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CUT ELIMINATION THEOREM FOR THE
SECOND ORDER ARITHMETIC WITH THE

Π_1^1 -COMPREHENSION AXIOM

AND THE ω -RULE

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CUT ELIMINATION THEOREM FOR THE SECOND ORDER
ARITHMETIC WITH THE \prod_1^1 -COMPREHENSION AXIOM
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Introduction.

In [1] Schütte introduced the constructive ω -rule to the first order arithmetic and proved the (complete) cut elimination theorem for the first order arithmetic, by translating it into a cut-free subsystem of the system with the constructive ω -rule. Takeuti extended this idea in [6] and showed that the second order arithmetic with the \prod_1^1 -comprehension axiom can be translated into a cut free subsystem of the second order arithmetic with the \prod_1^1 -comprehension axiom and the constructive ω -rule. This was done by modifying his consistency proof of the system SINN (cf. [5]), using the same system of ordinal diagrams.

In this article we shall prove the (complete) cut elimination theorem for the second order arithmetic with the \prod_1^1 -comprehension axiom and the (general) ω -rule. The proof of the theorem indicates that the reduction method which is used for the consistency proof of SINN works for the system with an infinite rule as well, although the system of ordinal diagrams which corresponds

¹Part of this work was done while the author was at the University of Bristol.

to the latter is no longer constructive.

At the end, we remark that if we restrict the ω -rule to the constructive one, then the cut elimination theorem holds within the system with the constructive ω -rule.²

§1

The Formulation of the System

In this section the system of the second order arithmetic with the \prod_1^1 -comprehension axiom and the ω -rule is formulated. It is a modification of the system SINN in [5] and shall be called the system W.

1.1. The Language and the Rules of Inference. (cf. Chapters 1 and 2 of [5]). The language and the formulas of W are those of SINN. The sequences are defined as those of SINN except that we admit only those sequences which do not have any occurrence of a free t -variable (a first order variable). If a formula or a sequence has no free t -variable, then it may be called 't-closed'.

The beginning sequences of W are the t-closed beginning sequences of SINN and the rules of inference of SINN except the induction and the \forall right on a t -variable are adopted in W. W has also the following rule of inference, called the ' ω -rule':

²The author thanks Dr. J. Cleave and Professor G. Takeuti for their valuable discussions.

ω -rule

$$\frac{\Gamma \rightarrow \Delta, F(i) \quad i < \omega}{\Gamma \rightarrow \Delta, \forall x F(x)}$$

where ' $\Gamma \rightarrow \Delta, F(i) \quad i < \omega$ ' expresses the fact that $\Gamma \rightarrow \Delta, F(i)$ is given for every natural number i . Each $\Gamma \rightarrow \Delta, F(i)$ is called the i -th upper sequence and $\Gamma \rightarrow \Delta, \forall x F(x)$ is called the lower sequence of an ω -rule. $F(i)$ is called a subformula and $\forall x F(x)$ is called the principal formula of the rule.

Following Schütte's terminology [1], we shall call the inferences weakening, exchange and contraction 'weak inferences' and others 'strong inferences'.

1.2. Proof-figures. The tree form proof-figure of W is defined like the proof-figure of SINN (cf. 13.3 of Chapter 1, [5]), changing the concept of inferences to the one in 1.1. The concepts concerning the proof-figures of SINN may be translated into the concepts concerning the proof-figure of W in an obvious manner. For example, an ω -rule is implicit if a descendant of its principal formula is a cut formula, and also an ω -rule can be a boundary inference. A sequence is said to be W -provable if it is the end sequence of a proof-figure of W .

In the following a 'proof' or a 'W-proof' means a proof-figure of W .

1.3. The ω -complexity of a W -proof P , which is given as a

countable ordinal and is denoted by $\omega(P)$, is defined as follows.

1) If P consists of a beginning sequence only, then $\omega(P) = 0$.

2) Let P be of the form $\frac{P_1}{S}$ or $\frac{P_1 P_2}{S}$. Then $\omega(P) = \omega(P_1)$ or $\omega(P) = \max(\omega(P_1), \omega(P_2))$ accordingly, where $\max(\delta_1, \delta_2)$ is the maximum of δ_1 and δ_2 in the sense of ordinal arithmetic.

3) Let P be of the form $\frac{P_i \quad i < \omega}{S}$, where ' $P_i \quad i < \omega$ ' expresses that a proof P_i is given for every natural number i . Then $\omega(P) = \sup_{i < \omega} \omega(P_i)$, where $\sup_{i < \omega} \delta_i$ is the supremum of δ_i for all $i < \omega$ in the sense of ordinal arithmetic.

It is obvious that $\omega(P) = 0$ if and only if P has no ω -rule and, if Q is a subproof of P , then $\omega(Q) \leq \omega(P)$.

If Q is a subproof of P and S is the end sequence of Q , then $\omega(Q)$ is sometimes denoted by $\omega(S : P)$.

1.4. W_Ω -Proofs. Let Ω be a countable (non-zero) ordinal. Let P be a proof of W which satisfies $\omega(P) < \Omega$. Then P is called a W_Ω -proof and the end sequence of P is said to be W_Ω -provable. It is obvious that every W -proof is a W_Ω -proof for some Ω .

§2

Cut Elimination Theorem

In this section Ω is arbitrary (countable ordinal) but fixed. Our main purpose is to prove the following.

Theorem. If a sequence S is W_Ω -provable, then S is W_Ω^- -provable without cut.

We prove the theorem in a more generalized form.

2.1. The System W'_Ω . First we introduce a rule of inference, called 'substitution' (cf. 3.1 of Chapter 2 in [5]), to W . Substitution is a rule of inference of the form

$$\frac{A_1, \dots, A_m \rightarrow B_1, \dots, B_n}{A_1(\overset{V}{\alpha}), \dots, A_m(\overset{V}{\alpha}) \rightarrow B_1(\overset{V}{\alpha}), \dots, B_n(\overset{V}{\alpha})},$$

where V is an arbitrary (t -closed) semi-isolated variety and is substituted for all occurrences of α in the concerning sequence. The definition of the proof in 1.1 is changed so that the substitution is allowed as a rule of inference. The ω -complexity $\omega(P)$ of a proof with substitutions is defined as in 1.3.

Let P be a proof in the present extended sense. P is called a W'_Ω -proof if:

- 1) $\omega(P)$ is less than Ω ;

2) there is no logical inference (including the ω -rule) in P under a substitution.

Part 2) implies that every substitution is in the end piece of P and hence the number of substitutions in a proof is finite.

The system W'_Ω is the collection of W'_Ω -proofs and the end sequence of a W'_Ω -proof is said to be W'_Ω -provable. It is easily seen that a W'_Ω -proof is a W_Ω -proof if and only if it has no substitution.

The substitution is redundant in W_Ω .

2.2. In order to prove our theorem (stated at the beginning of §2), we shall first define the W'_Ω -proof with degree in the manner that every W_Ω -proof is a W'_Ω -proof with degree, and prove the following.

Proposition. Let P be a W'_Ω -proof with degree. Then there is a cut-free W_Ω -proof of the end sequence of P .

Theorem then follows immediately: Let S be provable with a W_Ω -proof P . Then, as a special case of the above proposition, there is a cut free W_Ω -proof of S .

2.3. The Definition of W'_Ω -Proofs with Degree and the System of Ordinal Diagrams $O(\omega + 1, \Omega \times \omega^3)$. The γ -degree, the grade and the degree are defined like in Chapter 2 of [5]. Notice that the number of free f -variables which are used as eigen variables

of the inferences V right on an f -variable under a sequence is finite. Also, the degree is well defined since the number of substitutions in a W'_Ω -proof is finite. A W'_Ω -proof P is called a W'_Ω -proof with degree if there is a degree for P which satisfies the conditions in 4 of Chapter 2 in [5].

Let $\Omega \times \omega^3$ be the cartesian product of Ω and ω^3 which is ordered lexicographically. Then the system of ordinal diagrams (abbreviated to o.d.s.) $O(\omega + 1, \Omega \times \omega^3)$ is defined as in [4], which we sometimes denote by $O(\Omega)$. For the sake of simplicity, we call the o.d.s. of $O(\Omega)$ simply the o.d.s. The o.d.s. are mainly denoted by a, b, c, \dots . The elements of $\Omega \times \omega^3$ are denoted by $[u, a]$ etc., where $u < \Omega$ and $a < \omega^3$.

An o.d. of $O(\omega + 1, \Omega \times \omega^3)$ is assigned to every W'_Ω -proof with degree, like in Chapter 2 of [5]. Preceding the assignment of o.d.s., we define $s(a)$ for every o.d. a as follows. If a is $[u, a]$, then $s(a)$ is u . If a is $(j; [u, a], b)$, then $s(a)$ is $\max(u, s(b))$. If a is $a_1 \# \dots \# a_i$, then $s(a)$ is

$$\max(s(a_1), \dots, s(a_i)).$$

Let P be an arbitrary W'_Ω -proof with degree. The grade of an occurrence of a formula D in P , defined as in 2.3 of Chapter I in [5], is denoted by $g(D : P)$ (or $g(D)$ when P is fixed). We first assign o.d.s. of $O(\Omega)$ to the sequences in P .

- 1) The o.d. of a beginning sequence (in P) is $[0, 0]$,

2) If S_1 and S_2 are the upper sequence and the lower sequence of a weak inference, then the o.d. of S_2 is identical with that of S_1 .

3) If S is the lower sequence of one of the inferences \neg left, \forall left on a t-variable, \forall right on an f-variable and explicit \forall left on an f-variable, then the o.d. of S is $(\omega; [0,0], a)$, where a is the o.d. of the upper sequence.

4) If S is the lower sequence of an inference right, then the o.d. of S is $(\omega; [0,0], a \# b)$, where a and b are the o.d.s. of upper sequences.

5) If S is the lower sequence of an implicit \forall left on an f-variable of the form

$$\frac{F(V), \Gamma \rightarrow \Delta}{\forall \varphi F(\varphi), \Gamma \rightarrow \Delta},$$

then the o.d. of S is $(\omega; [0, g(F(V)) + 2], a)$, where a is the o.d. of the upper sequence.

6) If S is the lower sequence of a cut, then the o.d. of S is $(\omega; [0, m + 1], a \# b)$, where m is the grade of the cut formula, and a and b are the o.d.s. of the upper sequences.

7) Let S be the lower sequence of an ω -rule, and let $a_0, a_1, \dots, a_i, \dots, i < \omega$ be the o.d.s. assigned to its upper sequences. Then the o.d. of S is $(\omega; [\sup_{i < \omega} s(a_i), 0], [0, 0])$.

8) If S is the lower sequence of a substitution with degree i , then the o.d. of S is $(i : [0,0],a)$, where a is the o.d. of the upper sequence.

The o.d. of a sequence S in a W'_Ω -proof with degree, say P , is denoted by $w(S : P)$ or, sometimes abbreviated to $w(S)$. The o.d. of P is defined as the o.d. of the end sequence of P , which is sometimes denoted by $w(P)$.

2.4. Some Consequences of the Definition in 2.3. The following are obvious from the definition.

Corollary. 1) Let S be in a W'_Ω -proof P . Then $\omega(S;P) = s(w(S;P))$.
(See 1.3 for $\omega(S;P)$.)

2) Define the index elements of an o.d. as follows. $[u,a]$ has no index element; the index elements of $(j : [u,a],b)$ are j and those of b ; the index elements of $a_1 \# \dots \# a_i$ are those of a_1, \dots, a_i . If there is no substitution above a sequence S in P , then all index elements of $w(S : P)$ are ω .

3) If S_1 is under S_2 in a proof P , then

$$w(S_2 : P) \leq_0 w(S_1 : P).$$

\leq_0 holds if and only if there is a strong inference between S_1 and S_2 .

Note. Due to 1) above, we could have defined $w(S : P)$ using $\omega(Q)$ for subproofs Q of P instead of using $s(a)$. It is,

however, more convenient to use $s(a)$ in stating and proving certain lemmas for the o.d.s. (See below.)

$\omega(P)$ or, equivalently, $s(w(P))$ is sometimes denoted by $s(P)$.

7) of the definition in 2.3 makes sense since, by 1) of the corollary,

$$\sup_{i < \omega} s(a_i) = \sup_{i < \omega} \omega(S_i : P) = \omega(S : P) < \Omega,$$

where $S_0, S_1, \dots, S_i, \dots$ are the upper sequences of S .

The following lemmas are useful for the proof of Proposition in 2.2.

Lemma. 1) If there is a component of an o.d. b of the form $(i : [u, b], d)$, then u is called an outermost second element of b . Let a and b be o.d.s. whose index elements (if there are any) are all ω . If v is the maximum of the outermost second elements of b and $s(a) < v$, then $a <_j b$ for all j ($j \leq \omega$ or j is ∞).

2) Let a, b and c be o.d.s. such that there exist three finite lists of o.d.s.,

$$\{a_0 (= a), a_1, \dots, a_m\},$$

$$\{b_0 (= b), b_1, \dots, b_m\},$$

$$\{c_0 (= c), c_1, \dots, c_m\},$$

satisfying the following conditions.

(1) a_i ($i < m$) is of one of the forms $(k; [0, a], a_{i+1})$, $(k; [0, a], a_{i+1} \# d)$ and $(k; [0, a], d \# a_{i+1})$ and b_i and c_i are of the corresponding forms, i.e. b_i is $(k; [0, a], b_{i+1})$, $(k; [0, a], b_{i+1} \# d)$, or $(k; [0, a], d \# b_{i+1})$, and similarly for c_i .

(2) c_m is of the form $(l; [0, a], a_m \# b_m)$.

Then $s(c) = \max(s(a), s(b))$.

Note. We may omit b in the above definition. In that case the conclusion is $s(c) = s(a)$.

3) Let a^i , $i < \omega$, and c be o.d.s. such that there exist finite lists of o.d.s.

$$\{a_0^i (= a^i), a_1^i, \dots, a_m^i\}$$

for every $i < \omega$ and

$$\{c_0 (= c), c_1, \dots, c_m\}$$

satisfying the following conditions.

(1) c_j ($j < m$) is of one of the forms $(k; [0, a], c_{j+1})$, $(k; [0, a], c_{j+1} \# d)$ and $(k; [0, a], d \# c_{j+1})$, and a_j^i has a corresponding form for each $i < \omega$.

(2) c_m is $(\omega; [\sup_{i < \omega} s(a_m^i), 0], [0, 0])$.

Then $s(c) \geq \sup_{i < \omega} s(a^i)$.

4) Let a and b be o.d.s. such that there exist two finite lists of o.d.s.

$$\{a_0 (= a), a_1, \dots, a_m (= c)\}$$

and

$$\{b_0 (= b), b_1, \dots, b_m (= c)\}$$

satisfying the following conditions.

a_i ($i < m$) is of one of the forms $(k; [0, a], a_{i+1})$, $(k; [0, a], a_{i+1} \# d)$, $(k; [0, a], d \# a_{i+1})$ and $(k; [u, 0], [0, 0])$, where $u > s(a_{i+1})$ and b_i has a corresponding form, namely one of the forms $(k; [0, b], b_{i+1})$, $(k; [0, b], b_{i+1} \# d)$, $(k; [0, b], d \# b_{i+1})$ and $(k; [u, 0], [0, 0])$, where $u > s(b_{i+1})$, and $b \leq a$. Then $b \leq_j a$ for all j and, for every $j < \omega$, for every j -section of b , say e , there exists a j -section of a , say e' , such that $e \leq_j e'$.

5) (cf. Lemma 1 of Appendix to 10.1.1.2 of §4 in [5].)

Let p be any natural number, and let c and d be o.d.'s such that there exist two finite lists

$$\{c_0 (= c), c_1, \dots, c_m\}$$

and

$$\{d_0 (= d), d_1, \dots, d_m\}$$

of o.d.s. satisfying the following conditions (1)-(4).

(1) Every c_l ($l < m$) is of one of the forms $(k; [0, 0], c_{l+1})$,

where $k \geq p$, $(\omega; [0, a+1], c_{\ell+1} \neq e)$ and $(\omega; [0, a+1], e \neq c_{\ell+1})$.

(2) Every d_ℓ ($\ell < m$) is $(k; [0, 0], d_{\ell+1})$ or $(\ell; [0, a+1], d_{\ell+1} \neq e)$ or $(\ell; [0, a+1], e \neq d_{\ell+1})$ or $(\omega; [0, a+1], e \neq d_{\ell+1})$ according as c_ℓ is $(k; [0, 0], c_{\ell+1})$, or $(\omega; [0, a+1], c_{\ell+1} \neq e)$ or $(\omega; [0, a+1], e \neq c_{\ell+1})$.

(3) $d_m <_j c_m$ for any j such that $p \leq j \leq \omega$.

(4) For any j such that $p \leq j < \omega$, and for any j -section a of d_m , there exists a j -section b of c_m such that $a \leq_j b$.

Then, $d <_j c$ for any j such that $p \leq j \leq \omega$: and for any j such that $p \leq j < \omega$, and for any j -section a of d , there exists a j -section b of c such that $a \leq_j b$.

Proof. The proof is by induction on $m(a, b)$, where $m(a, b)$ is the sum of the numbers of ()'s and #'s in a and b .

1⁰. $m(a, b) = 0$. Let a be $[u, a]$ and b be $[v, b]$. Then $u < v$ by hypothesis. Therefore $a <_j b$ for all j (by definition). Suppose $m(a, b) > 0$.

2⁰. a is $(\omega; [u, a], c)$ and b is $[v, b]$.

2.1⁰. $a <_{\infty} b$ if and only if $[u, a] < [v, b]$. But from the hypothesis $u < v$.

2.2⁰. $a <_{\omega} b$ if $c <_{\omega} [v, b]$ and $a <_{\infty} b$.

The latter is true from 2.1⁰ and $c <_{\omega} [v, b]$ since $s(a) < v$ implies $s(c) < v$ and, as $m(c, b) < m(a, b)$, the inductive hypothesis holds.

2.3⁰. $j < \omega$. Since all index elements of a and b are ω , there is no j -section in either a or b if $j < \omega$. Therefore, $a <_j b$ if $a <_\omega b$, which is 2.2⁰.

3⁰. a is $[u, a]$ and b is $(\omega; [v, b], d)$. Similarly.

4⁰. a is $(\omega; [u, a], c)$ and b is $(\omega; [v, b], d)$.

4.1⁰. $a <_\infty b$ since $[u, a] < [v, b]$.

4.2⁰. $a <_\omega b$ if $c <_\omega b$ and $a <_\infty b$. The latter is 4.1⁰ and $c <_\omega b$ holds since $s(a) < v$ implies $s(c) < v$ and, as $m(c, b) < m(a, b)$, the inductive hypothesis holds.

4.3⁰. $a <_j b$ for $j < \omega$ from 4.2⁰.

5⁰. a is of form $a_1 \# \dots \# a_k$, $k > 1$. Obvious from the inductive hypothesis.

The proofs of 2) and 3) are omitted.

4) Prove the following for every $i \leq m$ by induction on $m - i$:

(*) $b_i \leq_j a_i$ for all j and, for every $j < \omega$, for every j -section of b_i , say d , there exists a j -section of a_i , say d' , such that $d \leq_j d'$.

1⁰. $i = m$. Both a_m and b_m are c . So (*) trivially holds.

2^o. Assume (*) for $i + 1$. As an example, take the case where a_i is $(k; [0, a], a_{i+1} \# d)$ and b_i is $(k; [0, b], b_{i+1} \# d)$.

2.1^o. $a = b$ and $a_{i+1} = b_{i+1}$. Then $a_i = b_i$ and the second part of (*) follows from a property of general theory of o.d.s.

2.2^o. $b < a$. $b_i <_{\infty} a_i$ since $b < a$.

1) $k = \omega$. $b_i <_{\omega} a_i$ since $b_{i+1} \# d \leq_{\omega} a_{i+1} \# d$ (by inductive hypothesis) $<_{\omega} a_i$ (an ω -section), and $b_i <_{\infty} a_i$.

Suppose $j < \omega$. If e is a j -section of b_i , then e is either a j -section of b_{i+1} or d . If e is a j -section of d , then e is a j -section of a_i . If e is a j -section of b_{i+1} , then by inductive hypothesis there is a j -section of a_{i+1} , say e' , such that $e \leq_j e'$. $e' <_j a_i$ and so, $e <_j a_i$. Let j_0 be the least ℓ such that $\ell > j$ and ℓ is an index of b_i and/or a_i . Then $b_i <_{j_0} a_i$ by inductive hypothesis. Therefore $b_i <_j a_i$.

2) $k < \omega$. For $j > k$, $b_i <_j a_i$ since $b_i <_{\infty} a_i$. $b_i <_k a_i$ since $b_{i+1} \# d \leq_k a_{i+1} \# d$ (by inductive hypothesis) $<_k a_i$ (k -section), and $b_i <_{\infty} a_i$. $b_{i+1} \# d$ is the only k -section of b_i and $a_{i+1} \# d$ is a k -section of a_i . For $j < k$, the argument in 1) for $j < \omega$ goes through.

5) See the proof of Lemma 1 of Appendix to 10.1.1.2 of §4 in [5].

2.5. Proof of Proposition in 2.2. The proposition is proved by transfinite induction on the o.d.s. of W'_Ω -proofs along the ordering $<_0$ of o.d.s. (cf. 3) of Corollary in 2.4.)

We more or less follow the consistency proof of Chapter 2 of [5]. Hence, we shall demonstrate the detailed proofs only for a few cases. At each step we show that the cut free W -proof P' (hence without substitution) which is obtained as a result of the reduction satisfies the condition $\omega(P') \leq \omega(P)$, which implies that P' is a W_Ω -proof.

First we introduce another rule of inference, 'term replacement', to W'_Ω -proofs. (cf. 8.1 of Chapter 2 in [5].)

The o.d.s. of the upper sequence and the lower sequence of a term replacement are identical. A term replacement is redundant in W'_Ω .

In the following, an o.d. which is placed above a sequence denotes the o.d. of that sequence in the proof under consideration.

1⁰. There is an explicit logical inference in the end piece of P . Let I be a last such inference.

1.1⁰. I is an ω -rule. Let P be of the form

$$\begin{array}{c}
 \vdots \\
 \Gamma \frac{a_i \quad \theta, F(i) \quad i < \omega}{(\omega; [\sup_{i < \omega} s(a_i), 0], [0, 0])} \\
 I \\
 \Gamma \rightarrow \theta, \quad \forall x F(x) \\
 \vdots \\
 \prod \frac{b}{\Lambda} ,
 \end{array}$$

where Λ contains $\forall x \tilde{F}(x)$. (\tilde{A} is either A itself or is obtained from A by one or more substitutions.) Define P_i for each $i < \omega$, copying P , as follows.

$$\frac{\begin{array}{c} \vdots \\ a_i \\ \Gamma \rightarrow \theta, F(i) \end{array}}{\Gamma \rightarrow F(i), \theta, \forall x F(x)} \quad \text{weakening, exchange}$$

$$\prod \begin{array}{c} \vdots \\ c_i \\ \rightarrow F(i), \Lambda \end{array}$$

To each substitution in P_i the same degree as to the corresponding substitution in P is assigned.

First, $a_i <_j (\omega; [\sup_{i < \omega} s(a_i), 0], [0, 0])$ holds for all j by 1) of Lemma in 2.4. (Recall that all index elements of a_i , if there is any, are ω : cf. Corollary 2) in 2.4.) Therefore, by letting a_i and $(\omega; [\sup_{i < \omega} s(a_i), 0], [0, 0])$ be d_m and c_m respectively, and c_i and b be c and d respectively, (1)-(4) in 5) of Lemma in 2.4 hold. (There is no j -section of a_i if $j < \omega$.) Thus $c_i <_0 b$ from 5) of Lemma in 2.4, and hence, by induction hypothesis, there is a cut free W_Ω -proof P_i' of $\prod \rightarrow \Lambda, \tilde{F}(i)$, such that $\omega(P_i') \leq \omega(P_i) (= s(c_i))$. Define P' as

$$\begin{array}{c}
 P_i' \quad i < \omega \\
 \hline
 I' \quad \prod \rightarrow \Lambda, \forall x \tilde{F}(x) \\
 \hline \hline
 \prod \rightarrow \Lambda \quad \text{exchange, contraction}
 \end{array}$$

Since no substitution and no cut are introduced P' is a cut free W -proof and $\omega(P') = \sup_{i < \omega} \omega(P_i') \leq \sup_{i < \omega} \omega(P_i) = \sup_{i < \omega} s(c_i) \leq s(P)$. (cf. Corollary 1) in 2.4 and 3) of Lemma in 2.4, where a is c_i and c is $w(P)$ here.)

1.2⁰. I is V left on an f -variable. Let P be of the form

$$\begin{array}{c}
 \vdots \\
 F(V), \Gamma \xrightarrow{a} \theta \\
 \hline
 I \quad (\omega; [0,0], a) \\
 \\
 \forall \varphi F(\varphi), \Gamma \rightarrow \theta \\
 \vdots \\
 \prod \xrightarrow{b} \Lambda.
 \end{array}$$

Define Q from P :

$$\begin{array}{c}
 \vdots \\
 F(V), \Gamma \xrightarrow{a} \theta \\
 \hline \hline
 \forall \varphi F(\varphi), \Gamma, F(V) \rightarrow \theta \\
 \prod, F(V) \xrightarrow{b} \Lambda
 \end{array}$$

Since $a <_j (\omega; [0,0], a)$ for any $j \leq \omega$ and there is no j -section of a if $j < \omega$, the conditions in 5) of Lemma in 2.4 hold for $a, c, (\omega; [0,0], a)$ and b . Therefore $c <_0 b$, and hence, by induction hypothesis, there is a cut free W_Ω -proof of $\prod, F(V) \rightarrow \Lambda$, say Q' , such that $\omega(Q') \leq \omega(Q)$. Define P' as

$$\frac{\frac{Q'}{\hline \forall \varphi F(\varphi), \prod \rightarrow \Lambda}}{\hline \hline \prod \rightarrow \Lambda} .$$

Then $\omega(P') = \omega(Q') \leq \omega(Q) = s(c) = s(b) = \omega(P)$ (cf. Corollary 1) in 2.4), and hence P' is a cut free W_Ω -proof.

1.3^o. I is \forall right on an f -variable. Similarly to 1.2^o. Use 4) of Lemma in 2.4.

2^o. The case where there is no explicit logical inference in the end piece of P but there is an equality axiom as a beginning sequence in the end piece of P . The reduction for this case is carried out like in 8.4 of Chapter II in [5].

3^o. The case where there is no explicit logical inference and equality axiom in the end piece of P , but there is a logical beginning sequence in the end piece of P . The reduction is

carried out like in 8.5 in [5].

4^o. Elimination of weakenings in the end piece of P . We may assume besides the conditions in 3^o that the end piece of P does not contain any logical beginning sequences. We can define another W'_Ω -proof with degree, say P^* , eliminating weakenings in the end piece of P by mathematical induction on the number of inferences in the end piece of P . (Note that, although P may be an infinite proof, the end piece of P is now finite under the above conditions.) The elimination of weakenings is carried out exactly like 8.6 in Chapter 2 of [5]. As a consequence, we can show that for every j -section a of $w(P^*)$ there is a j -section b of $w(P)$ such that $a \leq_j b$ for $0 \leq j < \omega$, and $w(P^*) \leq_j w(P)$ for $0 \leq j \leq \omega$. In particular, $w(P^*) \leq_0 w(P)$.

If $w(P^*) <_0 w(P)$, then apply inductive hypothesis to P^* and obtain a cut free W_Ω -proof $P^{*'}$ of the same end sequence. P' is defined by

$$\frac{\frac{P^{*'}}{\text{weakening, exchange}}}{\text{the end sequence of } P}$$

$s(P') = s(P^{*'}) \leq s(P^*) \leq s(P)$. If $w(P^*) = w(P)$, then proceed to the next step.

5^o. Essential Reduction. In the following we shall assume that the end piece of a W'_Ω -proof with degree contains none of the explicit logical inference, the beginning sequences and the weakening. We may also assume that P is distinct

from its end piece.

The existence of a suitable cut is proved like in 9 of Chapter 2 of [5], since the end piece of P is finite under the assumption of 5^0 .

Now we shall define the essential reduction according to the outermost logical symbol of the cut formula of a suitable cut. We shall find a W'_Ω -proof with degree (say Q) of the end sequence of P such that $w(Q) <_0 w(P)$ and $s(Q) \leq s(P)$. Then, by induction hypothesis, there is a cut free W'_Ω -proof Q' of the end sequence of Q such that $s(Q') \leq s(Q)$. Thus, taking Q' as P' , we complete the proof. The reduction of P to Q is carried out exactly like in 10 of Chapter 2 in [5] except the case where the outermost logical symbol of the cut formula is \forall on a t -variable, which shall be treated separately. The required properties on the o.d.s. are easily proved. (In applying Lemmas in Appendix to 10.1.1.2 in [5], read $[0, a]$ instead of a .)

The case where the outermost logical symbol is \forall on a t -variable is treated as follows. P is of the form

$$\begin{array}{ccc}
 \begin{array}{c} \Downarrow \\ a_i \\ \Gamma_1 \rightarrow \theta_1, F_1(i) \quad i < \omega \end{array} & & \begin{array}{c} \Downarrow \\ b \\ F_2(s), \Gamma_2 \rightarrow \theta_2 \end{array} \\
 \hline
 (\omega; [\sup_{i < \omega} s(a_i), 0], [0, 0]) & & (\omega; [0, 0], b) \\
 \\
 \begin{array}{c} \Gamma_1 \rightarrow \theta_1, \forall x F_1(x) \\ \Downarrow \\ \prod_1 \vDash \Lambda_1, \forall x F(x) \end{array} & & \begin{array}{c} \forall x F_2(x), \Gamma_2 \rightarrow \theta_2 \\ \Downarrow \\ \forall x F(x), \prod_2 \vDash \Lambda_2 \end{array}
 \end{array}$$

$$(\omega; [0, g(\forall x F(x)) + 1], c \# d)$$

$$\prod_1, \prod_2 \rightarrow \Lambda_1, \Lambda_2$$

$$\begin{array}{c} \Downarrow \\ \Sigma \vDash \Delta \end{array}$$

There is an i such that $s = i$ is true. Define P_1 and P_2 as follows, and then Q is defined in terms of P_1 and P_2 . In the following two figures P_1 and P_2 , the o.d.s. above the sequences are relative to Q .

$$\begin{array}{c}
 P_1: \\
 \frac{\Gamma_1 \stackrel{a_1}{\rightarrow} \theta_1, F_1(i)}{\Gamma_1 \stackrel{a_1}{\rightarrow} F_1(i), \theta_1, \forall x F_1(x)} \\
 \frac{\Pi_1 \stackrel{c}{\rightarrow} F(i), \Lambda_1, \forall x F(x)}{\forall x F(x), \Pi_2 \stackrel{d}{\rightarrow} \Lambda_2}
 \end{array}$$

$(\omega; [0, g(\forall x F(x)) + 1], c' \# d)$

$$\frac{\Pi_1, \Pi_2 \rightarrow F(i), \Lambda_1, \Lambda_2}{\Pi_1, \Pi_2 \rightarrow \Lambda_1, \Lambda_2, F(i)}$$

$$\frac{\Pi_1, \Pi_2 \rightarrow \Lambda_1, \Lambda_2, F(i)}{\Pi_1, \Pi_2 \rightarrow \Lambda_1, \Lambda_2, F(s)}$$

term replacement

$$\begin{array}{c}
 P_2: \\
 \frac{F_2(s), \Gamma_2 \stackrel{b}{\rightarrow} \theta_2}{\forall x F_2(x), \Gamma_2, F_2(s) \stackrel{b}{\rightarrow} \theta_2} \\
 \frac{\Pi_1 \stackrel{c}{\rightarrow} \Lambda_1, \forall x F(x)}{\forall x F(x), \Pi_2 \stackrel{d'}{\rightarrow} \Lambda_2}
 \end{array}$$

$(\omega; [0, g(\forall x F(x)) + 1], c \# d')$

$$\frac{\Pi_1, \Pi_2, F(s) \rightarrow \Lambda_1, \Lambda_2}{F(s), \Pi_1, \Pi_2 \rightarrow \Lambda_1, \Lambda_2}$$

Q:

$$\begin{array}{c}
 P_1 \quad P_2 \\
 \hline
 \Pi_1, \Pi_2, \Pi_1, \Pi_2 \rightarrow \Lambda_1, \Lambda_2, \Lambda_1, \Lambda_2 \quad \text{cut} \\
 \hline \hline
 \Pi_1, \Pi_2 \rightarrow \Lambda_1, \Lambda_2 \\
 \vdots \\
 \exists e' \Delta.
 \end{array}$$

Every substitution in Q is given the same degree as the degree of the corresponding substitution in P .

$s(Q) \leq s(P)$ is obvious from the way Q is constructed.

The proof of $e' <_0 e$ goes as follows. Let us call the sequence $\Pi_1, \Pi_2 \rightarrow \Lambda_1, \Lambda_2$ in P S_1 and the $\Pi_1, \Pi_2, \Pi_1, \Pi_2 \rightarrow \Lambda_1, \Lambda_2, \Lambda_1, \Lambda_2$ in Q S_2 . $a_i <_j (\omega; [\sup_{i < \omega} s(a_i), 0], [0, 0])$ for all $0 \leq j \leq \omega$

by 1) of Lemma in 2.4. (Recall that all index elements of a_i are ω , as there is no substitution above $\Gamma_1 \rightarrow \theta_1, F_1(i)$ in P . cf. Corollary 2) in 2.4.) Therefore a_i and $(\omega; [\sup_{i < \omega} s(a_i), 0], [0, 0])$

satisfy the condition for d_m and c_m in 5) of Lemma in 2.4.

((4) holds trivially, since a_i has no j -section if $j < \omega$.)

Hence $c' <_j c$ for $0 \leq j \leq \omega$ and, for every j -section of c' , say f , where $0 \leq j < \omega$, there is a j -section of c , say g , such that $f \leq_j g$. Thus

$$(\omega; [0, g(\forall x F(x)) + 1], c' \# d) <_j (\omega; [0, g(\forall x F(x)) + 1], c \# d)$$

for all $0 \leq j \leq \omega$ and, for $0 \leq j < \omega$, for every j -section of $(\omega; [0, g(\forall x F(x)) + 1], c' \# d)$, say f , there is a j -section of $(\omega; [0, g(\forall x F(x)) + 1], c \# d)$, say g , such that $f \leq_j g$.

By the definition of Q ,

$w(S_2 : Q)$

$$= (\omega; [0, g(F(s)) + 1], (\omega; [0, g(\forall x F(x)) + 1], c' \# d) \# (\omega; [0, g(\forall x F(x)) + 1], c \# d')),$$

while $w(S_1 : P) = (\omega; [0, g(\forall x F(x)) + 1], c \# d)$. $w(S_2 : Q) <_{\infty} w(S_1 : P)$ is obvious, since $g(F(s)) < g(\forall x F(x))$. (There is no V right on an f -variable under those sequences in either P or Q .)

$w(S_2 : Q) <_{\omega} w(S_1 : P)$, since each component of the ω -section of $w(S_2 : Q)$, say f , satisfies $f <_{\omega} w(S_7 : P)$ from above and $w(S_2 : Q) <_{\infty} w(S_1 : P)$. Suppose $0 \leq j < \omega$. If f is a j -section of $w(S_2 : Q)$, then it is a j -section of $(\omega; [0, g(\forall x \tilde{F}(x)) + 1], c' \# d)$ or of $(\omega; [0, g(\forall x \tilde{F}(x)) + 1], c \# d')$. In any case, there is a j -section of $w(S_1 : P)$, say g , such that $f \leq_j g$. Therefore, $f <_j w(S_1 : P)$. So by induction hypothesis, $w(S_2 : Q) <_j w(S_1 : P)$. Therefore, by 5) of Lemma in 2.4, $e' <_0 e$.

§3. Remark on the System with the Constructive ω -Rule.³

3.1. The Definitions of the System and the ω -Complexity.

The system of the second order arithmetic with the \prod_1^1 -comprehension axiom and the constructive ω -rule is defined by an inductive definition in terms of Godel numbering (see [2] and [6]). We shall call this system Z (which is actually a set of numbers). In particular, the constructive ω -rule is described as follows.

Let e be (Godel number of) a recursive function such that $\{e\}(i)$ gives a proof of a sequence of the form $\Gamma \rightarrow \theta, F(i)$ for every i . Then we may conclude $\Gamma \rightarrow \theta, \forall xF(x)$.

We shall use the notations $\ulcorner A \urcorner$, $\ulcorner P \urcorner$ etc. in order to denote the concepts of a formula A , a proof P , etc., though actually we have only the numbers.

The ω -complexity of a proof of Z , say $\ulcorner P \urcorner$, is defined like in 1.3, and it is easily shown that $\omega(\ulcorner P \urcorner) < \omega_1$ for every proof $\ulcorner P \urcorner$ of Z , where ω_1 is the first non-constructive ordinal. Thus, for the Ω in 1.4, we only have to consider $\Omega < \omega_1$. In fact we can give the ω -complexities in the set O_1 (a linearly ordered subset of the set O of constructive ordinals which has

³It should be noted that, like the case of the first order arithmetic (cf. [2]), the constructive ω -rule is adequate for any second order arithmetic. This has been proved by Takahashi in [3]. Hence, mathematically, it suffices to deal with the system with the constructive ω -rule.

the order type ω_1).⁴

The subsystem of Z which consists of all the proofs $\ulcorner P \urcorner$ such that $\omega(\ulcorner P \urcorner) <_0 \Omega$ for an Ω in O_1 is denoted by Z_Ω , where $<_0$ is the ordering of O .

3.2. We may extend Z so that the a rule of inference. The condition on the degree is recursive since the number of substitutions in a proof is finite (cf. 2.3.). Thus we can define the set of proofs with degree, say Z' , like in 2.3. The grade of a formula $\ulcorner A \urcorner$ in a Z' -proof $\ulcorner P \urcorner$ is defined as a recursive function of $\ulcorner A \urcorner$ and $\ulcorner P \urcorner$. It is easily shown as before that Z is a subset of Z' .

3.3. The concept of 'a cut free proof of Z' ' is defined in an obvious manner.

Lemma. There exists a partial recursive function f such that f is defined for all proofs with degree (of Z') and, if $\ulcorner P \urcorner$ is a member of Z' , then $f(\ulcorner P \urcorner)$ is a cut free Z -proof of the end sequence of $\ulcorner P \urcorner$. Moreover

$$\omega(f(\ulcorner P \urcorner)) \leq_0 \omega(\ulcorner P \urcorner).$$

From the lemma follows the

Theorem. (Cut Elimination Theorem). If a sequence is Z -provable

⁴In fact, the length of any proof in Z is less than ω_1 ; more precisely, it can be defined in O_1 .

then it is Z-provable without cut.

We only outline the proof of the lemma.

3.4. The function f is defined by examining the reductions which are carried out in 2.5. Let $q(e,p)$ be a partial recursive function of e and p such that if e actually gives the function f and p denotes a proof of Z' , then $q(e,p)$ gives the result of the reduction.

The crucial cases are 1.1^0 and 5^0 (of 2.5; the cases where the outermost logical symbols are \forall on a t -variable). For 1.1^0 $q(e,p)$ is expressed as $\xi(r(e,p),p)$, where $r(e,p)$ corresponds to a recursive function which produces the cut free proof of $\prod_i \rightarrow F(i)$, \wedge for every i and ξ is a recursive function (cf. 1.1^0 of 2.5). For 5^0 , $q(e,p)$ is expressed as $\{e\}(\tau_i(p))$, where $\tau_i(p)$ corresponds to the Q in 5^0 of 2.5 and i can be found recursively from p .

Thus, by recursion theorem, there is a number e_0 such that

$$\{e_0\}(p) \simeq q(e_0,p).$$

The partial recursive function which is represented by e_0 shall be called f .

3.5. We may define the system of o.d.s. $O(\omega + 1, O_1 \times \omega^3)$ and the well orderings $<_j$ for $j \leq \omega$ and $<_\infty$, where $O_1 \times \omega^3$ is ordered lexicographically. If p is in Z' , then we can assign an o.d. of the above system to p , say $w(p)$, as in 2.3

in terms of the degree and the grade (cf. 3.2). We can then prove the lemma in 3.3 for the function f , which has been defined in 3.4, by transfinite induction on $w(p)$ along $<_0$ of the above system. The computations on the o.d.s. and the ω -complexities are carried out like in 2.5, using the lemmas in 2.4. We shall only remark that $r(e_0, p)$ indeed represents a required recursive function, for: let η be a recursive function such that $\eta(i, \ulcorner P \urcorner) = \ulcorner P_i \urcorner$ in 1.1^o. Then $r(e_0, \ulcorner P \urcorner)$ is defined as $\Lambda i(\{e_0\}(\eta(i, \ulcorner P \urcorner)))$ where $\Lambda i(\{e_0\}(\eta(i, \ulcorner P \urcorner)))$ represents the Godel number of a function of i whose value is $\{e_0\}(\eta(i, \ulcorner P \urcorner))$ for each i . On the other hand, $w(\eta(i, \ulcorner P \urcorner)) <_0 w(\ulcorner P \urcorner)$ holds, and hence $\{e_0\}(\eta(i, \ulcorner P \urcorner))$ is defined for every i by induction hypothesis.

3.6. We could state the lemma in 3.3 as follows.

For any Ω in O_1 , there exists a partial recursive function f such that f is defined for all proofs with degree whose ω -complexities are less than Ω , and, for such a $\ulcorner P \urcorner$, $f(\ulcorner P \urcorner)$ is a cut free proof of Z_Ω .

The above statement is proved by using the system of o.d.s. $O(\omega + 1, C(\Omega) \times \omega^3)$, where $C(\Omega) = \{\mu / \mu \in O_1 \wedge \mu <_0 \Omega\}$. In this case, $O(\omega + 1, C(\Omega) \times \omega^3)$ is a recursively enumerable set and $<_i$ for $i \leq \omega$ and $<_{00}$ are partial recursive relations.

3.7. A Translation of the System SINN. The system SINN is translated into $Z_{\tilde{\omega}}$, where $\tilde{\omega}$ is the notation for ω in O_1 .

Proposition. Let S be a t -closed sequence (of SINN). If S

is SINN-provable, then S is Z_{ω} -provable.

Proof. A proof-figure of SINN is called regular if it satisfies the following conditions: all eigen variables are distinct from one another and if a variable a (α) is the eigen variable of a V right on a t -variable (f -variable), say I , then a (α) does not occur under I and in any string which does not contain the upper sequence of I . It suffices to prove the proposition for regular proofs (of SINN).

Let P be a proof-figure of SINN. Let $\pi(S;P)$ and $\pi(P)$ be defined as follows. If S is a beginning sequence in P , then $\pi(S;P) = 1$. If S is the lower sequence of a V right on a t -variable, and S_1 is its upper sequence, then

$$\pi(S;P) = \pi(S_1;P) + 1.$$

If S is the lower sequence of other inferences, then $\pi(S;P) = \pi(S_1;P)$ or $= \max(\pi(S_1;P), \pi(S_2;P))$ respectively, where S_1 and S_2 are upper sequences. $\pi(P)$ is defined as $\pi(\text{the end sequence of } P;P)$. ($\pi(P) < \omega$ is obvious.)

Now we shall prove the proposition in a stricter form:

(*) Let $P(b_1, \dots, b_k)$ be an arbitrary regular proof-figure of SINN, where b_1, \dots, b_k in P indicate all occurrences of free t -variables in P which are not used as eigen variables.

Then there is a recursive function \emptyset of k arguments such that for an arbitrary k -tuple of natural numbers i_1, \dots, i_k ,

$\emptyset(i_1, \dots, i_k)$ is $Z_{\pi(P(b_1, \dots, b_k))}$ -proof whose end sequence is

(Gödel number of) that of $P(i_1, \dots, i_k)$, where $P(i_1, \dots, i_k)$ is obtained from $P(b_1, \dots, b_k)$ by replacing b_1, \dots, b_k by i_1, \dots, i_k respectively.

First we introduce the rule 'term replacement' to the system and prove (*) by mathematical induction on the number, say ι , of two rules of inference, \forall right on a t -variable and induction in P .

0) $\iota = 0$, i.e. P has neither induction nor \forall right on a t -variable. Define \emptyset as $\emptyset(i_1, \dots, i_k) = \ulcorner P(i_1, \dots, i_k) \urcorner$ for all (i_1, \dots, i_k) . It is easily seen that, for an arbitrary (i_1, \dots, i_k) , $\emptyset(i_1, \dots, i_k)$ is a Z_1 -proof.

In the following $\iota > 0$ is assumed and, in order to simplify the notation, we shall assume $k = 1$ and denote b_1 and i_1 simply by b and i respectively. There are three cases.

1) There is an inference I in P which has two upper sequences and satisfies the following.

(a) There is neither induction nor \forall right on a t -variable under I .

(b) Let P be of the form

$$I \quad \frac{P_1(b) \quad P_2(b)}{R(b)}$$

where P_1 and P_2 are subproofs of P and R is the part of P under I . Then both P_1 and P_2 have either induction

or \forall right on a t -variable.

From (b) the number of inductions and \forall rights on a t -variable in each of P_1 and P_2 is less than ℓ , so that, by the inductive hypothesis, there are recursive functions $\emptyset_1(i)$ and $\emptyset_2(i)$ corresponding to P_1 and P_2 respectively. Let $\emptyset_j(i) = \ulcorner P_j'(i) \urcorner$ for $j = 1, 2$. Then define $\emptyset(i)$ as the Godel number of

$$\frac{P_1'(i) \quad P_2'(i)}{R(i)} .$$

Evidently \emptyset is recursive. That $\emptyset(i)$ is a $Z_{\pi}(P(b))$ -proof follows from the induction hypothesis.

2) 1) is not the case and the lowermost inference in P , say I , which is either induction or \forall right on a t -variable is induction. Let P be of the form

$$Q(a, b) \quad \left\{ \begin{array}{l} \text{I} \quad \frac{F(a), \Gamma \rightarrow \theta, F(a')}{F(0), \Gamma \rightarrow \theta, F(s)} \\ \text{S} \end{array} \right.$$

We may assume that s does not have a . The number of inductions and \forall rights on a t -variable in $Q(a, b)$ is less than ℓ , and hence inductive hypothesis applies. Namely, there is a recursive function ψ corresponding to $Q(a, b)$ and, for each (n, i) $\psi(n, i)$ is a $Z_{\pi}(Q(a, b))$ -proof whose end sequence is $\ulcorner F(n, i), \Gamma(i) \rightarrow \theta(i), F(n', i) \urcorner$, where $\Gamma(i)$ etc. is obtained from Γ etc. by replacing b by i ,

and $F(n,i)$ is an abbreviation of $F(n)(i)$. In particular, for all i , for an n fixed, this is so. Let s^* be obtained from a term s by replacing b by i . As s^* is closed, there is a numeral m such that $s^* = m$ is true. Using the above facts and the inductive hypothesis, $\emptyset(i)$ is defined as the Gödel number of the reduction of a proof-figure with respect to an induction for the consistency proof (cf. 8.3 in Chapter 2 of [5]).

3) 1) is not the case and the lowermost such inference is an \forall right on a t -variable. Let P be of the form

$$Q(a,b) \left\{ \begin{array}{l} \vdots \\ \Gamma \rightarrow \theta, F(a) \end{array} \right. \\ \hline R(b) \left\{ \begin{array}{l} \Gamma \rightarrow \theta, \forall xF(x) \\ \vdots \\ S \end{array} \right.$$

The number of such inferences in $Q(a,b)$ is less than ι , and hence the inductive hypothesis applies. Namely, there is a recursive function ψ corresponding to $Q(a,b)$ and, for every (n,i) , $\psi(n,i)$ is a $Z_{\pi}(Q(a,b))$ -proof. Let $\psi(n,i) = \ulcorner Q'(n,i) \urcorner$. The Gödel number of

$$Q'(n,i) \left\{ \begin{array}{l} \vdots \\ \Gamma(i) \rightarrow \theta(i), F'(n,i) \quad n < \omega \end{array} \right. \\ \hline \Gamma(i) \rightarrow \theta(i), \forall xF(x,i)$$

is given as $3.5 \wedge n \psi(n,i) . 7 \ulcorner \Gamma(i) \rightarrow \theta(i), \forall xF(x) \urcorner$, where $\Gamma(i)$, etc.

means the substitution of i for b . This is a recursive function of i , which we call $\chi(i)$. $\emptyset(i)$ is defined in terms of χ , by adding the part $R(i)$. \emptyset is recursive and $\emptyset(i)$ is a $W_{\pi}(P(b))$ -proof of $\lceil S(i) \rceil$.

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