > 0$ such that $a < \lambda(q) < 1 - a$ for all $q \in C$. Let $G = \operatorname{int} C$ and C_1 be a compact neighborhood of p such that $C_1 \subset G$. Let φ be a continuous function on $X \to [0,1]$ such that $\varphi(C_1) \subset \{1\}$ and $\varphi(X \sim G) \subset \{0\}$.

Let the functions $g_i : X \to E$ be defined by $g_1(q) = f(q) + \varphi(q) \, a \, [e^1_q - e^2_q]$ and $g_2(q) = f(q) - \varphi(q) \, a \, [e^1_q - e^2_q]$. Since for $q \in C$ $e^i_q \in \operatorname{Ext} M$, $i = 1,2$ and $[e^1_q, e^2_q] \subset S_E$ the preceeding Lemma implies that $\frac{e^1_q - e^2_q}{2} = p$ for all $q \in C$ where p is a fixed vector. Thus the functions g_i, $i = 1,2$ are continuous and $g_1(q) = g_2(q) = f(q)$ if $q \notin G$ and $g_i(q) \in [e^1_q, e^2_q] \subset S_E$ if $q \in G$. Thus $\| g_1 \| = \| g_2 \| = 1$, $g_1 \neq g_2$ and $f = \frac{g_1 + g_2}{2}$. Hence $f \notin \operatorname{Ext} U_X$ contradicting the hypothesis. The proof of the Lemma is complete.

Lemma 9. A set $P \subset C(X,E)$ is a M-set if and only if there is a M-set $M \subset E$ and a point $p \in X$ such that

$$P = \{ f \mid f \in C(X,E), \ \|f\| = 1 \ \text{and} \ f(p) \in M \}$$

Thus each M-set P in $C(X,E)$ could be represented as $p = (M,p)$ where M and p are chosen as above and two M-sets P_1, P_2 where $P_1 = (M_1, p_1)$ and $P_2 = (M_2, p_2)$ are equal if and only if $M_1 = M_2$ and $p_1 = p_2$.

This lemma is an immediate consequence of theorem 4.1 and lemma 4.3 in [8]. Hence the details of a proof are omitted.

Before proceeding to the main theorem of this section we note if X,Y are two compact Hausdorff spaces and *j* : X -* Y is a homeomorphism on X onto Y then the operator T: *C(Y,E)* -* C(X,E) defined by T(f)(p) = f(rp) for f e *C(Y,E),* p e X is verified to be a linear isometry on C(Y,E) onto *C(X,E).* Thus in the subsequent discussion we consider only the converse question.

Theorem 4. Let E be a smooth Banach space such that Ext U_E 4= 0 . Let X,Y be two first countable compact Hausdorff spaces such that there is a linear isometry on C(X,E) onto C(Y^E) . Then X and Y are homeomorphic.

Proof. Let x be an extreme point of U_, . Let T be a linear isometry on C(X,E) onto C(Y,E) . Let T* be the adjoint operator. Then T* is a linear isometry on (C(Y,E))* onto (C(X,E))*. Let $T_{JL} = T^{*-1}$.

Let *I* be the functional in E* supporting U_E at x_o . Since x_o is a smooth point of U_{bj} it is verified that *I* £ Ext U_E* . Thus if p e X it follows from Lemma 1 that e*(I,*p) £ Ext U_{r}* . Since T* preserves extreme points there is an extreme point A e Ext U_Y* such that T*A = *e(l,p).*

Thus from Lemma 1 it follows that there is an extreme point $\ell_p \in \text{Ext } U^*_E$ and a point $p' \in Y$ such that $T^* e(\ell_p, p') = e(\ell, p)$ i.e. for all $\ell(f(p)) = \ell_p(Tf(p'))$ for all $f \in C(X,E)$. Since T^* is 1-1 if $p' = \tau(p)$ it is verified τ is a function on X into Y. We next verify that $\tau(X) = Y$.

Let $q' \in Y$. Let K_{x_0} be the function in $C(X,E)$ defined by $K_{x_0}(q) = x_0$ for all $q \in X$. Since $K_{x_0} \in \text{Ext } U_{\hat{X}}$ and T is a linear isometry $TK_{x_0} \in \text{Ext } U_{\hat{Y}}$. Thus from Lemma 2 it follows that $\|TK_{x_0}(q')\| = 1$. Let $TK_{x_0}(q') = x'_0$. Let $m' \in E^*$ be such that $m'(x'_0) = \|m'\| = 1$. Since E is smooth such a functional $m' \in \text{Ext } U^*_E$. Hence $e(m', q') \in \text{Ext } U^*_{\hat{Y}}$. Then $e(m', q') \in \text{Ext } U^*_{\hat{Y}}$. Hence there is an extreme point $e(\ell_1, p_1)$ of $U^*_{\hat{X}}$ such that $T^*e(m', q') = e(\ell_1, p_1)$. Thus $\ell_1(K_{x_0}(p_1)) = m'(TK_{x_0}(q')) = 1$. Since $K_{x_0}(p_1) = x_0$ and $\|\ell_1\| = 1$, ℓ_1 supports U_E at x_0. Since x_0 is a smooth point of U_E, $\ell_1 = \ell$. Hence $T^*e(m', q') = e(\ell, p_1)$ i.e. $\tau(p_1) = q'$. Thus $\tau(X) = Y$.

(Continued)

Next we verify that T is 1 - 1 . If possible let p,q € X be such that r(p) = T(q) = r' . "Thus there are functionals M'_1, M^ in E* of unit norm such that -t(f(p)) = M^Tf(r')) and *(f(q)) = M£(Tf(r')) 'for all f eC(X,E) . Thus in particular 1 = $I(x_o)$ = $I(K_{x_o}(p))$ = $M'_X(T K_{x_o}(r^1))$ = 1 • Similarly it is verified that $M_i(TK_{x_o}(r^1))$ = 1 . Since $||T K_{x_o}(r')||$ = 1 and E is smooth it follows that M^ = Mj . Thus it is verified that T*e (M'_1, r') = *e(t,p) = e(l,q)* . Since T* is an isometry it follows that p = q . Hence T is 1 - 1 .

We proceed now to show that T is a continuous mapping. Since X and Y are first countable Hausdorff spaces it is enough to verify that if $\{p_n\}_{n\geq 1}$ is a sequence in X converging to the point p in X then $T(p_n)$ "• T(p) • Let us denote for convenience $r(p_n) = p'_n$ and T(P) = p^1 • Let T*e $(l^*_n, p^\wedge)$ = $e(£,p_n)$ for all n where we note that $[V_n\}_{n\geq 1}$ c Ext U_E^ . If p^ -» p^1 since X is a first countable compact Hausdorff space there exists a convergent subsequence $\{p_{n_i}'\}$ of $\{p_n\}$ such that $\lim p'_{n_i}$ = q' ≠ P^1 • Consider the function f^R_o as defined in the preceding paragraph.

Since $\|f^P_{x_0}(p_{n_i}) - f^P_0(p)\| \to 0$ it follows that $*(f^P_0(p_{n_i})) \to$ lit$*_0$(P)) • Thus $V_1(Tf^P_0(p'_{n_1})) \to 1$. Since $\|*'_{n_1}\| = 1$, $\|Tf^P_{x_0}(p'_{n_i}) - T f^P_{x_0}(q')\| \to 0$ and $\|T f^P_0(q')\| \leq 1$ it is verified that $\|T f^P_{x_0}(q')\| = 1$. From the equations

$l(f^l_0(p)) = 1 = V(T f^P_0(p'))$ and $\|*'\| = 1 = \|T f^P_0\|$ it follows that $\|T f^p_0(p')\| = 1$ • From Theorem 3 it is seen that $f^P_{x_0}$ is a smooth point of U_V. Hence $T f^P_{x_0}$ is a smooth point of U. Further $\|f^P_0\| = \|T f^P_0\| = 1$. Since $\|T f^P_{x_0}(p')\| = 1$ once again appealing to Theorem 3 we conelude that $p' = q'$. Thus every convergent subsequence of $\{r(p_n)\}$ converges to T(p) . Since Y is a first countable compact Hausdorff space it is verified that the sequence $\{r(P_n)\}$ converges to -r(p) .

Next we proceed to the case when the unit cell U_E of E is quasi-cylindrical. As noted in § 1 the T- sets are not necessarily discrepant and thus Jerison's theorem does not apply to this case. However we show below that if X^Y are as in the preceding theorem and U_ is quasi-cylindrical and the linear isometry T fulfills an additional condition then we have an analogue of the preceding theorem.

We shall denote the constant function in $C(X,E)$ with range in $\{x\}$ by K .

Theorem 5. Let E be a Banach space with a quasi-cylindrical unit cell U_E and X,Y be first cour. :able compact Hausdorff spaces. Let T be a linear isometry on $C(X,E)$ onto $C(Y^E)$ such that corresponding to each point $t \in X$ there are at least two points $x_n,,x_o \in \text{Ext } U$ for which $T K_{x_1}(t) \wedge T K_{x_2}(t)$. Then X is homeomorphic with Y.

Proof. If the T- sets in E are discrepant then Jerison's theorem applies and the conclusion follows. Thus we can assume there exist pairs of non-discrepant T-sets in E . Hence as noted in Remark 5 if M is a M-set in E then Card $M \geq 2$. Thus since M is a convex subset of S_E it follows that $M \sim \text{Ext } U_E \neq \$$ • Hence from the definition of a quasi-cylinder it follows that there are points $x \in M$ such that U_{Ei} is smooth at x . Hence corresponding to a M- set M there is one and only one functional m in $E^*(\|m\| = 1)$ such that m supports $U\_$ along M . From the uniqueness of m it is further verified that such a functional m is in $\text{Ext } U^*_E$.

Let M_1,M_2 be the pair of antipodal M- sets (guaranteed by (2) in Definition 1) such that $\text{Ext } U_{Ci} \subset M_1 \cup M_2$. Let

$-I_1$ ($f_2 = -f_1$) be the functionals in E^* supporting U_E along M_1 (M_2). Then as shown in the proof of the preceding theorem there exists a function $\tau : X \to Y$ such that $T^*e(\xi, p) = e(\xi', \tau(p))$ where ν is an extreme point of U_{E^*} depending only on p. We proceed to show that τ maps X homeomorphically onto Y.

We verify first that $\tau(X) = Y$. Let $p' \in Y$. Let $\dim E \geq 3$. As a initial step we assert that there are at least two points $y, z \in \operatorname{Ext} U_E \cap M_1 = \operatorname{Ext} M_1$ such that $T K_y(p')$, $T K_z(p')$ are in the same M- set. Since $\dim E \geq 3$ and interior of M_1 relative to the hyperplane $f_1^{-1}(1)$ is nonempty it follows that $\operatorname{Ext} M_1 = \text{rel-bd } M_1$ is a infinite set. (See Lemma 5). Let $x \in \operatorname{Ext} U_E \cap M_1$. Thus $K_x \in \operatorname{Ext} U_A$. Hence $T K_x \in \operatorname{Ext} U_Y$. Thus from Lemma 8 we conclude that $T K^x(p') \in \operatorname{Ext} U_E$. Thus $T K(p')$ is in M_1 or M_2. Let $T K_x(p') \in M_1$. If for some $y \in \operatorname{Ext} U_E \cap M_1$ $T K^y(p') \in M_1$ then the assertion is verified at once. If for all $y \in \operatorname{Ext} M_1 \sim \{x\}$, $T K^y(p') \in M_2$ then since $\operatorname{Ext} M_1$ is an infinite set the assertion is verified. The case when $T K(p') \in M_2$ is similarly dealt. If $\dim E = 2$ then since U_E is a quasi-cylinder S^E is a parallelogram and there are only two pairs of antipodal M-sets in E and each M-set is a non-degenerate line segment. If M^i $(i = 1,2,3,4)$ are the four M-sets of E it is directly verified that there is an i_o such that if $\{y, z\} = \text{Ext } M_{i_o}$ then $TK_y(p')$, $TK_z(p')$ are in the same M-set. Without loss of generality we can assume that $i_o = 1$, since M_{i_o} and its antipodal set could be taken for the sets M_1 and M_2 considered in the second paragraph above.

Let y,z be two points in Ext M_1 with the property asserted in the preceding paragraph. From Lemma 6 it follows that there is only one M-set, say M', such that $\{TK_y(p'), TK_z(p')\} \subset M'$. Consider the M-set (M', p') of $C(Y,E)$. For the definition of (M', p') we refer to Lemma 8. Since T is a linear isometry $T^{-1}(M', p')$ is an M-set of $C(X,E)$. Thus from Lemma 8 it follows that there exists exactly one M-set L in E and a unique point $p \in X$ such that $T^{-1}(M', p') = (L, p)$. We verify that $L = M_1$. Since $\{TK_y(p'), TK_z(p')\} \subset M'$, $\frac{T(K_y + K_z)}{2}(p') = TK_{\frac{y+z}{2}}(p') \in M'$. Since $T^{-1}(M', p') = (L,p)$, $K_{\frac{y+z}{2}}(p) \in L$. Since $\frac{y+z}{2} \in M_1 \sim \text{Ext } U_E$ and U_E is a quasi-cylinder $\frac{y+z}{2}$ is a smooth point of S_E. Thus M_1 is the only M-set such that $\frac{y+z}{2} \in M_1$. Hence $M_1 = L$. Thus $T(M_1, p) = (M', p')$. We proceed to show that if m' is the functional in E^* supporting U_E along M' and

$f \in C(X,E)$ then $e(l_1,p)(f) = e(m_1',p')(Tf)$. Let $\frac{y+z}{2} = v$. As already shown in paragraph 3 of the proof of preceding theorem there exists a function $f_v^p \in C(X,E)$ such that $f_v^p(p) = v$ and $\|f_v^p(q)\| < 1$ if $q \in X \sim \{p\}$. Further as shown there the norm in $C(X,E)$ is G-differentiable at f_v^p and the norm in $C(Y,E)$ is G-differentiable at $T f_v^p$. Since $f_v^p(p) \in M_1$ and $T(M_1,p) = (M',p')$ it is verified that $T f_v^p(p') \in M'$. Thus the linear functionals $e(l_1,p)$ and $e(m_1', p')$ are the Gateux gradients of the norms in $C(X,E)$ and $C(Y,E)$ at f_v^p and $T f_v^p$ respectively. Thus arguing as in paragraph 3 of the proof of Theorem 4 we conclude that if $f \in C(X,E)$ then $e(l_1, p)(f) = e(m_1', p')(Tf)$ i.e. $T^*e(m_1', p') = e(l_1,p)$. Hence $T(P) = p'$ and r maps X onto Y.

Next we proceed to verify that r is 1 - 1. Let p,q be two points in X such that $r(p) = T(q) = t$. From the definition of T it follows there exist two functionals m_1, m_2 in Ext U_E* such that (a) $T^*e(m_1, t) = e(l_1,p)$ and (b) $T^*e(m_2, t) = e(l_2,q)$. Let $T(M_1, p) = (M_1', p')$ and $T(M_1, q) = (M_2', q')$. Now let y be a smooth point in M_1.

Consider a function f^p_y e C(X,E) such that $f^p_y(p)$ = y and $||f^p_y(s)|| < 1$ for s e X ~ {p} . Since f^p_y e (M_1, p) , $Tf^p_y(p')$ e (M'_1 , p') . Since f^p_y is a smooth point in U_X T f^p_y is a smooth point in U_Y . Hence from Theorem 3 it is concluded that p' is the only point in Y such that $||Tf^p_y(p')||$ = 1. Further $Tf^p_y(p')$ is a smooth point of U_E . The equation (a) implies $m_1(Tf^p_y(p'))$ = -t(f^p_y(p)) = 1. Since ||m_1|| = 1 and ||T f^p_y || =1 it follows from the preceding equations $||Tf^p_y(t)||$= 1 . Thus p' = t . Hence $m_1(Tf^p_y(t))$= 1. Since $Tf^p_y(t)$ = $Tf^p_y(p')$ is a smooth point of S_{U_E} in M'_1 m'_1 is the functional in E* supporting U_E along M'_1 . Similarly it is verified that m'_2 is the functional in E* supporting U_E along M'_2 . From the additional hypothesis on the linear isometry T in the statement of the theorem it follows that there are two points x_1 ,x_2 e Ext M_1 , $x_1 \neq x_2$ such that T K_{x_1}(t) = T K_{x_2}(t) . Since K_{x_i} e (M_1 , p) for i = 1,2 from our choice of M_1 and M_2 it is verified that T K_{x_i} e (M'_j , t) for i = 1,2 and j = 1,2 . Thus T K_{x_i}(t) e M'_1 n M'_2). Hence from Lemma 6 it is inferred that $M'_1 = M'_2$. Since there are smooth points in a M- set of E and m'_i support U_E along

M^ we conclude that m^ = m^ . Thus T*e (m^ , t) = *e{l₁* , p) = e(£ᵢ , q) . Since e is 1 - 1 it follows that p = q and T is 1 - 1 .

The proof of the part that *r* is a continuous mapping is exactly same as the corresponding assertion in the proof of the preceding theorem after choosing for x_o a fixed smooth point of U_{X_J} in M_X . Thus the details of a proof are omitted.

As in the preceding theorem it follows that T maps X homeomorphically onto Y . The proof of the theorem is completed.

In conclusion we mention the following unsettled problems 1) If E is a smooth Banach space and Ext U_E 4 0 then must it be true that the T - sets in E are discrepant. 2) Is it necessary for the linear isometry T to fulfill the additional hypothesis in Theorem 5 for the conclusion of the Theorem.

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