BIMEASURABLE MAPS

by

John G. Michaels

Report 69-11

March, 1969

1. INTRODUCTION. All spaces considered in this paper are assumed to be metrizable, all ordinals considered are to be countable, and k will denote an infinite cardinal.

A map f between two absolute Borel (metric) spaces is <u>bimeasurable</u> if both f and f^{-1} preserve absolute Borel sets. R. Purves [6] has shown the following:

THEOREM 1.1. If f is a bimeasurable map between two separable absolute Borel metric spaces, then $f^{-1}(y)$ is countable except for at most countably many points in the range of f.

The purpose of this paper is to obtain generalizations of this theorem for non-separable spaces. In place of countability we are led to considerations of the cardinality and cr-discreteness of the sets $f^{-1}(y)$. Summarizing Theorems 4.3, 4.4, 5.1, and 5.2, we obtain the following (definitions are given in Section 2):

THEOREM. Let f be an a-bimeasurable map defined on an absolute Borel space X of weight k. Let

$$B = \{y \in f(X) : f^{-1}(y) \text{ not cr-discrete}\}$$

and let

$$B^* = \{y \in f(X) : \text{card } f^{-1}(y) > k\}.$$

Then

(i) card B $\leq$ k,

(ii) card B* $\leq$ k,

(iii) if B is absolutely K_o-analytic, then B is cr-discrete,

(iv) if B* is absolutely N_o-analytic, then B* is CT-discrete.

Each of the four conclusions in this theorem reduces to the theorem of Purves if the spaces in question are separable, i.e. if $k = N_o$.

In Section 3 we prove that the set B is absolutely k-analytic. This result (Theorem 3.3) partially generalizes a classical theorem of Mazurkiewicz and Sierpinski [5]. We use both this result and Theorem 1.1 to prove the main theorems in Section 4 (cf. (i) and (ii) above).

In Section 5 we put additional hypotheses on the sets B and B* (cf. (iii) and (iv) above). We also study the bimeasurability of projection maps and discuss the problem of finding sufficient conditions that a measurable map be bimeasurable.

2. DEFINITIONS AND NOTATION. The terminology follows [8], and we assume the reader is familiar with the definitions and basic properties of the Borel classification of sets and measurable (= Borel-measurable) maps, as in [8]. For future reference and to fix notation we repeat some of the definitions in this section.

A (metric) space X is an <u>absolute Borel set</u> if X is Borel in any metric space in which X is embedded, or, equivalently if X is Borel in some complete metric space. For a (countable) ordinal α, a bimeasurable map f is <u>α-bimeasurable</u> if f is measurable of class α.

If the range and domain spaces are absolute Borel, then every Borel isomorphism (1-1 map which is measurable in both directions) between them is bimeasurable and every generalized homeomorphism (1-1 map which is measurable of bounded class in both directions) between them is α-bimeasurable for some α. Also every measurable map from an absolute Borel space onto a σ-discrete space is bimeasurable. (A (metric) space A is <u>σ-discrete</u> if $A = \bigcup_{i=1}^{\infty} A_i$ where each set A_i is relatively discrete, i.e. each point of A_i is isolated in A_i.)

We have the following result concerning measurability and weight (assuming the generalized continuum hypothesis):

THEOREM 2.1. If f is a measurable map from a space X onto a space Y and if X is of weight k, then Y has weight at most k.

Proof. Let m be the weight of Y, and assume $m > k$. Then Y has 2^m Borel subsets [7,p.106]. By considering the inverse images under f of these sets, we conclude that X has at least 2^m Borel subsets. But X has 2^k Borel subsets, and under the generalized continuum hypothesis $2^k < 2^m$. Hence a contradiction, and therefore $m \leq k$.

Baire space $B(k)$ is the countable product of discrete spaces, each of which is of cardinal k. $B(k)$ is given the product topology; and if $t = (t_1, t_2, \ldots) \in B(k)$, a typical basic neighborhood of t in the product topology is

$$V(t_1, \ldots, t_n) = \{s = (s_1, s_2, \ldots) \in B(k) : s_1 = t_1, \ldots, s_n = t_n\}.$$

The space $B(k)$ is metrizable, and may be given the metric d where, for distinct $s, t \in B(k)$, we define $d(s,t) = 1/n$ if $s_1 = t_1, \ldots, s_{n-1} = t_{n-1}, s_n \neq t_n$. The Cantor set $C(\aleph_0)$ is the countable product of two-point discrete spaces. We regard the Cantor set as a subspace of any space $B(k)$, and so has as its metric the restriction of the metric for $B(k)$.

The following three theorems of A. H. Stone will be used often subsequently:

THEOREM 2.2 [9,p.660]. If X is an absolute Borel set, then one and only one of the following alternatives is true: X is σ-discrete, or X contains a subset homeomorphic to $C(\aleph_0)$.

THEOREM 2.3 [8,p.10]. If X is an absolute Borel set of weight $\leq k$, then there is a continuous generalized homeomorphism from a closed subset of $B(k)$ onto X.

THEOREM 2.4 [9,p.661]. If X is an absolute Borel set which is Borel isomorphic to a σ-discrete space, then X is σ-discrete.

Assume a space X and a cardinal k are given. Write $B(k) = \prod_{n=1}^{\infty} T_n$ and assume that for each finite sequence $t_1, \ldots, t_n$ ($t_i \in T_i$) a closed set $F(t_1, \ldots, t_n) \subset X$ is given. For $t = (t_1, t_2, \ldots) \in B(k)$ let $H(t) = \bigcap_{n=1}^{\infty} F(t_1, \ldots, t_n)$, and let $A = \bigcup\{H(t) : t \in B(k)\}$. Then A is a k-analytic subset of X. A space is absolutely k-analytic if it is k-analytic in any metric space in which it is embedded, or, equivalently [8,p.36] if it is k-analytic in some complete metric space. In particular, if $k = \aleph_0$ and the space X is complete and separable, the classical analytic sets are obtained.

3. A THEOREM ON a-DISCRETENESS. The following theorem is due to Mazurkiewicz and Sierpinski [5]: If f is a continuous function defined on a separable, absolutely analytic space X, and if B = (yef(X): card f~^{1}(y) > K_Q}, then B is absolutely analytic.

In this section a partial generalization of this theorem is obtained for non-separable spaces. This extension is obtained in three steps. The result is then used in Section 4 to obtain the main theorem of the paper.

LEMMA 3.1. If f is a continuous map defined on a complete space X of weight k, and if

B = {yef(X): f~^{1}(y) not cr-discrete},

then B is absolutely k-analytic.

Proof. Let $B_{\pm}$ = {yef(X): f"^{1}^) contains a

dense-in-itself sequence of distinct points}.

We shall show that B = B_1 and then that B^ is absolutely k-analytic.

Let yeB. Then f~^{1}(y) is a non-cr-discrete absolute Borel set and hence by Theorem 2.2 contains a homeomorph C of $C(N_o)$. Since C is separable, then C contains a countable dense set D„ Then D contains no isolated points, and hence yeB_1.

Conversely, if yeB_1, then say x_1, x_2, ... is a dense-in-itself sequence contained in f~^{1}(y). Let E be the closure in f (y) of this sequence. Then E is dense-in-itself since it is the closure of a dense-in-itself set. Since f is continuous then $f\sim^{1}(y)$ is complete; and hence by *[2,* p.444], E contains a homeomorph of $C(N_o)$. Hence (by Theorem 2.2) $f^{-1}(y)$ is not cr-discrete, and so yeB.

Thus $B = B_{1}$, and it only remains to show that B_1 is absolutely k-analytic.

Now let Y be the completion of f(X) and let W be the product space formed by taking the product of X with itself N_o times. Define

B_2= {(y,(x_1,x_2,...))eYXW: x_1,x_2,... is a dense-in-itself sequence],

B_3 = [(y, (x_1,x_2, . . .)) eYXW: f(x_x)= f(x_2)= ... = y),

and

B_4 = {(y,(x_1,x_2,...))€YxW: if m^n, then x_m/ x_n) .

Then each of the sets B_2,B_3, and *B^* is Borel in YXW (for a proof that B_2 is Borel, see *[2,* p.368]), and hence so is B_2n B_3n B_4. Note that W is of weight $\leq$ k, and also Y is of weight $\leq$ k by Theorem 2.1. Hence YXW is of weight £ k. Finally note that B_1 = ir(B_2n B_3n B^) where *ir* is the projection from YXW onto Y. Therefore B_1 is the continuous image of an absolute Borel set of weight £ k and so B_1 is absolutely k-analytic by [8, p.37].

Hence B is absolutely k-analytic, and the lemma is proved.

We now extend this lemma to continuous maps whose domains are absolute Borel sets.

LEMMA 3.2. If f is a continuous map defined on an absolute Borel set X of weight k, and if B = {yef(X) : f~^{1} (y) not cr-discrete), then B is absolutely k-analytic.

Proof. Let g be a continuous generalized homeomorphism from a closed subset A of B(k) onto X (Theorem 2.3). Let

$$B_1 = \{yefog(A): (f»g)"^{1} (y) \text{ not a-discrete}).$$

Using Theorem 2.4 we see that $B = B_1$. Applying Lemma 3.1 to the continuous map fog defined on the complete space A, we obtain that B is absolutely k-analytic.

We now extend this lemma to obtain the main theorem of this section.

THEOREM 3.3. If f is a measurable map of bounded class defined on an absolute Borel set X of weight k, and if B = {yef(X): f~^{1} (y) not a-discrete}, then B is absolutely k-analytic.

Proof. Let Y denote the completion of f(X), and let T c XXY be the graph of f. By [2,p.384], *T* is absolutely Borel (since f is measurable of bounded class). Let *w* be the projection from *T* into Y and let B_1= [yeTr(r): TT^{1} (y) not cr-discrete]. Lemma 3.2, applied to *T* and TT, yields that B_1 is absolutely k-analytic. But it is easily seen that $B = B_1$. Hence the theorem is proved.

4. NECESSARY CONDITIONS FOR *a*-BIMEASURABILITY. In this section we obtain two extensions of Theorem 1.1. We first prove two lemmas, and then use these together with Theorem 3.3 to obtain the main theorems 4.3 and 4.4.

LEMMA 4.1. If f is a bimeasurable map defined on an (absolute) Borel set X ^ B(k), and if for every yef(X) the set $f^{-1}(y)$ contains a homeomorph of $C(N_o)$, then $f(X)$ is cr-discrete.

Proof. The proof proceeds by contradiction. If $f(X)$ is not cr-discrete, then by Theorem 2.2 $f(X)$ contains a homeomorph of the Cantor set, say C. Index C by an index set G to obtain C = {y^: aeG] . By the hypothesis of the lemma, each set $f^{-1}(y_\alpha)$ contains a homeomorph, say C_α, of the Cantor set. Let D = U{ca: aeG).

There are two cases to consider:

(a) ir_n (D) is countable for all positive integers n (where w_n is the projection from B(k) onto its n-th coordinate space T_n);

(b) v_n (D) is uncountable for some n.

We shall show that in either case we are led to a contradiction.

In case (a), let E = $\prod_{n=1}^{\infty}$ ftr$_n$(D)) and let A = f^{-1}(C) H E. Since for every n, TT^n (D) is Borel in the discrete space T^n, then the countable product, E, is Borel in B(k). Hence A is Borel in X, and therefore absolutely Borel. Now let g be the restriction of f to A. Then g is bimeasurable. Since E is the countable product of separable spaces (each factor v_n(D) is countable by hypothesis), then E and hence A is separable. Also, g(A) = C. Finally note that if yeC, say y = y_{α}, then C_{α}<= <3^1 (y_{α}) > an(3 hence the inverse image of every point in the uncountable set C is uncountable. But this contradicts the theorem of Purves (Theorem 1.1).

We now proceed to obtain a contradiction in case (b).

Let m be a positive integer such that TT_m(D) is uncountable; let T = TT_m (D) . For each ueT choose a point t (u) eD such that IT_m (t(u)) = u; say t (u) eC . Since C Ħ Co = 0 if a and j8 are distinct indices in G, then a(u) is uniquely determined. Let & = (a(u): ueT}. Since T is uncountable and since TT_m (C_{α}) is finite for all *a* (because each "nMntC/y) ^s a compact subset of a discrete space), then IB is uncountable. For each jSefe, choose ug€T such that |8 = a (up) . Let F = {t(ug) : j8€B} .

Let h be the restriction of the map f to F. Then h is 1-1 and bimeasurable, hence, a Borel isomorphism. Also note that F is cr-discrete; in fact any two distinct points in F are of distance at least 1/m apart. By Theorem 2.4 h(F) is also a-discrete. Since IB is uncountable then F is uncountable, and therefore so is h(F). But h(F) is a subset of the Cantor set C, and hence we have a contradiction since any cr-discrete subset of a separable space is countable.

Cases (a) and (b) both lead to contradictions, and therefore

$f(X)$ is in fact a-discrete.

We now extend this lemma from Borel subsets of $B(k)$ to arbitrary Borel sets.

LEMMA 4.2. If f is a bimeasurable map defined on an (absolute) Borel set X, and if for every $y\epsilon f(X)$ the set $f^{-1}(y)$ contains a homeomorph of $c(K_o)$, then $f(X)$ is a-discrete.

Proof. Let X be of weight k, and by Theorem 2.3 let g be a generalized homeomorphism from a closed subset H of $B(k)$ onto X. If $y\epsilon f\circ g(H)$, then $(f\circ g)^{-1}(y)$ contains a homeomorph of $C(N_o)$ by Theorems 2.3 and 2.4. By applying Lemma 4.1 to the map $f\circ g$ and the set H, we obtain that $f(X)$ $(= f\circ g(H))$ is a-discrete.

We now prove the main theorems of this section.

THEOREM 4.3. If f is an a-bimeasurable map defined on an absolute Borel space X of weight k, and if $B = \{y\epsilon f(X)\colon f^{-1}(y)$ not a-discrete$\}$, then $\operatorname{card} B \leq k$.

Proof. By Theorem 3.3, B is absolutely k-analytic. By Theorem 2.1, $f(X)$, and hence B, has weight at most k. If $\operatorname{card} B > k$, then by [8,p.37], B contains a closed subset D homeomorphic to a Baire space of cardinal k^{**o}. Hence D contains a set C homeomorphic to $C(N^o)$. Let g be the restriction of f to the Borel set $f^{-1}(C)$. Then if $y\epsilon C$, $g^{-1}(y)$ contains a homeomorph of $C(K^o)$ by Theorem 2.2. Hence

Lenuna 4.2 applied to the map g and the set $f"^{1}(C)$ yields the result that $g(f\sim^{1}(O) = C$ is CT-discrete, a contradiction since the Cantor set C is not cr-discrete. Therefore card B $\leq$ k.

THEOREM 4.4. If f is an a-bimeasurable map defined on an absolute Borel space X of weight k, and if B* = {yef(X): card $f"^{1}(y) > k$), then card B* $\leq$ k.

Proof. We shall show that B*c B, where B is the set defined in the hypothesis of Theorem 4.3. If yeB*, then by [8,p.37] $f"^{1}(y)$ contains a homeomorph of $C(N_o)$. Therefore $f\sim^{1}(y)$ is not CT-discrete and hence yeB. Thus B*c B. If card B* > k, then card B > k also——a contradiction of Theorem 4.3. Hence card B* $\leq$ k.

Note that Theorems 4.3 and 4.4 each reduce to the theorem of Purves if the spaces are separable. For if $k = N_o$, then the property of being a-bimeasurable for some ordinal *a* is equivalent to bimeasurability; also B = B* = [yef(X): $f\sim^{1}(y)$ uncountable] since cr-discreteness and countability are equivalent in separable spaces.

Note also that Theorems 4.3 and 4.4 can be extended from a-bimeasurable to bimeasurable maps if it can be shown that Lemma 3.2 holds for continuous maps defined on absolutely J^*_o- analytic sets. For it is not difficult to verify that the graph of any measurable map is ^-analytic, and using this in connection with the suggested extension of Lemma 3.2 would yield a new

Theorem 3.3 valid for any measurable map (of bounded class or not).

Finally note that neither Theorem 4.3 nor 4.4 yield any result regarding the Borel structure of the sets B and B*. In the next section we study cases in which they do have a strong Borel structure.

5. FURTHER GENERALIZATIONS AND APPLICATIONS. In this section we study the sets B and B^* of Theorems 4.3 and 4.4. We also consider the bimeasurability of projection maps and the preservation of cr-discreteness under bimeasurable maps. We then discuss the problem of finding a sufficient condition for bimeasurability.

THEOREM 5.1. If f is a bimeasurable map defined on an absolute Borel space X of weight k, and if $B = \{y \in f(X) : f^{-1}(y) \text{ not cr-discrete}\}$ is absolutely N_o-analytic, then B is cr-discrete.

Proof. The proof turns on a theorem of El'kin [l,p.874] which extends Theorem 2.2 of Stone from the class of absolute Borel spaces to the class of absolutely N_o-analytic spaces.

If B is not cr-discrete, then by the theorem of El'kin B contains a homeomorph C of $C(N_o)$. Let g be the restriction of f to the space $f^{-1}(c)$. Applying Lemma 4.2 to g and $f^{-1}(C)$ yields that $g(f^{-1}(C)) = C$ is cr-discrete — a contradiction. Therefore B is cr-discrete.

THEOREM 5.2. If f is a bimeasurable map defined on an absolute Borel space X of weight k, and if $B^* = (y \in f(X)$: card $f^{-1}(y) > k\}$ is absolutely N_Q-analytic, then B^* is cr-discrete.

Proof. If B^* is not cr-discrete, then the theorem of El'kin gives a homeomorph C of $C(N_o)$ in B^*. By [8,p.37], if yeC then f~^{1}(y) contains a homeomorph of C(N_o). Now restrict f to f~^{1}(C) and use Lemma 4.2 to obtain a contradiction.

If additional assumptions are placed on the map, we obtain the following:

THEOREM 5.3. If f is a closed, O-bimeasurable map defined on an absolute Borel space X of weight k, and if B = (yef(X): f~^{1} (y) uncountable], then card B $\leq$ k.

Proof. The proof is basically a piecing-together of Theorem 4.3 and the following theorem of Las'nev [3,p.1505]: if f is a closed, continuous map defined on a metric space X and mapping into a T_1-space, then f~^{1}(y) is compact except for a a-discrete set of points in f(X).

Using this theorem we write f(X) = A_1 U A_2 where A_2 is a-discrete and if yeA_1 then f~^{1}(y) is compact. Applying Theorem 4.3 to the restriction of f on the absolute Borel set f~^{1} (A_1) yields that A_1 = A_3U A_4 where card A_3 $\leq$ k and if yeA_4 then f~^{1}(y) is a-discrete. Hence if yeA_4 then f^{-1}(y) is both compact and a-discrete; therefore f"^{1}(y) is countable. Hence B <= A_2U A_3. But A_2 is a a-discrete subset of f(X), a space of weight $\leq$ k by Theorem 2.1, hence A_2 is of cardinal $\leq$ k. Since the cardinal of A_3 is also £ k, then card B < k.

We now study the bimeasurability of projection maps.

THEOREM 5.4. Let X and Y be absolute Borel spaces and let *w* be the projection map from the product space XXY onto X. If *ir* is bimeasurable, then either X or Y is cr-discrete.

Proof. Assume that Y is not cr-discrete. Let B = {xeX: ir^{-1} (x) not cr-discretej . Since w^{-1} (x) is homeomorphic to Y for all xeX, and since Y is not a-discrete, then B = X. Applying Theorem 5.1 yields that X is cr-discrete.

As a partial converse we have the following:

THEOREM 5.5. Let X and Y be absolute Borel spaces and let *ir* be the projection from the product space XXY onto X. If X is cr-discrete, then *ir* is bimeasurable.

Proof. Since *ir* is continuous, it is measurable. Since X is cr-discrete then every subset of X is absolutely Borel **by** [9, p. 660] . Hence *ir* is bimeasurable.

Piecing together the results of both Theorems 5.4 and 5.5 we have the following:

COROLLARY 5.6. Let X be an absolute Borel space and let *v* be a projection from the product space XXX onto X. Then *v* is bimeasurable if and only if X is cr-discrete.

In Theorem 5.5 if Y is cr-discrete but X is not, then *v* need not be bimeasurable. To see this, let X be the Cantor set and let A be a non-Borel subset of X. Let Y be the set A with the discrete topology. If B = {(x,x): xeY], then B is Borel in XXY, but *ir* (B) = A which is not Borel. Hence *ir* is not bimeasurable.

THEOREM 5.7. If f is a bimeasurable map defined on a cr-discrete space X, then f(X) is cr-discrete.

Proof. If not, then since f(X) is absolutely Borel, f(X) contains a homeomorph of $C(\aleph_0)$ by Theorem 2.2 and hence a non-Borel set, say A. Then $f^{-1}(A)$ is not Borel — a contradiction since every subset of the cr-discrete space X is Borel.

THEOREM 5.8. If f is a continuous map defined on a space X such that f(X) is cr-discrete and $f^{-1}(y)$ is cr-discrete for all yef(X), then X is cr-discrete.

Proof. Write $f(X) = \bigcup_{n=1}^{\infty} Y_n$, where each Y_n is relatively discrete. For yef(X), write $f^{-1}(y) = \bigcup_{m=1}^{\infty} X_m(y)$ where $X_m(y)$ is relatively discrete. Let $X_{mn} = \bigcup\{X_m(y): yeY_n\}$. Then $X = \bigcup\{X_{mn}: m,n=1,2,...\}$. To show that each set X_{mn} is relatively discrete, let xeX_{mn}. Let $y_0 = f(x)$. Then y_0 belongs to the relatively discrete set Y_n, and so there is an open set U such that $y_0 eU$ and $U \cap Y_n = \{y_0\}$. Since $xeX_m(y_0)$ which is relatively discrete, there is an open set V such that xeV and $V \cap X_m(y_0) = \{x\}$. Then x belongs to the open set $f^{-1}(U) \cap V$, and $(f^{-1}(U) \cap V) \cap X_{mn} = \{x\}$. Hence X_{mn} is relatively discrete, and therefore X is cr-discrete.

We finally comment on the open problem of finding non-trivial sufficient conditions that a measurable map be bimeasurable.

If the spaces in question are separable and absolutely Borel, such a condition is known and is due to Lusin [4]: namely, that the inverse image of every point be countable. In fact it is easily seen that this condition may be weakened so that the condition in Purves' theorem is both necessary and sufficient.

This "countable-to-one" condition fails if the spaces are not separable: let D be the Cantor set with the discrete topology and let f be the identity map from D onto the Cantor set. Then f is continuous but not bimeasurable. Hence even a 1-1 continuous map need not be bimeasurable.

Note that an open, closed, continuous map need not be bimeasurable. For if *IT* is the projection from the unit square onto the unit interval, then TT is open, closed, and continuous, yet by Corollary 5.6. *ir* is not bimeasurable.

BIBLIOGRAPHY

[1] A.C. El'kin, "A-sets in Complete Metric Spaces," Soviet Math. Dokl., 8 (1967), pp. 874-877.

[2] K. Kuratowski, Topology Volume I (J. Jaworowski, transl.), Academic Press, New York, 1966.

[3] N. Lašnev, "Continuous Decompositions and Closed Mappings of Metric Spaces," Soviet Math. Dokl., 6 (1965), pp. 1504-1506.

[4] N. Lusin, Leçons sur les Ensembles Analytiques et leurs applications, Gauthier-Villars, Paris, 1930.

[5] S. Mazurkiewicz and W. Sierpinski, "Sur un problème concernant les fonctions continues," Fund. Math., 6 (1924), pp. 161-169.

[6] R. Purves, "Bimeasurable Functions," Fund. Math., 58 (1966), pp. 149-157.

[7] A.H. Stone, "Cardinals of Closed Sets," Mathematika, 6 (1959), pp. 99-107.

[8]___________, "Non-separable Borel Sets," Rozprawy Matematyczne 28, Warsaw, 1962.

[9]___________, "On a-discreteness and Borel Isomorphism," Amer. Jour, of Math., 85 (1963), pp. 655-666.