EXTREME POINTS OF THE UNIT CELL IN LEBESGUE-BOCHNER FUNCTIONS SPACES II

by

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The extremal structure of the unit cell in $C_B(X)$, the Banach space of continuous functions on a compact Hausdorff space X into a Banach space B with the supremum norm has received considerable attention in recent years. For these and related results see the papers of Blumenthal, Lindenstrauss and Phelps [1], Phelps [2] and Peck [3]. As pointed out in [1] and [2] part of the motivation to the problem stems from an attempt to characterize the extreme operators on a Banach space B into C(X), the space of real valued continuous functions on a compact Hausdorff space X with the usual supremum norm. A similar problem of interest is to characterize the extreme operators on Banach spaces $L_1(X, \Sigma, \mu)$ into a reflexive Banach space. From the representation theorems for such operators in Dunford and Schwartz [4] it is easily verified that the problem mentioned above is related to the extremal structure of the unit cell in Lebesgue-Bochner function spaces. For a detailed account of these spaces we refer to Bochner and Taylor [5], Edwards [6], Hille and Philips [7] and Bogdanowicz [8] and Dinculeanu [9].

With the above motivation we consider in this paper the problem of characterising the extreme points of the unit cell in Lebesque-Bochner function spaces L^p_E, $1 \le p \le \infty$, which are abstract generalisations of the classical L^p spaces. The case when E is

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finite dimensional has been considered in an earlier paper, Sundaresan [10] .

We adhere to the following notation throughout the paper. (X, T%|u) is a fixed measure space with X a locally compact Hausdorff space, £ the cr-ring of Borel sets in X and ju a regular positive measure. For a definition of these terms see Halmos [11] . As there is certain divergence in the terminology concerning vector valued measurable functions we wish to point out the terminology adopted here is the same as in the Chapter 8, Edwards [6]. Thus if E is a locally convex vector space then a function f: X -• E is measurable if f has the Lusin property, i.e. if K is a compact subset of X then for each € > 0 there exists a compact set C c K such that u(C~K) < *e* and the restriction of f to C, f|c, is continuous. The definition of a vector valued measurable function in [9] differs from the above definition. However when the range of f is metrizable then either of the defintions implies the other. If E is a Banach space then a function f: X -• E is measurable if it is measurable with respect to the strong topology on E. A function f: X -> E is w(w*) measurable if f is measurable with respect to the weak (weak* topology if appropriate) on E. If E is a Banach space the linear space of measurable functions f on X to E such that the function x - ||f(x) |]^p (1<£<«>) is ju-suramable is denoted by L^p_E. After the usual identification of functions agreeing a.e. it is verified that L^p_E is a Banach space when equipped with the norm $\|f\| = [\int_X \|f(x)\|^p d\mu]^{1/p}$. Likewise L_E^∞ is the Banach space of essentially bounded measurable functions f on X to E with the norm $\|f\| = \text{ess} \sup_{x \in X} \|f(x)\|$. We denote the norm in $L_E^p (1 \leq p \leq \infty)$ and the norm in E by the same symbol $\| \ \|$ as there is no occasion for confusion. U_E^p is the unit cell in L_E^p and $U_E(S_E)$ is the unit cell (unit sphere) in E. If f is a measurable function S_f is the set $\{x | f(x) \neq 0\}$ and $P(f)$ is the measurable function defined by $P(f)(x) = \frac{f(x)}{\|f(x)\|}$ if $x \in S_f$ and $P(f)(x) = 0$ if $x \notin S_f$. If A is a set X_A is the characteristic function of A and if C is a convex set ext C is the set of extreme points of C.

Before proceeding to the main results we deal with the simpler case of L_E^1 and then state a preliminary theorem of considerable use in the subsequent discussion.

Proposition 1. A function $f \in L_E^1$ is an extreme point of U_E^1 if and only if there exists an atom $A \subset X$ and a point $e \in \text{Ext } U_E$ such that $f(x) = \pm \frac{X_A(x) e}{\mu(A)}$ for x a.e..

Proof. As a first step we verify if there exist two disjoint measurable sets $C_1, C_2 \subset S_f$ of positive measure then f is not an extreme point of U_E^1. If such a pair of measurable sets exist and $C = C_1 \cup C_2$ let λ_i, $0 < \lambda_i < 1$, $i = 1,2$ be the numbers defined by

$$\int_{C_i} \|f(x)\| d\mu = \lambda_i \int_C \|f(x)\| d\mu.$$

Let g_i, $i = 1,2$ be the functions defined by

$$g_i = f\chi_{X\sim C} + (1+\lambda_j)f\chi_{C_i} + (1-\lambda_i)f\chi_{C_j}$$

where $j = 2$ if $i = 1$ and $j = 1$ if $i = 2$. It is verified that $g_i \in L^1_E$, $\|g_i\| = 1$, $f = \frac{g_1+g_2}{2}$ and $g_1 \neq g_2$. Thus $f \notin \text{Ext } U^1_E$.

From the observation in the preceding paragraph and the regularity of the measure μ it follows that if $f \in \text{Ext } U^1_E$ then S_f does not contain any measurable set M of positive measure, $\mu(M)<\mu(S_f)$. Since $\|f(x)\|$ is summable it is verified that S_f is an atom. Hence there exists a real number c and a vector $e \in E$ such that $f(x) = c\chi_{S_f}(x)e$ for x a.e.. Since $f \in \text{Ext } U^1_E$ it is verified that $c = \pm \frac{1}{\mu(S_f)}$ and $e \in \text{Ext } U_E$. This completes the proof of the " only if" part. Since the " if" part is easily verified the proof of the proposition is complete.

Theorem 1. If $1 < p < \infty$ then a function $f \in L^p_E$ with $\|f\| = 1$ is an extreme point of U^p_E if and only if $\frac{P(f\chi_M)}{[\mu(M)]^{1/p}} \in \text{Ext } U^p_E$ for every Borel set M such that $0 < \mu(M) < \infty$ and $M \subset S_f$.

Proof. Before proceeding to the proof of the " only if" part let us recall the well-known Clarkson inequalities for L^p_R where R is the real line. If $f,g \in L^p_R$ then

$$\|f+g\|^p + \|f-g\|^p \leq 2^{p-1}[\|f\|^p+\|g\|^p] \quad \text{if}$$

$2 \leq p < \infty$ and

$$\|f+g\|^q + \|f-g\|^q \leq 2[\|f\|^p+\|g\|^p]$$

if $1 < p \leq 2$ and $q = \frac{p}{p-1}$. Using these inequalities it is verified (*) if $f = \frac{g_1+g_2}{2}$ and $\|f\| = 1 = \|g_i\|$ then $\|f(x)\| = \|g_i(x)\|$, $i = 1,2$. Let now $f \in L^p_E$ with $\|f\| = 1$. Let M be a Borel set such that $M \subset S_f$ and $0 < \mu(M) < \infty$. Let $g = [\mu(M)]^{-1/p} P(f\chi_M)$. If possible let $g \notin \text{Ext } U^p_E$. It follows from (*) that there exist $g_i \in U^p_E$ such that $g = \frac{g_1+g_2}{2}$ and $\|g(x)\| = \|g_1(x)\| = \|g_2(x)\|$ a.e.. Thus if h_i, $i = 1,2$ are the functions defined by $h_i(x) = [\mu(M)]^{1/p} \|f(x)\| g_i(x)$ if $x \in M$ and $h_i(x) = f(x)$ if $x \notin M$ then $f = \frac{h_1+h_2}{2}$ and $\|h_i\| = 1$. Hence $f \notin \text{Ext } U^p_E$ completing the proof of the "only if" part.

Conversely if $f \in U^p_E$, $\|f\| = 1$ and $f \notin \text{Ext } U^p_E$ then there exist $g_i \in U^p_E$ such that $f = \frac{g_1+g_2}{2}$ and $g_1 \neq g_2$. Hence from (*) it follows that $\|f(x)\| = \|g_i(x)\|$ there exists a measurable set $M \subset S_f$, $0 < \mu(M) < \infty$ such that for all $x \in M$, $g_1(x) \neq g_2(x)$. It is verified that

$$\frac{P(f\chi_M)}{[\mu(M)]^{1/p}} = \frac{1}{2}\left[\frac{P(g_1\chi_M)}{[\mu(M)]^{1/p}} + \frac{P(g_2\chi_M)}{[\mu(M)]^{1/p}}\right].$$

Since the functions appearing in the right bracket are verified to be in U^p_E it follows that $\frac{P(f\chi_M)}{[\mu(M)]^{1/p}} \notin \text{Ext } U^p_E$ completing the proof of "if" part.

Remark 1. A characterization of extreme points of U^∞_E similar to the one provided in the preceding theorem is evident for if $f \in \text{Ext } U^\infty_E$ then $\|f(x)\| = 1$ a.e.. The verification of this assertion is as follows. If possible let $f \in \text{Ext } U^\infty_E$ and M be a Borel set, $0 < \mu(M) < \infty$ such that for $x \in M$, $\|f(x)\| < 1$. Since μ is a regular measure there exists a compact set $C \subset M$ of positive measure such that $f|C$ is continuous. Hence $f(C)$ is a compact set in the interior of U_E. Thus there exists a vector V in U_E (choose for V any vector with $0 < \|V\| < 1 - \underset{x \in C}{\mathrm{Max}}\|f(x)\|$) such that $\|f(x) \pm V\| \leq 1$ for all $x \in C$. Let now g_i, $i = 1,2$ be the functions $(f \pm V)X_C + fX_{X\sim C}$ where V is the constant function with $\{V\}$ as the range. It is verified that $\|g_i\| = 1$, $f = \frac{g_1+g_2}{2}$ and $g_1 \neq g_2$. Thus $f \in \mathrm{Ext}\, U_E^\infty$ completing the proof of the assertion

Next we proceed to the main results. We study the case when E is a separable conjugate Banach space (in particular when E is a separable reflexive Banach space) and show in this case the extreme points in U_E^p, $1 < p \leq \infty$ could be related to the extreme points in U_E.

Before proceeding to these we recall some facts required in the proof of the next theorem.

(a) The W*-topology relativised to the unit cell of E where E is a separable conjugate Banach space is metrizable. We can further assume that a metric d on U_E could be defined to satisfy the additional requirement $d(p,q) \leq \|p-q\|$ for all $p,q \in U_E$. For if $E = B^*$ then B is also a separable space. Thus there exists a countable dense subset $\{x_n\}_{n\geq 1}$ of the unit cell U_B with respect to the norm topology relativised to U_B. Let us define for $p,q \in U_E$, $d(p,q) = \sum_{n\geq 1} \frac{1}{2^n} |p(x_n)-q(x_n)|$. Then d has the required properties. See in this connection Theorem 1, p. 426 Dunford and Schwartz [4].

(b) Concerning set valued mappings we recall a definition and a useful theorem. Let X,Y be two topological spaces and 2^Y be the set of all closed subsets of Y. A function $F: X \to 2^Y$ is called upper semi continuous (u.s.c.) if the set $\{x \mid F(x) \subset G\}$ is open in X for all open sets G in Y. If X,Y are two topological spaces a function $f: X \to Y$ is said to be Borel measurable if $f^{-1}(G) \in \mathcal{F}$ for all open sets G in Y where $\mathcal{F}$ is the σ-ring generated by open sets in X. We state a theorem due to Engelking [12] in a form suitable for our purpose here.

Theorem [Engelking] Let X be a paracompact perfectly normal topological space and (Y,d) be a separable metric space which is d-complete. If $F: X \to 2^Y$ is a u.s.c. map then there exists a Borel measurable function $f: X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$.

Theorem 2. Let E be a separable conjugate Banach space and $1 < p < \infty$. If $f \in L^p_E$, $\|f\| = 1$ then $f \in \text{Ext}\, U_{L^p_E}$ if and only if $\frac{f(x)}{\|f(x)\|} \in \text{Ext}\, U_E$ for x a.e. in S_f. If $p = \infty$ and $f \in L^\infty_E$ with $\|f\| = 1$ then $f \in \text{Ext}\, U_{L^\infty_E}$ if and only if $f(x) \in \text{Ext}\, U_E$ for x a.e. in S_f.

Proof. Let $1 < p < \infty$. If $f \in L^p_E$, $\|f\| = 1$ and $\frac{f(x)}{\|f(x)\|} \in \text{Ext}\, U_E$ for x a.e. in S_f then it is verified that if M is a Borel set, $0 < \mu(M) < \infty$ and $M \subset S_f$ then $[\mu(M)]^{-1}(\chi_M f) \in \text{Ext}\, U_{L^p_E}$. This completes the proof of the "if" part.

To complete the proof of the "only if" part we start noting that since U_E is a compact convex metrizable subset in w*-topology the set $\text{Ext}\, U_E$ is a G_δ subset of U_E. See proposition 1.3, Phelps [13]. Hence Ext U_E is a Borel set in the norm topology of E. Now let $f \in$ Ext U^p_E and if possible let $\frac{f(x)}{\|f(x)\|} \notin$ Ext U_E for x a.e. in S_f. Since Ext U_E is a Borel set in U_E it follows that there exists a measurable set $M \subset S_f$ of positive measure such that $\frac{f(x)}{\|f(x)\|} \notin$ Ext U_E. Since g is a measurable function and μ is a regular measure there exists a compact set $C \subset M$, $0 < \mu(C) < \infty$ such that $g|C$ is continuous. Hence $g(C)$ is a compact set in S_E. We note that for $x \in C$ there exist $p_x, q_x \in S_E$, $p_x \neq q_x$ such that $g(x) = \frac{p_x + q_x}{2}$. For positive δ let C_δ be the subset of C of points x such that there exist $p_x, q_x \in S_E$ with $g(x) = \frac{p_x + q_x}{2}$ and $d(p_x, q_x) \geq \delta$ where d is the metric defined in (a) preceding the statement of the Theorem. We verify that C_δ is a Borel set in X, in fact a closed subset of the compact set C. For let $\{t_n | n \in D\}$ be a net in C_δ such that $t_n \to t$ for some $t \in C$. Let $g(t_n) = \frac{p_n + q_n}{2}$, $d(p_n, q_n) \geq \delta$ for some $p_n, q_n \in S_E$. Since (U_E, d) is a compact metric space there exist convergent sequences $\{p_{n_i}\}$, $\{q_{n_i}\}$ in $\{p_n\}$ and $\{q_n\}$ respectively. From the continuity of g it follows that $g(t) = \frac{p+q}{2}$ if $p_{n_i} \to p$ and $q_{n_i} \to q$. Further $d(p_{n_i}, q_{n_i}) \geq \delta$ implies $d(p,q) \geq \delta$. Thus $t \in C_\delta$ and C_δ is a closed subset of C. By considering the sequence of disjoint Borel sets $C_{\frac{1}{m}} \sim C_{\frac{1}{m-1}}$ for integers $m \geq 1$ and noting that $0 < \mu(C) < \infty$ it follows that there exists a positive number k, a compact set $C_0 = C_{2k} \subset C$ and two functions $g_i: C_0 \to S_E$ such that for all $x \in C_0$, $g(x) = \frac{g_1(x)+g_2(x)}{2}$ and $\|g_1(x)-g_2(x)\| \geq d(g_1(x), g_2(x)) \geq 2k$. Thus there exists a function f: g(Co) -• 2^{U_E} with the w*-topology on E relativised to XJE such that for all £ e 9 ($^c{}_o$^ ^{F}^ is the nonempty w*-closed set of points a in S_E satisfying the condition for some p e S_E, £ = 3$\frac{}{2}$e. and ||a-p|| 2 d(a,p)' 2 $^{2;k*}$ With the norm topology on E relativised to g(C_o) we proceed to verify that F is a u.s.c. map. Let G be an open set in (U_E,d) and let G^ = {£|F(£)cG}. Suppose that t e g(C_o) and that there is no neighbourhood N of t such that for all rj e N, F(T)) C G. Thus there exists a sequence {t_n'} in g(C_Q), t_R - t, P(t_n) £ g(C_Q) for all n ^ 1 which in turn implies the existence of a sequence {t^1_n} in S_{ht} with t^1_n e F(t_ny ~ G. Considering a sequence [t^2_n} in L_{hi} such that t_n = $\frac{t^1+t^2_n}{2}$—^^(t_n^x,t_n^2) ^ 2k by standard compactness arguments it follows that there exists a subsequence {$t^1_{n_1}$} in {t^1_n} converging to some point t^1 in the space (U_E,d) such that for some point t^2 € u_E, t = $\frac{t^1 \quad t^2}{2}$ and d(t^1,t^2) ^ 2k. Thus t^1 e F(t) c G. Since G is a neighbourhood of t^1 there exist t_{n_i} e G leading to a contradiction. Thus F is a u.s.c. map. It is verified that g(C_O) and (U_E,d) satisfy the conditions on X and Y respectively in Engelking's theorem stated earlier in the paper. Hence there exists a Borel measurable function h: g(C_O) -> ($U_£$,d) such that h(£) e F(£) for all £ e g(C_O). Let g^1: C_O -» U_E be the function defined by g_1 = hog. It is verified that g_1 is a w*-measurable function. Since E is a separable Banach space by the Theorem 3.5.5(2) on p. 74 in [7] it follows that g is a measurable function. Further from the definition of F and the choice of h it is inferred that there exist $g^2(x) \in S_E$ such that $g(x) = \frac{g^1(x)+g^2(x)}{2}$ and $\|g^1(x)-g^2(x)\| \geq d(g^1(x), g^2(x))^2 \geq 2k$. Since g, g^1 are measurable the function g^2 is also measurable and $g|C_o = \frac{g^1+g^2}{2}$. Now defining the functions $f^i: X \to U_E$, $i = 1,2$ by the equations $f^i(x) = g^i(x)$ if $x \in C_o$ and $f^i(x) = g(x)$ if $x \in X \sim C_o$ it is verified that $f^i \in U_E^p$, $g = \frac{f^1+f^2}{2}$ and $f^1 \neq f^2$. Thus $g \notin \text{Ext } U_E^p$ which in turn by Theorem 1 implies that $f \notin \text{Ext } U_E^p$ contradicting the choice of f. This completes the proof of Theorem 2 if $1 < p < \infty$.

The case $p = \infty$ is similarly dealt after noting the remark following Theorem 1 and the details are not supplied.

Corollary. If E is a separable reflexive Banach space and $f \in U_E^p$, $1 < p < \infty$ and $\|f\| = 1$ then $f \in \text{Ext } U_E^p$ if and only if for x a.e. in S_f, $\frac{f(x)}{\|f(x)\|} \in \text{Ext } U_E$.

The corollary follows from the preceding Theorem since a separable reflexive Banach space satisfies the condition on E in the Theorem.

We proceed to the case when E is a reflexive Banach space not necessarily separable. All subspaces in the rest of the paper are closed subspaces. If f is a measurable function then the range f is said to be essentially in the set M if $f(x) \in M$ a.e..

Theorem 3. If E is a reflexive Banach space and $1 < p \leq \infty$ then a function $f \in L_E^p$ with $\|f\| = 1$ is an extreme point of U_E^p if and only if $\frac{f(x)}{\|f(x)\|} \in \text{Ext } U_M$ a.e. in S_f for every separable subspace M essentially containing the range f.

Proof. Suppose $f \in \text{Ext } U_E^p$ and M is a separable subspace essentially containing the range f. If necessary redefining f on a set of measure zero it could be assumed that $f \in U_M^p$. Since $U_M \subset U_E$ clearly $f \in \text{Ext } U_M^p$. From the corollary to the Theorem 2 it follows that $\frac{f(x)}{\|f(x)\|} \in \text{Ext } U_M$. Conversely let there be a separable subspace M of E essentially containing the range f such that $\frac{f(x)}{\|f(x)\|} \in \text{Ext } U_M$ a.e. in S_f is false. Since it could be assumed that $f \in L_M^p$ and since $\|f\| = 1$ it follows from the Corollary to Theorem 2 that $f \notin \text{Ext } U_M^p$. Since $U_M^p \subset U_E^p$, $f \notin \text{Ext } U_E^p$. This completes the proof of the Theorem.

We next turn to some special Banach spaces E and study the extremal structure of U_E^p. We adopt the following notation. If Y is a compact Hausdorff space then $C(Y)$ is the Banach space of continuous real valued functions on Y with the supremum norm. If Γ is a discrete space $c_o(\Gamma)$ denotes the Banach space of real valued functions vanishing at ∞ with the supremum norm. In what follows p is either a real number $1 < p < \infty$ or $p = \infty$.

Theorem 4. If $E = C(Y)$ or $c_o(\Gamma)$ and $f \in L_E^p$, $\|f\| = 1$ then $f \in \text{Ext } U_E^p$ if and only if $\frac{f(x)}{\|f(x)\|} \in \text{Ext } U_E$ for x a.e. in S_f.

Proof. Let $E = C(Y)$. Let $f \in \text{Ext } U_E^p$. Let $g = P(f)$. Since $e \in \text{Ext } U_E$ implies $|e(x)|^2 = 1$ for all $x \in X$ it follows that $\text{Ext } U_E$ is a closed subset of U_E in the norm topology on E. Thus if $\frac{f(x)}{\|f(x)\|} \in \text{Ext } U_E$ for x a.e. in S_f is false then there exists a Borel set $M \subset S_f$, $\mu(M) > 0$ such that for all $x \in M$, $g(x) \notin \text{Ext } U_E$.

Hence by the regularity of ji there exists a compact set C c *M,* 0 < ;x(C) such that for all x € C, g(x) / Ext U_E and g|c is continuous. Let K = g(C). Thus K is a compact subset of S_w. Let F e K. Since F / Ext U_ there exists a t e Y and a positive number e_p such that -1 + 2e £ F(fc p) ^ 1 - 2€F#

If $F_\pm$ € K such that ||F-F_1|| <: €F then -1 + e_p £ ^ (t ^ £ 1 - e_p.

Since K is a compact set there exists a finite set [F } 1≤i≤n c K such that if B_p is the subset of K defined by {F| ||F-F_i||^e_p }
i i

then {B_F } 1 j£ i j£ n is a covering of K. Since g is a continuous function the family (g (B_pi) } l < l i ^ n is a finite family of compact subsets of C covering C. Hence one of the sets l g~ (B^) has a positive measure. For definiteness let u(g~ (B_))> 0. With 6 = €„ it is verified that *G e B^* implies
r1o r .

-1 + 6 < G(t_F) < 1 - 6. By Ascoli-Arzela's theorem on compact sets in C(Y) it follows that there exists an open neighbourhood U of t_f such that for all £ *e* U and for all G e B_p ,

-1 + *2"<G(^) < 1-6/2. Since Y is a compact Hausdorff space there exists a continuous function F_o: Y -+ [0,6/2] such that F_Q vanishes on Y ~ U and F_o^{-1}(6/2) is a nonempty subset of U. Let g_i, i = 1,2 be the functions on X to C(Y) defined by

9_X = X_X ^T g + X_T(g+F_Q) where T = g""1($B_{F>}$) and g_2 is the
1 1o

same as g_1 except that the constant function F is replaced by ~F_0* It: is verified that g_1 e U^ and g = —=g_2/2— and g_1 / g_2.

Thus g / Ext U^. Hence it follows by Theorem 1 that f / Ext U^ thus completing the proof of the "only if" part.

The proof of the "if" part is the same as the proof of the corresponding assertion in the Theorem 2. This completes the proof of the Theorem when $E = C(Y)$. The case when $E = C_0(T)$ is similarly dealt.

Remark 2. It is easily verified that the set $\operatorname{Ext} U_E = 0$ if $E = C_0(T)$. Thus it follows from the Proposition 1 and the preceding Theorem that $\operatorname{Ext} U = 0$ if $1 \le p \le \infty$ and $E = C_0(T)$.

We proceed to indicate an application of some of the results in the paper. More specifically we apply the corollary following the Theorem 2 to determine the extreme operators in the unit cell of $B(E_1, E_2)$, the Banach space of operators on E_1 to E_2 with the usual supremum norm when E_1 is the function space $L_1(X, \Sigma, \mu)$ and E_2 is a separable reflexive space. The measure space (X, Σ, μ) here is the same as the one considered in the introduction in addition to being σ-finite. We recall the following representation Theorem for such operators which is an easy corollary of the Theorem 10, p. 507 [4] after noting that every operator on a Banach space to a reflexive Banach space is weakly compact.

Theorem 5. If $B(L_1, E)$ is the Banach space of operators on the function space $L_1(X, \Sigma, \mu)$ into a separable reflexive Banach space E then the following map $\pi: B(L_1, E) \to L^{*}_{E}$ is an isometric isomorphism onto L^{*}_{E}. If $T \in B(L_1, E)$ then $\pi(T)$ is the function in L^{*}_{E} such that

$$Tf = \int_X \pi(T)(x) f(x)\, d\mu(x).$$

From the corollary to the Theorem 2 and the preceding theorem we obtain the following characterization.

Theorem 6. An operator $T \in B(L_1,E)$ with $\|T\| = 1$ is an extreme point of the unit cell of $B(L_1,E)$ if and only if $\frac{\pi(T)(x)}{\|\pi(T)(x)\|} \in \text{Ext } U_E$ a.e. in S_f.

Before proceeding to the conclusion we wish to make a remark concerning the hypothesis on the measure space (X,Σ,μ).

Remark 3. As already mentioned in the introduction (X,Σ,μ) is a topological measure space. More specifically X is a locally compact Hausdorff space. However this hypothesis is not too restrictive for by a well-known Theorem of Kakutani, see for example Theorem 2, p. 372 [9], if $1 < p < \infty$ and if (Y,Σ_1,μ_1) is a finite measure space then there exists a topological measure space (X,Σ,μ) of the type considered here such that $L_E^p(Y,\Sigma_1,\mu_1)$ is isometrically isomorphic with L_E^p.

In conclusion it might be mentioned that the characterization of $\text{Ext } U_E^p$ when E is an arbitrary (even separable) Banach space has not been dealt here. This case and generalisations of the results presented here to the class of Orlicz-Bochner Function Spaces with applications to the theory of operators and approximation will be dealt elsewhere.

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