EXTREME POINTS OF THE UNIT CELL LEBESGUE-BOCHNER FUNCTION SPACES I

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Several interesting results have been announced recently concerning the extremal structure of the unit cell in $C_B(X)$, the space of continuous Banach space valued functions on a compact Hausdorff space X with the supremum norm. For these and related results see Blumenthal, Lindenstrauss, and Phelps [1] (hereafter referred to as BLP), Phelps [2], Peck [3], and Cantwell [4]. The present paper is concerned with the extreme points of the unit cell of a space of Banach space valued functions which is an abstract analogue of the space L_p. For a detailed account of these spaces we refer to Bochner and Taylor [5], Bogdanowicz [6], Edwards [7] and Dinculeanu [8].

We adhere to the following notation; μ denotes the contraction of the Lebesgue measure to the unit interval $I = [0,1]$. X_E denotes the characteristic function of the set $E \subset I$. If C is a set then Ext C denotes the set of extreme points of C. If f is a Banach space valued function, $S_f = \{t | f(t) \neq 0\}$. If B is a Banach space with the norm $\| \ \|$ and f is a function on $I \to B$ then $P(f)$ is the function on $I \to B$ defined by

$$P(f)(t) = \frac{f(t)}{\|f(t)\|} \quad \text{if} \quad t \in S_f,$$

$$P(f)(t) = 0 \quad \text{if} \quad t \notin S_f.$$

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Definition. Let B be a Banach space. The class of all B-valued Lebesque measurable functions f on I such that the function $t \to \|f(t)\|$ is p-summable on $I (p \geq 1)$ is denoted by $L_p\{B\}$. Identifying the functions in $L_p\{B\}$ which agree a.e. and equipping the resulting linear space with the norm $\|f\| = [\int_I \|f(t)\|^p d\mu]^{\frac{1}{p}}$ we obtain a Banach space. We continue to denote this Banach space by $L_p\{B\}$.

Throughout the paper U is the unit cell in B, $U_p(B)$ is the unit cell in $L_p\{B\}$. Our first proposition concerns the cell $U_1(B)$. It is known when B is the real line $U_1(B)$ has no extreme points (see for example page 81, Day [11]).

Proposition 1. <u>The cell $U_1(B)$ has no extreme points</u>.

Proof. Let $f \in U_1(B)$ with $\|f\| = 1$. Since the function $x \to \int_0^x \|f(t)\| d\mu$ is a continuous function on I there exist a pair of disjoint measurable sets A_1 and A_2 such that $\int_{A_1} \|f(t)\| d\mu = \int_{A_2} \|f(t)\| d\mu \neq 0$. Let $B = A_1 \cup A_2$ and let g_i $i = 1,2$ be the functions on I defined by $g_1 = X_{I\sim B} f + (1+\epsilon) X_{A_1} f + (1-\epsilon) X_{A_2} f$ and g_2 is the same as g_1 except that ϵ is replaced by $-\epsilon$. With $0 < \epsilon < 1$ it is verified that $g_i \in U_1(B)$, $i = 1,2$, $g_1 \neq g_2$ and $f = \frac{g_1+g_2}{2}$. Thus Ext $U_1(B) = \phi$ as was to be shown.

Next we proceed to the case of $L_p\{B\}$, $1 < p < \infty$.

Theorem 1. <u>If $1 < p < \infty$ then a function $f \in L_p\{B\}$ with $\|f\| = 1$ is an extreme point of $U_p(B)$ if and only if</u>

$$\frac{P(f)}{[\mu(S_f)]^{\frac{1}{p}}} \in \text{Ext } U_p(B).$$

Proof. Let $f \in L_p(B)$ with $\|f\| = 1$. Let us recall the well-known Clarkson inequalities for $L_p(R)$. If $x, y \in L_p(R)$ then

$$\|x+y\|^p + \|x-y\|^p \leq 2^{p-1}[\|x\|^p + \|y\|^p] \quad \text{if} \quad 2 \leq p \quad \text{and}$$

$$\|x+y\|^q + \|x-y\|^q \leq 2[\|x\|^p + \|y\|^p] \quad \text{if} \quad 1 < p \leq 2 \quad \text{and} \quad q = \frac{p}{p-1} .$$

Using these inequalities it is verified (*) if $f = \frac{g_1+g_2}{2}$ and $g_1, g_2 \in U_p(B)$ then $\|f(t)\| = \|g_1(t)\| = \|g_2(t)\|$ a.e.. Thus, if $\frac{f}{[\mu(S_f)]^{\frac{1}{p}}} \notin \operatorname{Ext} U_p(B)$ then there exist $g_i \in U_p(B)$ $i = 1,2$ such that $g_1 \neq g_2$ and $\frac{f}{[\mu(S_f)]^{\frac{1}{p}}} = \frac{g_1+g_2}{2}$ where $\|g_1(t)\| = \|g_2(t)\| = [\mu(S_f)]^{-\frac{1}{p}}$ a.e.. Hence $f = \frac{h_1+h_2}{2}$ where $h_i(t) = [\mu(S_f)]^{\frac{1}{p}} \|f(t)\| g_i(t)$. It is verified that $f \notin \operatorname{Ext} U_p(B)$.

Conversely if $f \notin \operatorname{Ext} U(B)$ then there exist $g_i \in U_p(B)$ such that $f = \frac{g_1+g_2}{2}$, $g_1 \neq g_2$. Hence by (*) $\|f(t)\| = \|g_1(t)\| = \|g_2(t)\|$ a.e.. In particular $\chi_{S_f} = \chi_{S_{g_1}} = \chi_{S_{g_2}}$. With these observations it follows that $\frac{f}{[\mu(S_f)]^{\frac{1}{p}}} = \frac{a_1+a_2}{2}$, $a_1 \neq a_2$, where

$a_i = \frac{g_i}{[\mu(S_{g_i})]^{\frac{1}{p}}}$, $i = 1,2$. Since $a_i \in U_p(B)$, $\frac{f}{[\mu(S_f)]^{\frac{1}{p}}} \notin \operatorname{Ext} U_p(B)$ completing the proof.

It is natural to inquire whether $f \in \operatorname{Ext} U_p(B)(p>1)$ if and only if $\frac{f(t)}{\|f(t)\|} \in \operatorname{Ext} U$ for t a.e. in S_f. The results that follow show that this assertion is true if B is finite dimensional while the if part is always true.

Theorem 2. *If* $1 < p < \infty$ *then* $f \in \text{Ext}\, U_p(B)$ *iff* $\|f\| = 1$ *and* $\frac{f(t)}{\|f(t)\|} \in \text{Ext}\, U$ *for* t *a.e. in* S_f.

Proof. Let g be the function $\frac{f(t)}{\|f(t)\|\,[\mu(S_f)]^{1/p}}$. If $g \notin \text{Ext}\, U_p(B)$ then there exist $g_1, g_2 \in U_p(B)$ such that $g = \frac{g_1+g_2}{2}$ and $g_1 \neq g_2$. Since $1 < p < \infty$ as observed in (*) in the proof of Theorem 1 it follows that $\|g(t)\| = \|g_1(t)\| = \|g_2(t)\| = \frac{1}{[\mu(S_f)]^{1/p}}$ for t a.e. in S_f. Hence in particular $\chi_{S_f} = \chi_{S_{g_1}} = \chi_{S_{g_2}}$. Since $\frac{g_1+g_2}{2} = g = \frac{f(t)}{\|f(t)\|\,[\mu(S_f)]^{1/p}}$ for $t \in S_f$. Thus if M is the measurable set $\{t : g_1(t) \neq g_2(t)\}$ then $\mu(M) > 0$ and $\mu(M \cap S_f) = \mu(M)$ and for $t \in M \cap S_f$, $\frac{f(t)}{\|f(t)\|} \notin \text{Ext}\, U$ contradicting the hypothesis. Hence $g \in \text{Ext}\, U_p(B)$ but this implies $f \in \text{Ext}\, U_p(B)$ by Theorem 1.

Before proceeding to the converse of Theorem 2, we establish two useful lemmas. We state these lemmas in a more general setting than required.

Lemma 1. *Let* C *be a compact convex subset of a finite dimensional Banach space and* K *a compact subset of* I. *Let* $f: K \to C$ *be a continuous mapping such that for all* $t \in K$, $f(t) \notin \text{Ext}\, C$. *Then there exists a measurable set* $M \subset K$, $\mu(M) > 0$, *a positive number* δ *such that if* $t \in M$ *there exist* $Y_t, Z_t \in C$ *with the properties* $f(t) = \frac{Y_t+Z_t}{2}$ *and* $\|Y_t - Z_t\| \geq \delta$.

Proof. Since for $t \in K$ $f(t) \in C - \text{Ext}\, C$, for each $t \in K$ there exist $Y_t, Z_t \in C$ such that $f(t) = \frac{Y_t+Z_t}{2}$ and $\|Y_t - Z_t\| > 0$. Let $\epsilon > 0$ and M_ϵ be the set of all points t in K such that there exist $y_t, z_t \in C$, $f(t) = \frac{y_t+z_t}{2}$ and $\|y_t - z_t\| \geq \epsilon$. M_ϵ is a closed subset of K. For let $\{t_n\}$ be a sequence in M_ϵ such that $t_n \to t$ for some $t \in K$. Let $\{y_n\},\{z_n\}$ be sequences in C such that $f(t_n) = \frac{y_n+z_n}{2}$ and $\|y_n - z_n\| \geq \epsilon$. Since C is a compact set there exist convergent subsequences $\{y_{n_i}\}$ and $\{z_{n_i}\}$ in y and z respectively. Let $y_{n_i} \to y_o$ and $z_{n_i} \to z_o$. Since f is continuous $f(t_{n_i}) \to f(t) = \frac{y_o+z_o}{2}$. Further $\|y_o - z_o\| = \lim\|y_{n_i} - z_{n_i}\| \geq \epsilon$. Thus $t \in M_\epsilon$ and M_ϵ is a closed subset of K. Let $\{B_n\}$ be the sequence of Borel sets in K defined by $B_n = M_{\frac{1}{n+1}} \sim M_{\frac{1}{n}}$. Then $\{B_n\}$ is a measurable partition of K. Since $\mu(K) > 0$ there exists an integer m such that $\mu(B_m) > 0$. Thus choosing $\frac{1}{m+1}$ for δ and B_m for M the proof is completed.

Before proceeding to the next lemma, we recall a definition and a theorem concerning set valued functions. Let X,Y be two topological spaces and 2^Y be the set of closed sets in Y. A mapping $F : X \to 2^Y$ is called upper semicontinuous (u.s.c.) if the set $\{x | F(x) \subset G\}$ is open in X for all open sets $G \subset Y$. We state a selection theorem, Kuratowski and Ryll-Nardzewski [12] Theorem [Kuratowski and Ryll-Nardzewski]. Let $X, (Y,d)$ be two metric spaces and y d-complete and separable. If $F : X \to 2^Y$ is a u.s.c. map then there exists a Borel measurable function $f : X \to Y$ such that $f(x) \in F(x)$.

Lemma 2. rf C,K,f are as in the preceding lemma, then there exist two measurable functions f_1, f_2 cm K -> C such that

$f = \frac{X}{2}$ ^ and $M\{t | f_1(t) \wedge f_2(t)\} > 0$.

Proof. It follows from the preceeding lemma that there exist a compact set K_1 c: K^ $M(K_n) > 0$ and two functions g_1, g_2 on $K_\pm$ - C such that $f(t) = \frac{g^tj+g^t)}{}$^ and llSj^t) - 9 2 ^ " ≥ 26 for some positive number 6. Thus, there exists a function $F : K_1 - 2^C$, F(t) being the non-empty closed set of points ξ∈C such that for some TjeC, f(t) = ^j^1 and ||"f - r)\\ ≥ 26. Further F is a u.s.c. map as shown below. Let G be an open subset of C and $G_1 = \{x | F(x) \subset G\}$. Suppose that $X \in G_1$ and' that there exists no neighborhood N of x such that for all yeN, F(y) CIG. It follows that there exists a sequence $[x_n)$ in K_n, $x_n \to x^1$, such that $F(x_n) \setminus G$ for all n. Thus there exists a sequence $\{£_n\}$, $^_n GF(x_n)$ *, G. Considering a sequence $\{77_n)$ with $f(x_n) = \frac{n \quad n}{2}$ and $||t_n - 77_n|| \geq 26$, assured by the function F, it follows by straightforward compactness arguments that there exists a subsequence $[\%_{n_i}\}$ in $\{§_n\}$ $!>_{n_i}$ -• ^ for some ^eF(x). Since G is a neighborhood of x^1 there exists $*?_{n_i}$ GG contradicting the choice of ^_n. Thus F is a u.s.c. map. Hence by the Kur towski and Ryll-Nardzewski theorem there exists a measurable function f : K^^ - C with f(x)GF(x) for all $x \in K_1$. Let $f_2(x)$GC be such that $||f_L(x) - f_2(x)|| \geq 26$

and $f(x) = \frac{f_1(x)+f_2(x)}{2}$. Then the function f_2 is also measurable and $f_1(x) \neq f_2(x)$ for all $x \in K_1$ and the proof of the lemma is complete.

Since the unit cell of a finite dimensional Banach space is a compact convex set the preceding lemma implies the following theorem.

Theorem 3. If B is a finite dimensional Banach space then $f \in \text{Ext } U_p(B)$ $(1 < p < \infty)$ if and only if $\|f\| = 1$ and $\frac{f(t)}{\|f(t)\|} \in \text{Ext } U$ for t a.e. in S_f.

Proof. The if part is taken care of by Theorem 1. Conversely if $f \in \text{Ext } U_p(B)$ then clearly $\|f\| = 1$. Since Ext U is a G_δ subset of U (see proposition 1.3 [9]) and since $P(f)$ is measurable if f is measurable the set $\{t \mid \frac{f(t)}{\|f(t)\|} \notin \text{Ext } U, t \in S_f\}$ is measurable. Thus if $\frac{f(t)}{\|f(t)\|} \notin \text{Ext } U$ for t a.e. in S_f then there exists a measurable set $M \subseteq S_f$, $\mu(M) > 0$ such that for $t \in M$, $\frac{f(t)}{\|f(t)\|} \notin \text{Ext } U$. Since μ is regular there exists a compact set $K \subseteq M$ with $\mu(K) > 0$ such that the restriction of $g = [\mu(S_f)]^{1/p} P(f)$ to K is a continuous function into $[\mu(S_f)]^{1/p}(U \sim \text{Ext } U)$. Hence by lemma 2, there exist measurable functions g_i, $i = 1,2$ on K to $[\mu(S_f)]^{1/p}(U \sim \text{Ext } U)$ such that $g_1(t) \neq g_2(t) > 0$ and the restriction of g to $K = \frac{g_1+g_2}{2}$. Now defining $f_i : I \to B$ by $f_i(t) = g_i(t)$ if $t \in K$ and $f_i(t) = g(t)$ if $t \notin K$ it follows that $g_i \in U_p(B)$ and $g = \frac{f_1+f_2}{2}$ and $f_1 \neq f_2$. Thus $g \notin \text{Ext } U_p(B)$

which in turn by Theorem 1 implies $f \notin \text{Ext } U_p(B)$ contradicting our choice of f.

Remark. In view of theorem 1, p. 490 of [11], theorem 3 in BLP deals with the same question as our theorem 3, except that they consider $C_B(X)$ the space of continuous functions on a compact Hausdorff space into the space B with the supremum norm. It might be worthwhile to summarize this theorem in [1]. Denoting the unit cell of $C_B(X)$ by V the theorem states that $f \in \text{Ext } V$ if and only if $\|f(t)\| = 1$ for all $t \in X$ and $f(t) \in \text{Ext } U$ for t in a dense subset of X if $\dim B \leq 3$ or B is finite dimensional with a polyhedral unit cell. Even in the case when $X = I$ and B is 4-dimensional they provide a counter example by exhibiting a function $f \in \text{Ext } V$ but for all $t \in I$, $f(t) \notin \text{Ext } U$. Thus theorem 3 of this paper is in sharp contrast with theorem 3 in BLP [1].

Next we proceed to the case of the Banach space $L_\infty\{B\}$ of measurable functions f on I into B such that the function $t \to \|f(t)\|$ is essentially bounded with $\|f\| = \operatorname{ess\,sup}_{t \in I} \|f(t)\|$. Let $U_\infty(B)$ be the unit cell of $L_\infty\{B\}$.

<u>Theorem</u> 4. $f \in \text{Ext } U_\infty(B)$ if $f(t) \in \text{Ext } U$ a.e. Further if B is finite dimensional then the above condition is necessary and sufficient.

The proof of the more difficult part of the theorem i.e. the necessity of the condition, is essentially the same as that of theorem 3 and the details are not supplied.

In conclusion it might be mentioned that a complete characteri-zation of extreme points of $U_p(B)$ is not provided here when B is infinite dimensional and we hope to consider this question elsewhere.

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