AN EVALUATION OF THE O>-COMPLEXITY OF FIRST ORDER ARITHMETIC WITH THE CONSTRUCTIVE CO-RULE

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AN EVALUATION OF THE ω-COMPLEXITY OF FIRST ORDER ARITHMETIC WITH THE CONSTRUCTIVE ω-RULE[1]

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§0. Introduction. Concerning first order arithmetic with the restricted (constructive) ω-rule, Shoenfield showed the following in [5]. First we quote his definition.

For each ordinal σ, define a class S_σ of sentences (of arithmetic) as follows. S_0 is the class of provable sentences of Z_μ. $S_{\sigma+1}$ is the class of sentences which are provable from sentences of S_σ by the ω-rule, together with their logical consequences. If σ is a limit number,

$$S_\sigma = \bigcup_{\tau<\sigma} S_\tau .$$

He claims:

If we replace the ω-rule by the restricted ω-rule (in the above definition), then $S_{\omega}^{\wedge}$ is the class of true sentences of Z_μ.

He attained this result by analyzing his proof of the completeness of the restricted ω-rule and considerations of [3].

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[1] Part of this work was done while the author was at the University of Bristol.

(See also [1].) Here, we shall show that a subset of S_{ω^2} will do for all the true sentences of Z_μ. The argument is an application of Shoenfield's main result (the completeness of the restricted ω-rule) and the cut elimination theorem for the first order arithmetic with the constructive ω-rule (cf. [4]).

§1. The system and the ω-complexity. The first order arithmetic with the constructive ω-rule was formulated, for example, in [5]. Here, however, we adopt a Gentzen type formulation of arithmetic.

Definition 1. A formulation of the system Z. The formulas and the sequents are defined like in [2] except that we now permit only closed formulas (sentences) in the sequents. The rules of inference in [2] except '$\forall$ in the succedent', '$\exists$ in the antecedent' and 'VJ' are adopted. Instead of those three rules, we introduce the 'constructive ω-rule' into our system. Like in [5], we assume that Gödel numbers have been assigned to the formulas and the sequents, and to the partial recursive functions. We write $\ulcorner A \urcorner$ for the Gödel number of a formula A and $\ulcorner S \urcorner$ for the Gödel number of a sequent S. The notion of a number of a proof-figure in Z is defined naturally in terms of Godel numbering of the rules of inference in [2] (except '$\forall$ in the succedent', '$\exists$ in the antecedent' and 'VJ'). The ω-rule is formulated as follows.

Let $^{r}P.^{"1}_{1}$ be a number of a proof-figure in Z of a sequent F -• 0, F(i) for every natural number i, where i^ is the numeral which denotes i and *T* and 0 stand for finite sequences of formulas. If e is a number such that (e}(i) = \n for all i, then $3.5^{e}.7^{F \sim * 9> VxF(x)}$ is a number of a proof-figure in Z (of the sequent *T* -• 6, VxF(x)).

We say a sequent S is provable in Z if there is a number of a proof-figure in Z of S. A formula A is said to be provable in Z if -» A is provable in Z.

In order to simplify the presentation, we shall often say a !formula1, a fproof-figure!, etc., instead of !a number of1 a formula , a proof-figure , etc. Thus, we may simply say fP is a proof-figure of a sequent S in Zf; we may even omit 'in Z! . The co-rule shall then be expressed as follows.

$$\frac{P_{\pm} \quad i < \infty}{r - 0,\ VxP(x)}$$

where P. is a proof-figure of *T* - 0, F(i) for every natural number i, and there is a recursive function f such that f(i) produces P_i (or, f(i) = $^{fp}i^{"1}$) •

As in [5], we assume that definitions of all primitive recursive functions have been introduced in our formal system.

Definition 2. The ω-complexity of a proof-figure P, denoted by $\omega(P)$, which is a countable ordinal (cf. 1.3 of [6]) is defined as follows.

1) If P consists of a beginning sequent only, then $\omega(P) = 0$.

2) If P is of the form $\frac{P_1}{S}$ or $\frac{P_1 P_2}{S}$, then $\omega(P) = \omega(P_1)$ or $\omega(P) = \max(\omega(P_1), \omega(P_2))$ respectively.

3) If P is of the form $\frac{P_i \quad i < \omega}{S}$, then $\omega(P) = \sup_{i<\omega} \omega(P_i)$.

Definition 3. Let σ be a non-zero countable ordinal. S'_σ is defined as the set of all the sentences (of Z) which are provable with proof-figures whose ω-complexities are less than σ.

Note. Although there is a slight difference in the definition, our S'_{ω^2} is S_{ω^2} in [5].

§2. The theorem and some known results. Our purpose is to prove the following.

Theorem. S'_{ω^2} is the class of true sentences of arithmetic.

We shall prove this theorem by using the following well-known results. (The proof of the theorem shall be given in §4.)

Theorem 1. (cf. [5].) Any true sentence of arithmetic is provable in Z.

Theorem 2. (cf. [4].) There is a partial recursive function f such that if P is a proof-figure, then f(ʳP^) is defined and is a number of a cut free proof-figure of the end sequent of P.

Proposition. If A is a sentence of arithmetic, then there is a prenex normal form in alternating quantifiers, say B, such that A ^{s} B is provable with a proof-figure whose o>-complexity is finite (i.e. A = B belongs to S'_{ω}).

§3. Some lemmas.

Definition 4. A condition (*) on a sequent *T* -• 8 is the following.

(*) All (sequent) formulas of *T* are quantifier free and every (sequent-) formula of 6 is either quantifier free or in the alternating prenex normal form.

Definition 5. Suppose 8 satisfies the condition on 9 in (*) and there are k (sequent-) formulas in 6 which start with the universal quantifier. Then $8[n_1, \ldots, n_k]$ denotes a sequence of formulas which satisfies the following.

(1) If the j^{th} formula of 8 (from the left) is of the form VxA(x) and it is the i^{th} formula which starts with the universal quantifier, then $A(\underline{n_i})$ is the j^{th} formula of $8[n_1, \ldots, n_k]$.

(2) If the j^{th} formula of 8 does not start with the

universal quantifier, then it is the j^{th} formula of $\theta[n_1,\ldots,n_k]$.

(3) Every formula of $\theta[n_1,\ldots,n_k]$ is one of the formulas described in (1) and (2) above.

The number k as above shall be denoted by $k(\ulcorner\theta\urcorner)$, or $k(\ulcorner P\urcorner)$ if the $\Gamma \to \theta$ above is the end sequent of P.

Lemma 1. There is a recursive function h of two arguments which has the following property.

$$h(n, \ulcorner P\urcorner) = \begin{cases} \ulcorner P[n_1,\ldots,n_k]\urcorner & \text{if } P \text{ is a proof-figure whose end sequent, say } \Gamma \to \theta \text{, satisfies (*), } n = 2^{n_1+1}\cdot 3^{n_2+1}\cdots p_k^{n_k+1}\cdot \ell \text{, where } \ell \text{ has none of the factors } 2, 3, \ldots, p_k \text{ } (p_k \text{ is the k-th prime number}), \text{ and } k \geq k(\ulcorner\theta\urcorner) \text{, where } P[n_1,\ldots,n_k] \text{ is a proof-figure of } \Gamma \to \theta[n_1,\ldots,n_k]; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is obvious, since $\forall x F(x) \to F(\underline{i})$ is provable in Z for an arbitrary natural number i.

Lemma 2. There is a partial recursive function g such that $g(\ulcorner P\urcorner)$ $(=\ulcorner\tilde{P}\urcorner)$ is defined whenever P is a cut free proof-figure in Z whose end sequent, say $\Gamma \to \theta$, satisfies (*) and, in such a case, $\tilde{P}$ is a proof-figure of a sequent $\Gamma \to \tilde{\theta}$ which satisfies the following condition ($\sim$).

(~) (1) If a formula in 8 is of the form 3yA(y), then there are a finite number of terms s, ...,t such that A(s),...,A(t) are in #.

(2) If a formula in 9 does not start with the existential quantifier, then it is in *$.

(3) Only the formulas described in (1) and (2) above are in ^.

r - # is said to satisfy (~) for I' -> 6. We can actually specify the order of the formulas in *Tf* effectively, though we omit such details throughout. Notice also that *T* -* # again satisfies (*), and that g and P determine the terms s, ...,t (in the condition (~)).

Proof. First consider the following transformations (of P into $\tilde{P}$), according to the last inference in P, say I. It should be noted that, as P is cut free, every sequent in P satisfies the condition (*), and hence every subproof of P possesses the same property as P.

0) P consists of a beginning sequent only. Then take P itself as P, since P has no quantifier in this case*

1) I is an a>-rule. Let P be of the form

$$P_{\pm} \quad \left\{ \frac{\begin{array}{c} \vdots \\ \Gamma \rightarrow \Delta, F(i) \quad i < \omega \end{array}}{\Gamma \rightarrow \Delta, \forall x F(x)} \right.$$

Suppose $\tilde{P}_i$ is already defined for every i.

1.1) F(i) has no quantifiers. Then the end sequent of $\tilde{P}_i$ is of the form *T - *K,* F(i). Define $\tilde{P}$ as

$$\tilde{P}_i \left\{ \frac{\Gamma \rightarrow \tilde{\Delta},\ F(i) \qquad i < \omega}{F - \tilde{A},\ VxF(x)} \right.$$

1.2) F(i) is of the form 3yA(i,y). Then, the end sequent of $\tilde{P}_i$ is of the form *T* -• $\tilde{A}$, A(i,s),...,A(i,t), where s, ...,t depend on i. Define P as

$$\frac{\dfrac{\tilde{P}_{\pm} \quad [T - A,\ A(i,s),...,A(i,t) \qquad i < CO}{T - ff,\ 3yA(i,y) \qquad i < co}}{T - \tilde{A},\ Vx3yA(x,y)}$$

where ^ means that there are ' 3's in the succedent' applied to A(i,s),...,A(i,t), as well as some interchanges and contractions

2) I is a 3 in the succedent. Let P be of the form

$$Q \left\{ \frac{T - A,\ F(s)}{T - A,\ 3yF(y)} \right.$$

Suppose $\tilde{Q}$ is defined. Notice that F(s) does not start with ∃ and hence the end sequent of Q is *T* -+ A, F(s). Take $\tilde{Q}$ as $\tilde{P}$.

3) I is one of the inferences which introduce propositional connectives. We shall present only one such example -- I is a '$\wedge$ in the succedent'. Let P be of the form

$$\frac{P_1 \left\{ \begin{array}{c} \vdots \\ \Gamma \to \Delta, A \end{array} \right. \qquad P_2 \left\{ \begin{array}{c} \vdots \\ \Gamma \to \Delta, B \end{array} \right.}{\Gamma \to \Delta, A \wedge B} .$$

Suppose $\tilde{P}_1$ and $\tilde{P}_2$ are defined. Since $A \wedge B$ has no quantifier, $\tilde{P}$ may be defined as

$$\frac{\tilde{P}_1 \left\{ \begin{array}{c} \vdots \\ \Gamma \to \tilde{\Delta}, A \end{array} \right. \qquad \tilde{P}_2 \left\{ \begin{array}{c} \vdots \\ \Gamma \to \tilde{\Delta}, B \end{array} \right.}{\Gamma \to \tilde{\Delta}, A \wedge B} .$$

4) I is a contraction in the succedent. Let P be of the form

$$\frac{Q \left\{ \begin{array}{c} \vdots \\ \Gamma \to \Delta, D, D \end{array} \right.}{\Gamma \to \Delta, D} .$$

Suppose $\tilde{Q}$ is defined.

4.1) D does not start with the existential quantifier. Then the end sequent of $\tilde{Q}$ is of the form $\Gamma \to \tilde{\Delta}, D, D$. Define $\tilde{P}$ as

$$\frac{\tilde{Q} \left\{ \begin{array}{c} \vdots \\ \Gamma \to \tilde{\Delta}, D, D \end{array} \right.}{\Gamma \to \tilde{\Delta}, D} .$$

4.2) D is of the form 3yD(y). Then the end sequent of Q̃ is of the form *T* -* A, D(s..),...,D(s.), D(t;),«••,D(t) for
x ~*> ,x m
some &ı,•.•,&*, tı-,...,tm . Take Q and P.

5) I is a contraction in the antecedent. For this case an argument similar to 4.1) goes through.

6) I is a weakening in the succedent. Let P be of the form

$$\frac{Q\left\{\begin{array}{c}\vdots\\ r - A\end{array}\right.}{\Gamma \to \Delta,\ D}\ .$$

Suppose Q is defined.

6.1) D does not start with the existential quantifier. Define P̃ as

$$\frac{Q\ \mathrm{I}\ r - A}{r - A,\ D}\ .$$

6.2) D is of the form 3yD(y). Define P as

$$\frac{\begin{array}{c} f\ V \\ Q\ (\ r - A\end{array}}{r - \tilde{A},\ D(o)}\ .$$

7) I is a weakening in the antecedent. This case is treated similarly to 6.1).

Now define a partial recursive function q(r,⌜P⌝) according to the above transformation. We shall quote the case numbers j) in the above transformation.

q(r,V) s ⌜p⌝ if 0);

≡ $3.5^{e_1}.7^{\ulcorner \Gamma \to \tilde{\Delta}, \forall x F(x) \urcorner}$ if 1.1), where

e_1 = Ai({r}({e}(i))) and e is a number determined by ⌜P^) such that (e)(i) = ⌜P^ ;

^ . S * 2 / 1 ^ * ' VxF(x)1 if 1.2), where

e_2 - Ai(E({r)){e)(i)),{e}(i))), e is as above, and E(⌜R'^^R¹) is a recursive function which produces a proof-figure of *ir — £,* 3y^B(y) if the end sequent of R¹ is of the form *ir -> ci₉* B(s)^..^B(t) and gyB(y) is the last formula in the succedent of the end sequent of R;

≡ {r}(V) if 2);

≡ I({r}(⌜P₁^),{r}(⌜P₂"¹)) if 3), where iCl^"¹ / R ^) is a recursive function which produces a proof-figure of T - A, A A B if Rj and R_g are the proof-figures of *T* -• A, A and *T* -• A, B respectively;

≡ Like Case 3) for other propositional connectives;

≡ C({r}("Qⁿ)) if 4.1), where C(⌜R"¹) is a recursive function which produces a proof-figure from R by a contraction in the succedent:

$\equiv \{r\}(\ulcorner Q \urcorner)$ if 4.2);

$\equiv C'(\{r\}(\ulcorner Q \urcorner))$ if 5) for an appropriate $\ulcorner Q \urcorner$ and a recursive C':

$\equiv W(\{r\}(\ulcorner P \urcorner), \ulcorner P \urcorner)$ if 6.1), where $W(\ulcorner R \urcorner, \ulcorner P \urcorner)$ is a recursive function which produces a proof-figure of $\pi \rightarrow \Lambda, D$ from R by adding D as a weakening formula provided that $\pi \rightarrow \Lambda$ is the end sequent of R and D is the last formula in the end sequent of P;

$\equiv W_o(\{r\}(\ulcorner P \urcorner), \ulcorner P \urcorner)$ if 6.2), where $W_o(\ulcorner R \urcorner, \ulcorner P \urcorner)$ is a recursive function which produces a proof-figure of $\pi \rightarrow \Lambda, D(0)$ by a weakening of $D(0)$ provided that the end sequent of R is $\pi \rightarrow \Lambda$ and the last formula in the end sequent of P is $\exists y D(y)$.

$\equiv$ Similarly to 6) if 7).

By recursion theorem, there is a number r_o such that

$$\{r_o\}(\ulcorner P \urcorner) \simeq q(r_o, \ulcorner P \urcorner).$$

Let us call the partial recursive function which is represented by r_o g. Then

$$g(\ulcorner P \urcorner) \simeq q(r_o, \ulcorner P \urcorner).$$

It is easily seen that $g(\ulcorner P \urcorner) = \ulcorner \tilde{P} \urcorner$ under appropriate circumstances. Hence we can see that g is defined if P is a cut free proof-figure whose end sequent satisfies (*) and $g(\ulcorner P \urcorner)$ (or $\tilde{P}$) is a proof-figure of a sequent whose end sequent satisfies (~). The precise proof is carried out by transfinite induction on the length of $\ulcorner P \urcorner$ (which is less than ω_1 (cf. §3 of [6])). Notice that, if P is cut free and its end sequent satisfies (*), then all subproofs of P have the same property. Thus, if a $\{r\}(\ulcorner Q \urcorner)$ occurs in the definition of q, then the induction hypothesis applies since it can be easily proved that Q is a subproof of P and hence the length of Q is less than the length of P. It should be also noted that the cases 0) ~ 7) exhaust all the possibilities of the last inference of P. In cases 1.1) and 1.2), e_1 and e_2 respectively represent the constructive ω-rule, since $\Lambda i[\{r_o\}(\{e\}(i))]$ and $\Lambda i[E(\{r_o\}(\{e\}(i)),\{e\}(i))]$ represent partial recursive functions of i, and, if P is a proof-figure in Z, then they are defined for all i (by the definition of e and induction hypothesis).

Lemma 3. There is a partial recursive function of two arguments, say ν, such that $\nu(n, \ulcorner P \urcorner)$ $(= \ulcorner P[n_1,\ldots,n_k]^{\dagger} \urcorner)$ is defined if P is a proof-figure whose end sequent, say $\Gamma \rightarrow \theta$, satisfies (*), $k = k(\ulcorner \theta \urcorner)$ $(= k(\ulcorner P \urcorner))$, and $n = 2^{n_1+1} \cdot 3^{n_2+1} \cdots p_k^{n_k+1}$, where $P[n_1,\ldots,n_k]^{\dagger}$ is a proof-figure of a sequent which satisfies (~) for $\Gamma \rightarrow \theta[n_1,\ldots,n_k]$.

Proof. Let f be a partial recursive function which gives the transformation in Theorem 2 in §2. Thus, if P is a proof-figure, then $f(^{r}P^{n})$ is a cut free proof-figure of the same end sequent. Define

$$I/Cn/P^{1}) = g\text{-}f\text{-}h(n, {}^{r}P^{1}),$$

where h and g are the functions in Lemma 1 and Lemma 2 respectively. Then it is obvious that *v* is a partial re-cursive function which is defined if P is a proof-figure whose end sequent satisfies (*) and $n = 2^{n_1+1} * 3^{n_2+1} ** * p_k^{n_k+1} \#1$ for some k, where 1 has none of the factors $2, 3, \ldots, p_k$, and $k \geq k(^{r}p^{n})$. In particular

$$1/(2^{n_x+1} \cdot 3^{n_z+1} \cdots p_k^{n_k+1} \ulcorner, P \urcorner) * \stackrel{r}{=} \wedge[\wedge \cdots s \wedge] \dagger^{\wedge}$$

is well-defined if P is as above and $k = k(^{r}8^{1})$. The end sequent of $P[n_1, \ldots, n_k]'$ then satisfies (~) for F- $8[n_1, \ldots, n_k]$ by the definition of g.

Lemma 4. There is a partial recursive function *Æ* such that $M \wedge P^{1}) \; (= {}^{r}P^{\circ}"^{1})$ is defined if P is a proof-figure whose end sequent, say *T -* Q,* satisfies (*) and, in such a case, P° is a proof-figure of *T* -• 8 and co(p°) < o:*m, where m is the maximum among the numbers of quantifiers in the formulas of 8 (hence m may be denoted by m(P)).

Proof. The proof is carried out by mathematical induction on m.

We first give an intuitive idea of the construction of P°. Let $k = k(\Gamma \to \Theta)$ $(= k(P))$. Then, by Lemma 3, $P[n_1, \ldots, n_k]^\dagger$ is a proof-figure of a sequent, say $\Gamma \to \Theta[n_1, \ldots, n_k]^\dagger$, which satisfies (*) for $\Gamma \to \Theta[n_1, \ldots, n_k]$ for every k-tuple $(n_1, \ldots, n_k)$. It is easily seen that $m(P[n_1, \ldots, n_k]^\dagger) = m - 1 < m$. Furthermore, $\Gamma \to \Theta[n_1, \ldots, n_k]^\dagger$ also satisfies (*). Hence by induction hypothesis $(P[n_1, \ldots, n_k]^\dagger)^\circ$ is defined and its end sequent is $\Gamma \to \Theta[n_1, \ldots, n_k]^\dagger$.

Let Θ consist of $\forall x_1 A_1(x_1), \ldots, \forall x_p A_p(x_p)$, $\exists y_1 B_1(y_1), \ldots, \exists y_q B_q(y_q)$, $\forall z_1 \exists u_1 C_1(z_1, u_1), \ldots, \forall z_r \exists u_r C_r(z_r, u_r)$, Θ', where $A_1(x_1), \ldots, A_p(x_p)$ are quantifier free and Θ' consists of quantifier free formulas. Then $\Theta[n_1, \ldots, n_k]$ consists of $A_1(n_{j_1}), \ldots, A_p(n_{j_p})$, $B_1(s^1_1), \ldots, B_1(s^1_{g_1}), \ldots, B_q(s^q_1), \ldots, B_q(s^q_{g_q})$, $C_1(n_{l_1}, t^1_1), \ldots, C_1(n_{l_1}, t^1_k), \ldots, C_r(n_{l_r}, t^r_1), \ldots, C_r(n_{l_r}, t^r_k)$, Θ' correspondingly, where $n_{j_1}, \ldots, n_{j_p}$, $s^1_1, \ldots, s^1_{g_1}, \ldots, s^q_1, \ldots, s^q_{g_q}$, $n_{l_1}, \ldots, n_{l_r}$, $t^1_1, \ldots, t^1_k, \ldots, t^r_1, \ldots, t^r_k$ are determined by P and $(n_1, \ldots, n_k)$. P° is defined in terms of the following $Q(n_1, \ldots, n_k)$. First $Q(n_1, \ldots, n_k)$ is defined as follows:

$$(P[n_1, \ldots, n_k]^\dagger)^\circ$$

$$\vdots$$

$$\Gamma' \to \Theta[n_1, \ldots, n_k]^\dagger$$

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$\exists$'s in the succedent applied to t's in the C's, $\exists$'s in the succedent applied to s^*_j in the B's, 'interchanges' and 'contractions'

$$\Gamma \to \Theta', \exists y_1 B_1(y_1), \ldots, \exists y_q B_q(y_q), \exists u_1 C_1(n_{l_1}, u_1), \ldots, \exists u_r C_r(n_{l_r}, u_r)$$

for appropriate θ''. Note that

$$\omega(Q(n_1,\ldots,n_k)) = \omega((P[n_1,\ldots,n_k]^{\dagger})^{o}) < \omega\cdot(m-1)$$

Let $\forall x_1 D_1(x_1),\ldots,\forall x_k D_k(x_k)$ be all the formulas of θ which start with $\forall$ and suppose $\forall x_i D_i(x_i)$ corresponds to n_i, (Those are among $\forall x A(x)$'s and $\forall z \exists u C(z,u)$'s. Exactly one such formula corresponds to one n_i), and let θ^* be θ', $\exists y_1 B_1(y_1),\ldots, \exists y_q B_q(y_q)$. P^o is defined as the following Q_k.

$$
Q_k \left\{
\begin{array}{lll}
 & & Q(n_1,\ldots,n_k) \\
\hline\hline
Q_1(n_2,\ldots,n_k) & I_1 & \Gamma \to \theta^*,\ D_k(n_k),\ldots,D_1(n_1) \qquad n_1 < \omega \\
\hline
 & & \Gamma \to \theta^*,\ D_k(n_k),\ldots,\forall x_1 D_1(x_1) \\
\hline\hline
Q_2(n_3,\ldots,n_k) & I_2 & \Gamma \to \theta^*,\ \forall x_1 D_1(x_1),\ D_k(n_k),\ldots,D_2(n_2) \qquad n_2<\omega \\
\hline
 & & \Gamma \to \theta^*,\ \forall x_1 D_1(x_1),\ D_k(n_k),\ldots,\forall x_2 D_2(x_2) \\
 & & \vdots \\
 & I_k & \Gamma \to \theta^*,\ \forall x_1 D_1(x_1),\ldots,D_k(n_k) \qquad n_k < \omega \\
\hline
 & & \Gamma \to \theta^*,\ \forall x_1 D_1(x_1),\ldots,\forall x_k D_k(x_k) \\
\hline\hline
 & & \Gamma \to \theta
\end{array}
\right.
$$

where $I_1, I_2, \ldots, I_k$ are the only ω-rules under $Q(n_1,\ldots,n_k)$. Since $\omega(Q(n_1,\ldots,n_k)) < \omega\cdot(m-1)$, $\omega(Q_1(n_2,\ldots,n_k)) \leq \omega\cdot(m-1)$, $\omega(Q_2(n_3,\ldots,n_k)) \leq \omega\cdot(m-1)+1,\ldots$; in general $\omega(Q_j(n_{j+1},\ldots,n_k)) \leq \omega\cdot(m-1)+(j-1)$, $1 \leq j \leq k$. Thus $\omega(P^o) \leq \omega\cdot(m-1)+(k-1) < \omega\cdot m$.

The definition of the required function μ goes as follows.

First define recursive functions $\psi_0(i, \ulcorner Q \urcorner, \ulcorner P \urcorner)$, $\psi_1(\ulcorner Q \urcorner, \ulcorner P \urcorner)$, $\psi_2(\ulcorner Q \urcorner, \ulcorner P \urcorner)$, $\psi_3(\ulcorner Q \urcorner, \ulcorner P \urcorner)$, $\phi_4(e, \ulcorner P \urcorner)$, $\phi(e, \ulcorner P \urcorner)$, and $M_0(i, e, \ulcorner P \urcorner)$.

$\psi_0(0, \ulcorner Q \urcorner, \ulcorner P \urcorner) = Q$,

$$\psi_0(1, \ulcorner Q \urcorner, \ulcorner P \urcorner) = \ulcorner \begin{array}{c} Q \\ \hline\hline \Gamma \rightarrow V,\ C(s) \\ \hline \Gamma \rightarrow V,\ \exists y C(y) \end{array} \urcorner$$

if $C(s)$ is the right most formula among those which are in the end sequent of Q and which satisfy that there is a formula of the form $\exists y C(y)$, $C(y)$ being quantifier free, in the end sequent of P, while $C(s)$ is not in the end sequent of P;

$$= \ulcorner \begin{array}{c} Q \\ \hline\hline \Gamma \rightarrow V,\ C(n,s) \\ \hline \Gamma \rightarrow V,\ \exists y C(n,y) \end{array} \urcorner$$

if there is no $C(s)$ as above and $C(n,s)$ is the right most formula among those which are in the end sequent of Q and which satisfy that there is a formula of the form $\forall x \exists y C(x,y)$ in the end sequent of P, where n is a numeral, while $C(n,s)$ is not in the end sequent of P;

$= 0$ otherwise.

$\psi_o(i + 1, \ulcorner Q \urcorner, \ulcorner P \urcorner) = \psi_o(1, \psi(i, \ulcorner Q \urcorner, \ulcorner P \urcorner), \ulcorner P \urcorner)$

$\psi_1(\ulcorner Q \urcorner, \ulcorner P \urcorner)$ = the number of formulas C(s) or C(n,s) which satisfy the conditions in the definition of α_o.

$\psi_2(\ulcorner Q \urcorner, \ulcorner P \urcorner) = \psi_o(\psi_1(\ulcorner Q \urcorner, \ulcorner P \urcorner), \ulcorner Q \urcorner, \ulcorner P \urcorner)$.

$\phi_3(c, k, \ulcorner P \urcorner) = \Lambda n_1 \ldots \Lambda n_k \psi_2(\{c\}(n_1, \ldots, n_k), \ulcorner P \urcorner)$

$\phi_4(c, \ulcorner P \urcorner) = \phi_3(c, k(\ulcorner P \urcorner), \ulcorner P \urcorner)$

$\varphi(e, 0, \ulcorner P \urcorner) = 0$;

$\varphi(e, l, \ulcorner P \urcorner) =$ An₂...An_t(3-5 An_x({e)(n_x, ...,n.)) «. * -7^s)

r V; ⊣

if $l > 0$, {e)(n_r ...,11^) - J| ^' ^i ' F(n l) ,

s = [f - j,, VxF(x) and VxF(x) is in the end sequent of P.

Note. If $l = 1$, then there is no '$\Lambda n_2 \ldots \Lambda n_l$'.

$\mu_o(0, e, \ulcorner P \urcorner) = \varphi(e, k, \ulcorner P \urcorner)$;

$\mu_o(i + 1, e, \ulcorner P \urcorner) = \varphi(\mu_o(i, e, \ulcorner P \urcorner), k \dot{-} (i + 1), \ulcorner P \urcorner)$, where $k = k(\ulcorner P \urcorner)$.

$\tilde{\psi}(b, \ulcorner P \urcorner, k) = \Lambda n_1 \ldots \Lambda n_k(\{b\}(\nu(2^{n_1+1} \ldots p_k^{n_k+1}, r)))$.

$\phi_5(b, \ulcorner P \urcorner) = \tilde{\psi}(b, \ulcorner P \urcorner, k(\ulcorner P \urcorner))$.

$\phi(b, \ulcorner P \urcorner) = \mu_o(k(\ulcorner P \urcorner) \dot{-} 1, \phi_4(\phi_5(b, \ulcorner P \urcorner), \ulcorner P \urcorner), \ulcorner P \urcorner)$.

By recursion theorem, there is a number b_o such that

$$\{b_o\}(\ulcorner P \urcorner) \simeq \prod(b_o, \ulcorner P \urcorner).$$

Call the partial recursive function which is defined by b_o μ. We show by induction on $m(P)$ that μ is defined for all P which satisfy the condition in Lemma 4, $\mu(\ulcorner P \urcorner)$ is a proof-figure of the end sequent of P for such a P, and that $\omega(P) < \omega \cdot m$.

Suppose P satisfies the condition and $k = k(\ulcorner P \urcorner)$. Then $\nu(2^{n_1+1} \ldots p_k^{n_k+1}, \ulcorner P \urcorner) = \ulcorner P[\underline{n_1}, \ldots, \underline{n_k}]^{\dagger} \urcorner$ (cf. Lemma 3) and $m(P[n_1, \ldots, n_k]^{\dagger}) = m - 1 < m(P)$. Thus, by induction hypothesis, $\mu(\nu(2^{n_1+1} \ldots P_k^{n_k+1}, \ulcorner P \urcorner))$ is defined and is a proof-figure of the end sequent of $P[n_1, \ldots, n_k]^{\dagger}$ (hence is written as

$$\ulcorner (P[\underline{n_1}, \ldots, \underline{n_k}]^{\dagger})^{o} \urcorner).m((P[n_1, \ldots, n_k]^{\dagger})^{o}) < \omega \cdot (m - 1)$$

obviously holds. Observe the following.

$$\psi_2(\ulcorner (P[\underline{n_1}, \ldots, \underline{n_k}]^{\dagger})^{o} \urcorner, \ulcorner P \urcorner) = \ulcorner Q(\underline{n_1}, \ldots, \underline{n_k}) \urcorner .$$

$$\psi_5(b_o, \ulcorner P \urcorner) = \Lambda n_1 \ldots \Lambda n_k (\ulcorner (P[\underline{n_1}, \ldots, \underline{n_k}]^{\dagger})^{o} \urcorner).$$

$$\psi_4(\psi_5(b_o, \ulcorner P \urcorner), \ulcorner P \urcorner) = \Lambda n_1 \ldots \Lambda n_k \ulcorner Q(\underline{n_1}, \ldots, \underline{n_k}) \urcorner$$

where $k = k(\ulcorner P \urcorner)$.

$$\mu_o(0, \psi_4(\psi_5(b_o, \ulcorner P \urcorner), \ulcorner P \urcorner), \ulcorner P \urcorner)$$

$$= \varphi(\Lambda n_1 \ldots \Lambda n_k \ulcorner Q(\underline{n_1}, \ldots, \underline{n_k}) \urcorner, k, \ulcorner P \urcorner)$$

$$= \Lambda n_2 \ldots \Lambda n_k (3 \cdot 5^{\Lambda n_1(\ulcorner Q(n_2, \ldots, n_k) P \urcorner)} \cdot 7^s)$$

where $k = k(\ulcorner P \urcorner) > 0$ is assumed and

$s = \ulcorner \prod \to \Xi'(n_2, \ldots, n_k), \forall x D_1(x) \urcorner$ for appropriate $\prod$, Ξ, $D_1(x)$.

$$= \Lambda n_2 \ldots \Lambda n_k (\ulcorner Q_1(n_2, \ldots, n_k) \urcorner).$$

Suppose $i < k$ and

$$\mu_o(i, \psi_4(\psi_5(b_o, \ulcorner P \urcorner), \ulcorner P \urcorner), \ulcorner P \urcorner)$$

$= \Lambda n_{i+2} \ldots \Lambda n_k (\ulcorner Q_{i+1}(n_{i+2}, \ldots, n_k) \urcorner)$ holds where $k = k(\ulcorner P \urcorner)$. Then supposing $i + 1 < k$,

$$\mu_o(i + 1, \psi_4(\psi_5(b_o, \ulcorner P \urcorner), \ulcorner P \urcorner), \ulcorner P \urcorner)$$

$$= \varphi(\mu_o(i, \psi_4(\psi_5(b_o, \ulcorner P \urcorner), \ulcorner P \urcorner), \ulcorner P \urcorner), k \dot{-} (i + 1), \ulcorner P \urcorner)$$

$$= \Lambda n_{i+3} \ldots \Lambda n_k (3 \cdot 5^{\Lambda n_{i+2}(\ulcorner \Lambda n_{i+2} \ldots \Lambda n_k (\ulcorner Q_{i+1}(n_{i+2}, \ldots, n_k) \urcorner)\urcorner)} \cdot 7^s)$$

where $s = \ulcorner \prod \to \Xi'(n_{i+3}, \ldots, n_k), \forall x D_{i+2}(x) \urcorner$.

$$= \Lambda n_{i+3} \ldots \Lambda n_k (\ulcorner Q_{i+2}(n_{i+3}, \ldots, n_k) \urcorner).$$

$$= k(\ulcorner P \urcorner).$$

Thus

$$\mu(\ulcorner P \urcorner) = \prod(b_o, \ulcorner P \urcorner) = \mu_o(k \dot{-} 1, \psi_4(\psi_5(b, \ulcorner P \urcorner), \ulcorner P \urcorner), \ulcorner P \urcorner) = \ulcorner Q_k \urcorner$$

or, $\ulcorner P^{o} \urcorner = \mu(\ulcorner P \urcorner) = \ulcorner Q_k \urcorner$.

For the proof of $\omega(P^{o}) < \omega\cdot m$, see the preceding, intuitive description of P^{o}. Note. 1) It is easily seen that for each $i < k$, $\mu_o(i, \psi_4(\psi_5(b_o, \ulcorner P \urcorner), \ulcorner P \urcorner), \ulcorner P \urcorner)$ yields a constructive ω-rule. 2) In fact, μ_o should be defined so that it includes some necessary interchanges in order to obtain $Q_i(n_{i+1}, \ldots, n_k)$. We have omitted such details

§4. Proof of Theorem (see §2). From Theorem 1 and Proposition in §2, it suffices to show that all provable sentences (of Z) which are in the prenex form with alternating quantifiers are provable with the proof-figures whose ω-complexities are less than ω^2. If A is provable and is in prenex normal form with alternating quantifiers, then any proof of $\rightarrow A$ satisfies the condition on P in Lemma 4: i.e., $\rightarrow A$ satisfies the condition (*). Thus, from Lemma 4, A is provable with an ω-complexity less than ω^2, or A belongs to S'_{ω^2}. Therefore all true sentences belong to S'_{ω^2}. This completes the proof of our theorem.

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