A DISCONJUGACY CRITERION
FOR SELF-ADJOINT LINEAR
DIFFERENTIAL EQUATIONS

by

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A linear differential equation

$$y^{(n)} + p_{n-1}y^{(n-1)} + \ldots + p_o y = 0$$

with real continuous coefficients p_k is said to be disconjugate on the real interval $[a,\infty)$ if none of its nontrivial solutions have more than $n-1$ zeros in $[a,\infty)$ (where the zeros are counted with their multiplicities). The problem as to how the disconjugacy or non-disconjugacy of an equation is reflected in its coefficients in the case of general n has been treated by many authors (cf. [1], [4], [5], [6], [8], [9], [10], [11], [12], [14], [15], [16]); in the case $n = 2$ (and, to a lesser extent, the cases $n = 3,4$) there exists a very considerable body of literature, references to which can be found in the recent book of C. A. Swanson [17].

In the present paper we shall address ourselves to the particular differential equation

$$y^{(2n)} - (-1)^n p(x)y = 0, \qquad n \geq 1, \tag{1}$$

where $p(x)$ is nonnegative and piecewise continuous in the interval under consideration. The equation is self-adjoint, and this is essential for the success of the method we shall employ. However, the particularly simple form of the equation (1) is dictated by the desire to keep the analysis as simple as possible. With suitable modifications, the method may also be applied to equations of the form $L^*Ly - (-1)^n p(x)y = 0$, where L is a disconjugate operator of order n (i.e., the n-th order equation $Lu = 0$ is disconjugate) and L^* is the adjoint operator.

We now state our principal results.

<u>Theorem I. Let</u> $p(x)$ <u>be nonnegative and piecewise continuous on the interval</u> $[a,\infty)$, <u>and let</u> γ_n, δ_n <u>be the constants</u>

$$\gamma_n = \frac{1}{2}[1\cdot 3\cdot \ldots \cdot (2n-1)]^{\frac{1}{n}}, \quad n = 1,2,\ldots \tag{2}$$

$$\delta_n = \frac{\gamma_n^{1-n}}{n} \sum_{k=0}^{n-1} \frac{(n+k)!}{(2k+1)k!}, \quad n = 2,3,\ldots; \quad \delta_1 = \log 2. \tag{3}$$

<u>In order that equation</u> (1) <u>be disconjugate in</u> $[a,\infty)$, <u>it is necessary that</u>

$$\int_\alpha^\beta p^{\frac{1}{2n}}\,dx \le \gamma_n \log\frac{\beta-a}{\alpha-a} + \delta_n \tag{4}$$

<u>and sufficient that</u>

$$\int_\alpha^\beta p^{\frac{1}{2n}}\,dx \le \gamma_n \log\frac{\beta-a}{\alpha-a}. \tag{5}$$

<u>for all</u> α,β <u>such that</u> $a < \alpha < \beta < \infty$.

It is evident from the formulation of the theorem that the constant γ_n is the best possible in both cases. However, the value (3) for the constant δ_n in (4) is not the smallest possible, except in the case $n = 1$ in which, accordingly, the sharp inequality reads

$$(6) \qquad \int_\alpha^\beta \sqrt{p}\, dx \leq \log \frac{\beta-a}{\alpha-a} + \log 2.$$

Indeed, in the latter case it is possible to find the precise maximum for the left-hand side of (6) if α, β are given.

Theorem II. If the equation

$$y'' + p(x)y = 0$$

$(p(x) \geq 0)$ *is disconjugate in* $[a,\infty)$, *then*

$$(7) \qquad \int_\alpha^\beta \sqrt{p}\, dx \leq \frac{1}{2} \log \frac{\sqrt{\beta-a} + \sqrt{\beta-\alpha}}{\sqrt{\beta-a} - \sqrt{\beta-\alpha}} = \frac{1}{2} \log \frac{\beta-a}{\alpha-a} + \log \left(1+ \sqrt{\frac{\beta-\alpha}{\beta-a}}\right)$$

for $a < \alpha < \beta < b$. *The sign of equality in* (7) *is possible only if* $p \equiv 0$ *for* $x \in (a,\alpha)$ *and* $x \in (\beta,\infty)$, *and*

$$(8) \qquad p = [1 - (\beta-x)^2(\beta-a)^{-1}(\beta-\alpha)^{-1}]^{-1}$$

for $x \in (\alpha,\beta)$.

To prove Theorem I, we start out from the obvious fact that, under the assumptions made, the eigenvalue problem

$$y^{(2n)} - (-1)^n \lambda p(x) y = 0$$

(9)
$$y(a) = y'(a) = \dots = y^{(n-1)}(a) = y(\beta) = y'(\beta) = \dots$$
$$= y^{(n-1)}(\beta) = 0$$

cannot have a solution for $\lambda = 1$. By classical results, λ is positive, it increases for decreasing β, and $\lambda \to \infty$ as $\beta \to a$. We may therefore conclude that $\lambda > 1$ for all β in (a,∞). This, in turn, has the consequence that $\mu > 1$, where μ $(\mu < \lambda)$ is the lowest eigenvalue of the problem

$$y^{(2n)} - (-1)^n \mu p(x) y = 0$$

(10)
$$y(a) = y'(a) = \dots = y^{(n-1)}(a) = 0 = y^{(n)}(\beta) =$$
$$y^{(n+1)}(\beta) = \dots = y^{(2n-1)}(\beta)$$

(cf. [11], [3], [16], for the cases $n = 1$, $n = 2$, and general n, respectively). Next, we use the classical fact (cf., e.g., [2]) that μ is characterized by the minimum property

(11)
$$\mu = \min_y \frac{\int_a^\beta [y^{(n)}]^2 dx}{\int_a^\beta p y^2 dx},$$

where y ranges over the class of all the functions in $D^n[a,\beta]$ which satisfy the conditions

$$(12) \qquad y(a) = y'(a) = \dots = y^{(n-1)}(a) = 0.$$

The boundary conditions in (10) which refer to the point β do not have to be taken into account in minimizing the Rayleigh quotient (11).

Since the disconjugacy of equation (1) in $[a,\infty)$ implies that $\mu > 1$ for all $\beta \in (a,\infty)$, we may conclude from (11) that

$$(13) \qquad \int_a^\beta py^2 dx \le \int_a^\beta [y^{(n)}]^2 dx, \qquad \beta \in (a,\infty),$$

where y is any function of $D^n[a,\beta]$ which satisfies the conditions (12) for $x = a$. By choosing special functions y which possess these properties it is possible to obtain a great variety of necessary disconjugacy criteria for equation (1) [[11],[3],[16]). However, inequality (4) cannot be obtained by simply substituting a suitable function y in (13). As our proof will show, a certain amount of additional manipulation is required.

We denote by p_o the function

$$(14) \qquad p_o = \frac{\eta}{(x-a)^{2n}},$$

where

$$(15) \qquad \eta = 4^{-n}[1\cdot 3\dots(2n-1)]^2 = \gamma_n^{2n}$$

(γ_n being the constant (2)), and we define the function y by

$$p_o^{1-\frac{1}{2n}} y^2 = 1, \qquad x \in [\alpha,\beta], \tag{16}$$

where $\alpha \in (a,\beta)$, and all roots of positive numbers are taken to be positive. In the interval $[a,\alpha)$, y will be taken to be a polynomial of the form

$$y = (x-a)^n[a_o + a_1x + \ldots + a_{n-1}x^{n-1}] = (x-a)^n Q(x), \tag{17}$$

whose coefficients are to be determined by the conditions that $y, y', \ldots, y^{(n-1)}$ should be continuous at the point $x = \alpha$. Evidently, $y \in D^n[a,\beta]$, and y satisfies the conditions (12).

By the Hölder inequality, we have

$$\begin{aligned}\int_\alpha^\beta p^{\frac{1}{2n}}\,dx &= \int_\alpha^\beta (pp_o^{-1+\frac{1}{2n}})^{\frac{1}{2n}} (p_o^{\frac{1}{2n}})^{1-\frac{1}{2n}}\,dx \\ &\leq [\int_\alpha^\beta pp_o^{-1+\frac{1}{2n}}\,dx]^{\frac{1}{2n}} [\int_\alpha^\beta p_o^{\frac{1}{2n}}\,dx]^{1-\frac{1}{2n}}\end{aligned} \tag{18}$$

and thus, by (16),

$$\int_\alpha^\beta p^{\frac{1}{2n}}\,dx \leq [\int_\alpha^\beta p_o^{\frac{1}{2n}}dx]^{1-\frac{1}{2n}} [\int_\alpha^\beta py^2dx]^{\frac{1}{2n}}.$$

Using (13) and the fact that $p \geq 0$, we have

$$\int_\alpha^\beta py^2dx \leq \int_a^\beta py^2dx \leq \int_a^\beta [y^{(n)}]^2\,dx,$$

and the preceding inequality thus leads to

$$\int_\alpha^\beta p^{\frac{1}{2n}}\,dx \leq \left[\int_\alpha^\beta p_o^{\frac{1}{2n}}\,dx\right]^{1-\frac{1}{2n}} \left[B + \int_\alpha^\beta [y^{(n)}]^2\,dx\right]^{\frac{1}{2n}},$$

where

$$B = \int_a^\alpha [y^{(n)}]^2\,dx. \tag{19}$$

By (14) and (16),

$$y = \eta^{\frac{1}{4n}-\frac{1}{2}}(x-a)^{n-\frac{1}{2}} \qquad x \in (\alpha,\beta) \tag{20}$$

and therefore

$$y^{(n)} = \frac{\eta^{\frac{1}{4n}-\frac{1}{2}}[1\cdot 3\ldots(2n-1)]}{2^n(x-a)^{1/2}}.$$

Hence, by (15) and (14),

$$[y^{(n)}]^2 = \eta^{\frac{1}{2n}}(x-a)^{-1} = p_o^{\frac{1}{2n}}, \quad x \in (\alpha,\beta).$$

Substituting this in the last inequality, and using the relation between the geometric and arithmetic means, we obtain

$$\int_\alpha^\beta p^{\frac{1}{2n}}\,dx \leq \left[\int_\alpha^\beta p_o^{\frac{1}{2n}}\,dx\right]^{1-\frac{1}{2n}} \left[B + \int_\alpha^\beta p_o^{\frac{1}{2n}}\,dx\right]^{\frac{1}{2n}}$$

$$\leq \left(1-\frac{1}{2n}\right) \int_\alpha^\beta p_o^{\frac{1}{2n}}\,dx + \frac{1}{2n}\left[B + \int_\alpha^\beta p_o^{\frac{1}{2n}}\,dx\right],$$

i.e.,

$$\int_{\alpha}^{\beta} p^{\frac{1}{2n}}\,dx \le \int_{\alpha}^{\beta} p_o^{\frac{1}{2n}}\,dx + \frac{B}{2n}. \tag{21}$$

Since, by (14) and (15),

$$\int_{\alpha}^{\beta} p_o^{\frac{1}{2n}}\,dx = \gamma_n \log \frac{\beta-a}{\alpha-a}\,, \tag{22}$$

where γ_n is the constant (2), this will prove inequality (4) if we can show that $B = 2n\delta_n$, where δ is the constant (3).

The polynomial $Q(x)$ in (17) is determined by the conditions

$$[(x-a)^n Q(x)]^{(k)}\Big|_{x=\alpha} = y^{(k)}(\alpha), \quad k = 0,1,\dots,n-1, \tag{23}$$

where y is the function (20). Equivalently, these conditions may be stated in the form

$$Q^{(k)}(\alpha) = \eta^{\frac{1}{4n}-\frac{1}{2}}[(x-a)^{-\frac{1}{2}}]^{(k)}\Big|_{x=\alpha}, \quad k = 0,\dots,n-1.$$

Since

$$\begin{aligned}(x-a)^{-\frac{1}{2}} &= (\alpha-a)^{-\frac{1}{2}}\left[1 + \frac{x-\alpha}{\alpha-a}\right]^{-\frac{1}{2}}\\ &= (\alpha-a)^{-\frac{1}{2}} \sum_{\nu=0}^{\infty} c_\nu \left(\frac{x-\alpha}{\alpha-a}\right)^{\nu},\end{aligned}$$

where the c_ν are given by

$$(1+t)^{-\frac{1}{2}} = \sum_{\nu=0}^{\infty} c_\nu t^\nu, \tag{24}$$

it follows from (23) and Taylor's theorem that

$$(25) \qquad Q(x) = \eta^{\frac{1}{4n}-\frac{1}{2}}(\alpha-a)^{-\frac{1}{2}} \sum_{\nu=0}^{n-1} c_\nu \left(\frac{x-\alpha}{\alpha-a}\right)^\nu .$$

Integrating the right-hand side of (19) by parts, and observing that $y^{(2n)} \equiv 0$ in (a,α) and that $y(a) = y'(a) = \ldots = y^{(n-1)}(a) = 0$, we have

$$(26) \qquad B = [y^{(n)}y^{(n-1)} - y^{(n+1)}y^{(n-2)} + - \ldots + (-1)^{n-1}y^{(2n-1)}y]_{x=\alpha} .$$

Since $y = (x-a)^n Q(x)$ in (a,α), it follows from (25) that

$$y^{(n+k)}(\alpha) = \eta^{\frac{1}{2n}-\frac{1}{2}} c_k (n+k)!\,(\alpha-k)^{-k-\frac{1}{2}}$$

for $k = 0,1,\ldots,n-1$. Computing the derivatives of order $< n$ from (20), and using the value of c_k given by (24), we obtain

$$y^{(n+k)}(\alpha)\,y^{(n-k-1)}(\alpha) = (-1)^k \eta^{\frac{1}{2n}-1} \frac{(n+k)!}{k!}\left(\frac{2}{2k+1}\right) \frac{1\cdot 3 \ldots (2n-1)}{2^n}$$

for $k = 0,1,\ldots,n-1$. Substituting these expressions in (26), and using (15), we find that $B = 2n\delta_n$, where δ_n is the constant (3). Because of (21) and (22) this completes the proof of the inequality (4).

To show that the validity of (5) for all α,β such that $a < \alpha < \beta < \infty$ is a sufficient condition for the disconjugacy of equation (1) in $[a,\infty)$, we note that the equation

(27) $$y^{(2n)} - (-1)^n \frac{\gamma^{2n}}{(x-a)^{2n}} \, y = 0$$

is known to be disconjugate in (a,∞) [5]. Since, by (5),

$$0 < p(x) \leq \frac{\gamma_n^{2n}}{(x-a)^{2n}}, \qquad a < x < \infty,$$

it follows therefore from the generalization of the Sturm comparison theorem to equations of the form (1) ([8],[13]) that the disconjugacy of equation (27) implies that of equation (1). This completes the proof of Theorem I.

Turning now to the proof of Theorem II, we begin by discussing the more general problem of determining the supremum of

(28) $$\int_\alpha^\beta p^{\frac{1}{2n}} \, dx$$

under the assumption that equation (1) is disconjugate in $[a,\infty)$. Although we cannot solve this problem except in the case $n = 1$ we shall show that, in all cases, a complete answer can be obtained if we use a slightly more restrictive notion of disconjugacy, and if it can be shown that a certain nonlinear boundary value problem has a solution. We shall say that equation (1) is disconjugate on $[a,\infty)$ in the sense of Reid [15] if none of its nontrivial solutions can have two n-th order zeros. For $n = 1$, the two definitions of disconjugacy coincide, and the same is true -- for less obvious reasons -- for $n = 2$ (and equations of the form (1)) [7]. For general n, disconjugacy in the sense of Reid is evidently the more restrictive condition. However, it arises in a natural manner if the subject is approached from a calculus-of-variations point of view.

The relation between the problem of maximizing the expression (28) for disconjugate equations (1) and the possibility of solving a certain nonlinear boundary value problem is described in the following statement.

Suppose that the nonlinear differential equation

$$(29) \qquad u^{(2n)} - (-1)^n H_{\alpha,\beta}(x) u^{-\frac{2n+1}{2n-1}} = 0$$

(*where* $a < \alpha < \beta < \infty$ *and* $H_{\alpha,\beta}(x)$ *denotes the characteristic function of the interval* $[a,b]$) *has a solution* u *in* $[a,\beta]$ *which is positive in* $[\alpha,\beta]$ *and satisfies the boundary conditions*

$$(30) \qquad u(a) = u'(a) = \ldots = u^{(n-1)}(a) = 0 = u^{(n)}(\beta) = u^{(n+1)}(\beta) = \ldots = u^{(2n-1)}(\beta).$$

If the function p_o *is defined in* $[a,\infty)$ *by*

$$(31) \qquad \begin{aligned} p_o &= 0, \quad x \in [a,\alpha),\ x > \beta \\ p_o &= u^{-\frac{4n}{2n-1}}, \quad x \in [\alpha,\beta], \end{aligned}$$

then the linear differential equation

(32) $$u^{(2n)} - (-1)^n p_o u = 0$$

is disconjugate in the sense of Reid in $[a,\infty)$. *If*

(33) $$y^{(2n)} - (-1)^n py = 0$$

is another equation which is disconjugate in $[a,\infty)$, *then*

(34) $$\int_\alpha^\beta p^{\frac{1}{2n}}\,dx \le \int_\alpha^\beta p_o^{\frac{1}{2n}}\,dx.$$

The usefulness of this statement depends of course on the possibility of constructing a solution of (29) which has the specified properties. Unfortunately, we are unable to do this except in the case $n = 1$, in which this construction will provide the proof of Theorem II.

To establish the preceding statement, we first note that, because of the definition (31) of the function p_o, the solution u of (29) is also a solution of the linear equation (32) in $[a,\beta]$. Indeed, u is the first eigenfunction of the problem (10) (with p_o instead of p), with the eigenvalue $\mu = 1$. Since $p_o = 0$ for $x > \beta$, elementary considerations involving the Rayleigh quotient (11) show that μ will retain the value 1 if the interval $[a,\beta]$ is replaced by $[a,\gamma]$, where $\gamma > \beta$. Since

$\mu < \lambda$, where λ is the lowest eigenvalue of the problem (9) (with p_o, γ instead of p, β, respectively), we have $\lambda > 1$ for all γ, and it follows that (32) cannot have a solution with two n-th order zeros in $[a,\infty)$. The equation is thus found to be disconjugate in the sense of Reid.

To establish the extremal property (34), we use the inequality (18), where p_o now denotes the function defined in (31). This yields

$$\int_\alpha^\beta p^{\frac{1}{2n}}\,dx \le \left[\int_\alpha^\beta pu^2 dx\right]^{\frac{1}{2n}} \left[\int_\alpha^\beta p_o^{\frac{1}{2n}}\,dx\right]^{1-\frac{1}{2n}} . \tag{35}$$

Since $p \ge 0$ and equation (1) is disconjugate in $[a,\infty)$ we have, in accordance with (13),

$$\int_\alpha^\beta pu^2 dx \le \int_a^\beta pu^2 dx \le \int_a^\beta [u^{(n)}]^2 dx.$$

Computing the latter integral from (32) (and observing the boundary conditions (30)), we obtain

$$\int_\alpha^\beta pu^2 dx \le \int_a^\beta p_o u^2 dx = \int_\alpha^\beta p_o u^2 dx = \int_\alpha^\beta p_o^{\frac{1}{2n}} dx,$$

the last equality following from (31). Combining this with (35), we arrive at the inequality (34).

For $n = 1$, the boundary value problem (29)-(30) possesses the elementary solution

$$u = (x-a)[(\beta-a)(\alpha-a)]^{-\frac{1}{2}}, \quad x \in [a,\alpha],$$

$$u = [1-(\beta-x)^2(\beta-a)^{-1}(\beta-\alpha)^{-1}]^{\frac{1}{2}}, \quad x \in [\alpha,\beta].$$

By (31), we have therefore

$$p_o = [1-(\beta-x)^2(\beta-a)^{-1}(\beta-\alpha)^{-1}]^{-1}, \quad x \in [\alpha,\beta].$$

Using this expression to compute the integral on the right-hand side of (34), we obtain (7). The uniqueness assertion of Theorem II is evident from the way the Hölder inequality was used in the proof of (34).

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