EXPONENTIAL STABILITY FOR A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction.

In a number of recent papers ([1],[2],[3]) Coleman and Mizel have investigated evolution equations of a type arising in continuum mechanics. The form of these equations can be described as follows. One has an underlying Banach space $\mathfrak{B}$ of functions $\varphi, \psi, \ldots$ mapping $[0,\infty)$ into a finite dimensional space $\mathcal{E}$ and a continuous map f from $\mathfrak{B}$ into $\mathcal{E}$. One is given an element $\varphi \in \mathfrak{B}$. Then it is required to find a function x mapping $(-\infty, T)$, $T > 0$, into $\mathcal{E}$ such that:

(i) x is differentiable on $(0,T)$,

(ii) the functions x^t defined by $x^t(\tau) = x(t - \tau)$, $\tau \in [0,\infty)$, are in $\mathfrak{B}$ for each $t \in [0,T]$,

and such that x satisfies the equations,

$$\dot{x}(t) = f(x^t) \qquad t \in (0,T) \qquad (1.1),$$

$$x(t) = \varphi(-t) \qquad t \in (-\infty, 0] \qquad (1.2).$$

Coleman and Mizel studied the stability of processes of the above type. Suppose that f is zero for the zero function. Then the zero solution of (1.1) and (1.2) is called stable if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for $\|\varphi\|_{\mathfrak{B}} < \delta$ every solution of (1.1) and (1.2) has $\|x(t)\|_{\mathcal{E}} < \epsilon$ and a solution exists for arbitrary T. If, in addition, there is a $\delta > 0$ such that every solution of (1.1) and (1.2) with $\|\varphi\|_{\mathfrak{B}} < \delta$ exists for all t and obeys $\lim_{t \to \infty} \|x(t)\|_{\mathcal{E}} = 0$, then the zero solution is called asymptotically stable.

The paper [3] discusses the connection between stability and the ideas of [1] and [2]. The latter two works explore possible structures for the space $\mathcal{B}$ and the restrictions which thermodynamics places on the mapping $\underset{\sim}{f}$.

The purpose of this paper is to indicate some situations in which the stability is actually exponential, that is in which solutions of (1.1) and (1.2) decay exponentially in time. The major goal is to show that conditions for exponential stability can be related to the nature of $\mathcal{B}$ and $\underset{\sim}{f}$, as described in [1] and [2].

The basic tool in [3] was an analog of Lyapunov functions for (1.1). In contrast, our approach is via perturbation theory and thus a central role is played by the linearized version of (1.1). For the spaces $\mathcal{B}$ under consideration here, the linearized equation has the form,

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}\underset{\sim}{x}(t) + \int_0^\infty \underset{\sim}{B}(\tau)\underset{\sim}{x}(t - \tau)\,d\tau + \underset{\sim}{g}(t) \qquad (1.3),$$

where $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are linear transformations. Actually, we confine ourselves to single equations of the type arising in a discussion of the extension of filaments (see [3], section 6). In this case, $\mathcal{E}$ is just R and equations (1.1) and (1.3) become*,

$$\ddot{y}(t) = -f(y^t) \qquad (1.4),$$

$$\ddot{y}(t) = -L(y^t) + g(t) \qquad (1.5),$$

where,

*The minus signs are introduced to make the notation of section three correspond to that of [3].

$$L(\dot{y}^t) = G(0)y(t) + \int_0^\infty G'(\tau)y(t-\tau)\,d\tau \qquad (1.6).$$

Equations (1.4) and (1.5) could be put into the form (1.1) or (1.3) by introducing $\mathbf{y} = (y,\dot{y})$ and taking $\mathcal{B} = R$. It turns out to be unnecessary to do this, but a little care must be exercised in the prescription of initial conditions. If we did reduce to a system and then used (1.2) we would have to prescribe $y(t) = \phi_1(-t)$ and $\dot{y}(t) = \phi_2(-t)$ for $t \in (-\infty, 0]$. Actually, it suffices to prescribe $\dot{y}$ only at $t = 0$, as was observed in [3]. Thus our initial conditions for (1.4) or (1.5) will have the form,

$$y(t) = \phi(-t) \qquad t \in (-\infty, 0] \quad \phi \in B$$

$$\dot{y}(0) = \dot{y}_0 \qquad (1.7)^*.$$

Precise statements of our results require some discussion of the space B and are given in section 2. However, we can summarize them here. First, we require that B should be a history space (of L^p type) in the sense of [1]. Then we require that L, as defined in (1.6), be continuous on B. This introduces certain restrictions on G. We then impose conditions on L in order that it be compatible with thermodynamics in the sense of [2].

We impose a final restriction. This is rather technical to state but it implies that the influence function of B is an obliviator of all orders (see section 2). (This condition is introduced in [1] as a part of the study of retardation and its

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*One could also study solutions which are discontinuous at $t = 0$. That is, one could require that $y(t) = \phi(-t)$ on $(-\infty, 0)$ and $y(0^+) = y_0$. This would make only a minor change in equation (5.2) of Section 5.

appearance in stability theory we find interesting but difficult to understand.)

With $\mathfrak{B}$ and G subject to all the restrictions above we call L, as defined by (1.6), <u>exponentially stable</u>.

We can now outline our results. These are closely analogous to results for ordinary differential equations. Consider first the homogeneous equation corresponding to (1.5), that is $g = 0$. Then there will exist what might be called a <u>natural decay rate</u>. This is a number $\beta > 0$ such that <u>every</u> solution y of the homogeneous equation must satisfy the relation,

$$y(t) = O(e^{-\beta t}) \quad \text{as} \quad t \to \infty \tag{1.8}$$

independently of what the initial function ϕ may be.

For the nonhomogeneous equation (1.5) there exists a kind of dichotomy. If $g(t) = O(e^{-\beta' t})$ with $\beta' > \beta$ then the solution still obeys (1.8). If $g(t) = O(e^{-\beta' t})$ with $\beta' < \beta$ then (ordinarily) the solution y will satisfy $y(t) = O(e^{-\beta' t})$*.

For the nonlinear equation, we prove the following. Suppose that f in (1.4) vanishes at the zero function $0^{\dagger}$. Suppose further that f is differentiable at $0^{\dagger}$ and that its differential (which will have the form (1.5)) is exponentially stable. Then given any $\epsilon > 0$ there exists a δ such that if

$$\|\phi\|_{\mathfrak{B}} + |\dot{y}_0| < \delta \tag{1.9}$$

*The case in which $g(t) = O(e^{-\beta t})$ can also be treated but requires more detailed assumptions about g and we omit the discussion (see [6]).

then any solution of (1.4) and (1.7) satisfies

$$|y(t)| = O(e^{-(\beta-\epsilon)t}) \text{ as } t\to\infty$$

where β is the natural decay rate corresponding to the differential of f at 0.

It turns out that our methods for linear equations are also applicable to the equation of one-dimensional, linear viscoelasticity. In this case, one seeks a function u of x and t which satisfies the equation*,

$$u_{tt}(x,t) = L(u_x(x,\cdot)) + g(x,t). \qquad (1.10)$$

This equation is to hold on $0 < x < L$ and boundary conditions of the form,

$$u(0,t) = a(t), \quad u(L,t) = b(t) \qquad (1.11)$$

are imposed. Conditions (1.7) are replaced by,

$$u(x,t) = \varphi(x,-t) \text{ on } (-\infty,0], \quad u_t(x,0) = \dot{u}^0(x) \qquad (1.12)$$

The results for the above problem are of the same type as for (1.5). Again there exists a natural decay rate β for the homogeneous problem (not the same rate, however, as for (1.5)). This rate is still independent of φ and $\dot{u}^0$. Solutions of the nonhomogeneous problem will decay at the same rate if a,b, and g are all $O(e^{-\beta' t})$, $\beta' > \beta$, but will be $O(e^{-\beta' t})$ if a,b, or g is $O(e^{-\beta' t})$, $\beta' < \beta$. Thus the results resemble those of [6] for parabolic equations.

*Here we are assuming that L is as in (1.6). This corresponds to homogeneous viscoelasticity. In the nonhomogeneous case G can depend on x and this case could also be treated as we indicate in section 5.

It is to be noted that equation (1.10) is outside the scope of [3] since S is infinite dimensional and L contains an unbounded operator. We have no results on the very interesting problem of the nonlinear version of (1.10).

It appears that if one relaxes the obliviator condition we impose on B, then it is possible to find linear equations for which one has asymptotic stability which is not exponential. Thus, there appears to be an essential difference in this respect between our equations and ordinary differential equations. We hope to return to this question in the near future.

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2. Preliminaries and Statement of Results.

Our basic space will be one of past histories in the sense of [1]. The space B will consist of measurable functions defined on $[0,\infty)$. We require that $|\chi(0)| < \infty$ for each $\chi \in B$. Moreover, there is to be a $p \geq 1$ and a non-negative function k on $(0,\infty)$ such that for each $\chi \in B$ we have

$$\|\chi\|_r = \left(\int_0^\infty k|\chi|^p\right)^{1/p} < \infty.$$

We set

$$\|\chi\| = |\chi(0)| + \|\chi\|_r \qquad (2.1)$$

and then B becomes a Banach space. The function k is called the ~~influence function~~ of the space.

The work of [1] gives quite precise results concerning the restrictions placed on k by various physical considerations. We refer to [1] for details but we state that these considerations lead to the following assumptions concerning k:

$$k \in L^1(0,\infty) \qquad \text{(A.1)},$$

$$ae^{-bs} \le k(s) \le c \quad \text{a.e. on } (0,\infty) \qquad \text{(A.2)},$$

for some positive constants a,b,c

$$sk(s) \to 0 \ \text{ essentially as } \ s \to \infty \qquad \text{(A.3)}.$$

There are two further possible restrictions on k which are considered in [1]. These concern the quantity $N(a)$ defined by,

$$N(a) = \operatorname*{ess\,sup}_{s\in(0,\infty)} \frac{k(s+a)}{k(s)} .$$

The condition,

$$\lim_{y\to\infty} \Big(\sup_{a\in[y,\infty)} N(a) \Big) = c_0 < \infty \qquad (2.2),$$

is shown to be equivalent to what is called the relaxation property. The stronger condition,

$$\lim_{a\to\infty} N(a) = 0 \qquad \text{(A.4)},$$

is shown to imply that k is an obliviator of all orders, a condition which is important in retardation theory.

Once again we make no effort to expand on these notions other than to state that they have physical implications. We shall however impose on k the conditions (A.1)-(A.4) and then we can infer from [1] the following result (see theorem 10, p. 121 and its proof):

Lemma 2.1. If k satisfies conditions (A.1)-(A.4) then we must have,

$$N(\sigma) \le e^{-\gamma\sigma} \quad \text{for some} \quad \gamma > 0 \tag{2.3}$$

and,

$$k(\sigma) \le Me^{-p\gamma\sigma} \quad \text{for some} \quad M > 0 \tag{2.4}.$$

We consider next functionals defined on $\mathcal{B}$. These are maps $f: \mathcal{B} \to R$. f is continuous if $\chi_n \to \chi$ in $\mathcal{B}$ implies $f(\chi_n) \to f(\chi)$. It is easy to see that all linear functionals over $\mathcal{B}$ must have the form,

$$f(\chi) = A\chi(0) + \int_0^\infty K(\tau)\chi(\tau)\,d\tau \tag{2.5},$$

where A is a constant and,

$$K = k^{1/p}\mathcal{K}, \quad \text{where} \quad \mathcal{K} \in L^q, \; p^{-1} + q^{-1} = 1 \tag{2.6}.$$

A functional f on $\mathcal{B}$ is differentiable at χ if

$$f(\chi + \alpha) = f(\chi) + \Delta(\chi|\alpha) + o(\|\alpha\|) \tag{2.7},$$

where $\Delta(\chi|\cdot)$ is a linear functional over $\mathcal{B}$.

We are now ready to discuss evolution equations over $\mathcal{B}$. The ones we consider have the form

$$\ddot{y}(t) = -f(y^t) \tag{2.8},$$

and are to be interpreted as follows. f is assumed to be a continuous functional on $\mathcal{B}$. A solution of (2.8) on $(-\infty, T]$ with initial history $(\varphi, \dot{y}_0)$, is a function y defined on $(-\infty, T]$ and satisfying the conditions:

(i) If $y^t(\tau) = y(t - \tau)$, then $y^t \in \mathcal{B}$ for $t \in [0,T]$,

(ii) $y(t) = \varphi(-t)$ $t \leq 0$, $\varphi \in \mathcal{B}$, $\dot{y}(0) = \lim_{t \downarrow 0} \dot{y}(t) = \dot{y}_0$,

(iii) $y \in C^2(0,T)$,

(iv) y satisfies (2.8) on $(0,T)$.

Equations of the form,

$$u_{tt}(x,t) = f(u_{xx}(x,\cdot)) \tag{2.9}$$

with initial histories

$$\begin{aligned} u(x,t) &= \rho(x,-t) \text{ on } t \leq 0 \\ u_t(x,0) &= \dot{u}^0(x) \end{aligned} \tag{2.10}$$

can also be studied with obvious modifications of (i)-(iv).

Suppose that the functional f in (2.8) is differentiable at 0 and vanishes for $y^t(\tau) \equiv 0$. Then, by (2.5) and (2.7), equation (2.8) can be written in the form,

$$\ddot{y}(t) = -A\, y(0) - \int_0^\infty K(\tau) y(t - \tau) + F(y^t) \tag{2.11}$$

where

$$F(y^t) = o(\|y^t\|).$$

It follows immediately from (2.6) and (A.1) that $K \in L^1(0,\infty)$. Let us assume further that K is continuous in $[0,\infty)$. Then the function,

$$G(\tau) = A + \int_0^\tau K(\xi)\, d\xi$$

is differentiable on $[0,\infty)$, $G(\infty)$ exists, and (2.11) can be written,

$$\ddot{y}(t) = -L(y^t) + F(y^t) \tag{2.12}$$

where L has the form (1.6).

We are now ready to give a precise statement of our results. We impose one further condition for technical convenience:

From now on, it is to be understood that our basic space $\mathbb{B}$ has an influence function k which satisfies conditions (A.1)-(A.4). In addition, L will always denote a linear functional on $\mathbb{B}$. It will then have the form (1.6). We call such a linear functional *exponentially stable* if it satisfies the additional conditions:

G' uniformly continuous on $(0,\infty)$ and $G' \in L^1[0,\infty)$ (B.1)

$$G(\infty) > 0 \qquad \text{(B.2)},$$

$$G''(T) > 0 \text{ for all } T \qquad \text{(B.3)}.$$

Condition (B.1) is imposed for technical convenience while (B.2) and (B.3) connect our work with thermodynamics, as we show in the next section. We make the following observations. Since $G' \in L^1(0,\infty)$ and is uniformly continuous, we deduce that G' tends to zero as T tends to infinity. Then (B.3) yields the inequality,

$$G'(T) < 0 \text{ on } [0,\infty) \qquad (2.13).$$

We propose to establish the following results:

Theorem (1). Let L be exponentially stable. Then there exists a number $\beta > 0$ such that the following statements hold;

(i) *If $g(t) = O(e^{-\mu t})$ as $t \to \infty$ with $\mu > \beta$ then, for any $\varphi \in \mathbb{B}$ and y^0, there exists a unique solution of (1.5) and (1.7) and this solution satisfies $y(t) = O(e^{-\beta t})$.*

(ii) *If $g(t) = o(e^{-\mu t})$ as $t \to \infty$ with $\mu < \beta$ then, for any $\varphi \in \mathbb{B}$ and y^0, there exists a unique solution of (1.5) and (1.7)* and this solution satisfies $y(t) = O(e^{-\beta t})$ for any $\beta < \mu$.

Theorem (2). The results of theorem (1) hold for equations (1.10), (1.11) and (1.12) provided that a and b satisfy the same restrictions as g.

Theorem (3). Let f be differentiable and $f(0^+)$ be zero, so that (1.4) can be written in the form (2.12). Suppose that L is exponentially stable. Then there exists a $\delta > 0$ such that if (1.9) holds, then any solution of (1.4) and (1.7) on $(-\infty,T)$ can be extended to $(-\infty,\infty)$ and tends to zero exponentially as t tends to infinity.

Remarks; (1) The techniques used for the linear equations involve the Laplace transform and are similar to those in [6]. In analogy to the terminology used in that paper we could summarize the results of theorems (1) and (2) by saying that the solution »follows» g (or g, a and b) if $\mu > \beta$. in this connection, we remark that one need not require g, a or b to decay exponentially. They can decay algebraically or even tend to infinity and then the solutions will exhibit the same behavior (see [6]). What is important is that one can never hope to achieve a decay rate faster than $e^{-\mu t}$ no matter how rapidly g, a and b decrease.

(2) It might be suspected that the rate of decay β in theorems (1) and (2) is the same as the number γ of lemma (2.1). This is not so and in fact, as we show later by an example, β can be arbitrarily small.

3. Stability and Thermodynamics.

The goal of this section is to provide some motivation for conditions (B.2) and (B.3) and to establish some implications of those conditions. We begin by reviewing some ideas from [2] and [3].

A free energy functional for the equation,

$$\ddot{y}(t) = L(y^t) \qquad (3.1),$$

(L as in (1.6)) is a functional p of the following type. p should be differentiable and we write,

$$\Delta p(\chi|\alpha) = Dp(\chi)\,\alpha(0) + \delta p(\chi|\alpha) \qquad (3.2),$$

where $\delta p(\chi|\alpha)$ is a linear functional on the space $\mathcal{B}_r$. This space consists of restrictions to $(0,\infty)$ of functions of $\mathcal{B}$, with the norm $\|\alpha\|_r$ of section (2). It is required that

$$L(y^t) = Dp(y^t) \qquad (3.3),$$

and that,

$$\delta p(y^t|\dot{y}^t) \leq 0 \qquad (3.4),$$

for all regular histories. A regular history is one such that $y^t(\tau)$ is differentiable for all τ and such that $y^0(\tau)$ has compact support.

If such a free energy functional exists we say that the functional L is compatible with thermodynamics. The reason for this terminology is that it is shown in [2] that if (3.1) derives from a thermodynamic process, obeying the second law, then its free energy must be a functional satisfying (3.3) and (3.4).

One says that the zero solution of (3.1) is stable if given any e > 0 there exists a 6 > O such that if y is a solution of (3.1), with initial history (<P,y_Q) such that (\<p\\ + |y_Q|) < 6, then y exists for all t and satisfies |y(t)| < e for all t. If, in addition, there is a 6 such that each solution of (3.1)^ with initial history (<P, Y_Q) such that (\\<p\\ + |y_Q|) < &} satisfies $\lim_{t \to \infty} |y(t)| = 0$ one says that the zero solution is asymptotically stable. The results of [3], when specialized to (3.1), yield the following result.

Theorem (4). Let L be. compatible with thermodynamics. Let P]2§. its free energy functional and define p° on R by; p°(x) = p(x*), $\dot{x}$* (T) = x for all r. Tg p° has a strict local minimum at 0 then the zero solution of (3.1) JLS stable.

In order to insure asymptotic stability, we must impose a stronger condition on p, one that is not a consequence of thermodynamic considerations. p is called a strictly dissipative functional if whenever y is a solution of (3.1) the quantity

$$\Sigma(t) = \tfrac{1}{2}\,\dot{y}(t)^2 + p(y^t)$$

satisfies

with equality only if y is a constant. If y is a regular history then (3.3) and (3.4) show that (see [3], p. 261)

$$\frac{d\Sigma(t)}{dt} = \dot{y}(t)\ddot{y}(t) + L(y^t)\dot{y}(t).$$

Thus, for a strongly dissipative functional, strict inequality should hold in (3.4) for regular histories unless the history is identically zero.

We can connect these ideas with our conditions in section 2 provided that we impose a stronger condition on the function $K = G^?$ appearing in the linear functional L. This condition is,

$$p > 2 \quad \text{and} \quad K = k^{2//p}K,\ KGL^{\bar{q}},\ \bar{q}"^1 + 2p\sim^1 = 1 \qquad (A.5).$$

Note that (A.5.) implies (2.6). We emphasize that the assumption (A.5) is in no way necessary for the validity of theorems 1, 2 or 3. It is used here only in the construction of free energy functionals.

Theorem (5). Under conditions (A.I)-(A.5) and (B.1)-(B.3), L *jis* compatible with thermodynamics. Furthermore, for regular histories strict inequality holds in (3.4) unless y = 0.

Proof; We assert that p, defined by the equation,

$$p(\chi) = \backslash G(0)*(0)^2 + \%(0) \int_0^\infty G'(T)\%(T)dT - \frac{1}{2}\int_0^\infty (3'(T)7t^2(T)dT \qquad (3.5),$$

is a free energy functional and that p° has a strict local minimum at 0.

We let p^1 denote the first two terms on the right side of (3.5) and p^2 denote the last term. Since $G^1 = K$ we see immediately from (2.6) that p^1 is differentiable and that,

$$\Delta p^1(\chi|\alpha) = \left(G(0)\chi(0) + \int_0^\infty G'(\tau)\chi(\tau)\,d\tau\right)\alpha(0) + \chi(0)\int_0^\infty G'(\tau)\alpha(\tau)\,d\tau$$

$$= L(\chi)\alpha(0) + \chi(0)\int_0^\infty G'(\tau)\alpha(\tau)\,d\tau \qquad (3.6).$$

The functional p^2 is more complicated and it is here that we need condition (A.5). Let $\bar{q}$ be defined as in (A.5). Then we have, by Hölder's inequality and (2.13),

$$\int G'\chi^2 d\tau = \int K\chi^2 \leq \left(\int (Kk^{-2/p})^{\bar{q}}\right)^{1/\bar{q}}\left(\int (k^{2/p}\chi^2)^{p/2}\right)^{2/p}$$

$$= \left(\int \hat{K}^{\bar{q}}\right)^{1/\bar{q}}\|\chi\|_r^2 \qquad (3.7).$$

Hence p^2 is defined on $\mathcal{B}$. We have, moreover,

$$p^2(\chi + \alpha) = p^2(\chi) + 2\int_0^\infty K\chi\alpha + \int_0^\infty K\alpha^2 \qquad (3.8).$$

By (3.7) we see that

$$\int_0^\infty K\alpha^2 = O(\|\alpha\|^2).$$

Finally, we assert that the second integral on the right side of (3.8) is a bounded linear functional of α. As we have seen before this fact will be established if we show that $K\,k^{-1/p} \in L^q$. Once again we apply Hölder's inequality and obtain (note that $p/q = p - 1 > 1$, since $p > 2$),

$$\int (K\chi k^{-1/p})^q \leq \int (Kk^{-1/p})^q k^{-q/p} k^{q/p}\chi^q = \int (Kk^{-2/p})^q k^{q/p}\chi^q$$

$$\leq \left(\int (Kk^{-2/p})^{qs}\right)^{1/s}\left(\int (k^{q/p}\chi^q)^{p/q}\right)^{q/p} \leqq \left(\int \hat{K}^{qs}\right)^{1/s}\|\chi\|_r^q$$

where $\frac{1}{s} + \frac{q}{p} = 1$ or $qs = \bar{q}$.

We have thus shown that p^2 is a differentiable functional on $\mathcal{B}$. We have in fact,

$$\Delta p^2(\chi|\alpha) = -\int_0^\infty K(\tau)\chi(\tau)\alpha(\tau)\,d\tau \tag{3.9}$$

If we decompose Δp into two parts as in (3.2) we find from (3.6) and (3.9),

$$Dp(\chi) = L(\chi) \tag{3.10}$$

$$\delta p(\chi|\alpha) = -\int_0^\infty K(\tau)\,(\chi(\tau) - \chi(0))\,\alpha(\tau)\,d\tau \tag{3.11}$$

Equation (3.10) shows that p satisfies (3.2). If y is a regular history then we have from (3.11),

$$\begin{aligned}\delta p(y^t|\dot{y}^t) &= -\int_0^\infty K(\tau)\,(y(t-\tau) - y(t))\,\dot{y}(t-\tau)\,d\tau \\ &= \frac{1}{2}\int_0^\infty G'(\tau)\frac{d}{d\tau}(y(t-\tau) - y(t))^2\,d\tau\end{aligned} \tag{3.12}$$

An integration by parts in (3.12) yields (recall that $G'(\infty) = K(\infty) = 0$),

$$\delta p(y^t|\dot{y}^t) = -\frac{1}{2}\int_0^\infty G''(\tau)\,(y(t-\tau) - y(t))^2\,d\tau.$$

By (B.3) this is less than or equal to zero and equal to zero only if $y^t(\tau) \equiv y(t)$ that is $y \equiv 0$.

Finally, we note that

$$\begin{aligned}p^o(x) &= \frac{1}{2}G(0)x^2 + x^2\int_0^\infty G'(\tau)\,d\tau - \frac{x^2}{2}\int_0^\infty G'(\tau)\,d\tau \\ &= \frac{1}{2}G(\infty)x^2\end{aligned}$$

and this shows that $p^o(x)$ has a strict minimum at $x = 0$. This completes the proof of theorem (5).

The concept of dissipativity for linear evolution equations in one which arises often in network theory. In particular it was considered at some length, for functional differential equations, by König and Meixner [5]. One of the principal results of [5] is that dissipativity gives rise to conditions on the Laplace transform of G and we want to develop this idea here.

Consider again conditions (3.3) and (3.4). By the chain rule (see [3]) and (3.3) we have, for a regular history y,

$$\dot{£} p(y^{t}) - L(y^{t})\dot{y}(t) = Dp(y^{t})\dot{y}(t) + apfy\wedge\dot{y}^{t}) - L(y^{t})\dot{y}^{t}$$

$$= Spty\wedge\dot{Y}*) \leq 0 \qquad (3.13) .$$

If we integrate (3.13) from -co to T we find,

$$p(y^{T}) - \int_{-CD}^{T} L(y^{t})\dot{y}(t)dt \leq 0 \qquad (3.14).$$

Now let $\%^{t}(T) = jL(O)$ for all T. Then by (3.5) and (2.13) we have

$$p(\chi) - p(\chi^{\dagger}) = \int_{O}^{\infty} G'(T)\,(TC(O)JC(T) - X(0)^{2}) - \tfrac{1}{2}T \int_{O}^{\infty} G'(T)\,(X(T)^{2} - X(0)^{2})$$

$$- - \tfrac{1}{J} \int_{O}^{\circ\circ} G'(T)\,[\wedge(0) - \wedge(T)]^{2} \geq 0.$$

It follows that

$$P(Y^{T}) \geq P((y^{T})^{+}) = \backslash\, G(O)y(T)^{2} + \backslash\, y(T)^{2} \int_{O}^{\infty} j"G'(T)dT$$

$$= \backslash\, G(oo)y(T)^{2} \geq 0$$

by (B.I). Substituting this inequality into (3.14) we obtain the following result.

Lemma 3.1. For all regular histories y we have

$$\int_{-\infty}^{T} L(y^{t})\dot{y}(t)\,dt \geq 0 \qquad (3.15).$$

The formula (3.15) is the Konig and Meixner definition of dissipativity. They establish the following as a consequence of (3.15)*.

Lemma 3.2. Let L be of the form (1.5) and suppose that the Laplace transform Q, of G,

$$Q(s) = \int_{0}^{\infty} e^{-s\tau} G(\tau)\,d\tau,$$

exists in $\operatorname{Re} s > 0$. Then (3.15) holds only if

$$\operatorname{Re} Q(s) > 0 \quad \text{in} \quad \operatorname{Re} s > 0 \qquad (3.16).$$

Let us investigate the Laplace transform of our function G. By (2.6) and (2.4) we have,

$$K \leq Me^{-\gamma\tau}K, \quad K \in L^{q} \qquad (3.17),$$

and

$$G(\tau) = G(0) + \int_{0}^{\tau} K(\xi)\,d\xi = G(\infty) + H(\tau),$$

where

$$H(\tau) = -\int_{\tau}^{\infty} K(\xi)\,d\xi.$$

From (3.17) one deduces readily that for any $\gamma' < \gamma$ we have,

$$|H(\tau)| \leq M_{\gamma'} e^{-\gamma'\tau} \quad \text{for some constant } M_{\gamma'}.$$

Thus, we have the following result.

*In the terminology of [5] L in our case is a transformation of order ∞ on C_{∞} and is of slow growth.

Lemma 3.3. The Laplace transform $\mathcal{G}$ of G is analytic in $\text{Re } s > -\gamma$ except for a simple pole of residue $G(\infty)$ $(\neq 0)$ at $s = 0$. That is

$$\mathcal{G}(s) = \frac{G(\infty)}{s} + \mathcal{H}(s),$$

where $\mathcal{H}(s)$ is analytic in $\text{Re } s > -\gamma$.

We shall have need of a sharpened form of lemma 3.2 and we prove this now.

Lemma 3.4. The transform $\mathcal{G}$ of G satisfies the condition

$$\text{Re } \mathcal{G}(s) > 0 \quad \text{in} \quad \text{Re } s \geq 0 \tag{3.18}$$

(Note that Lemma (3.2) follows from lemma (3.4), the maximum principle and the fact that $\mathcal{G} \to 0$ in $\text{Re } s \geq 0$.)

Proof: By (3.16) we need consider only the case $\text{Re } s = 0$. Further since $G(\infty)/s$ is imaginary on $\text{Re } s = 0$ we need consider only $\mathcal{H}$. We have, for $\eta > 0$,

$$\text{Re } \mathcal{H}(i\eta) = \int_0^\infty \cos \eta\tau (G(\tau) - G(\infty))d\tau$$

$$= -\frac{1}{\eta} \int_0^\infty \sin \eta\tau \, G'(\tau)d\tau.$$

Now we write,

$$\int_0^\infty \sin \eta\tau \, G'(\tau)d\tau = \sum_{k=0}^\infty \left(\int_{2k\pi/\eta}^{(2k+1)\pi/\eta} \sin \eta\tau \, G'(\tau)d\tau + \int_{(2k+1)\pi/\eta}^{(2k+2)\pi/\eta} \sin \eta\tau G'(\tau)d\tau \right)$$

$$= \sum_{k=0}^\infty \int_0^{\pi/\eta} \sin \eta\tau \left(G'\left(\tau + \frac{2k}{\eta}\right) - G'\left(\tau + \frac{2k}{\eta} + \frac{\pi}{\eta}\right)\right) d\tau$$

and this quantity is less than zero by (B.3). Hence

$$\text{Rett(irj)} > O \quad \text{for} \quad 77 > 0 \qquad (3.18).$$

But W is real for real s hence ^(-iTj) = $\overline{\#(i7j)}$ so that (3.18) holds also for 77 < 0. For *rj* = 0 we have

$$\text{H}(O) = \int_{O}^{\infty} (G(\tau) - G(oo))dr > 0,$$

since G! is negative.

The inequality (3.18) is the key to stability of linear equations as will be seen in subsequent sections.

The final result in this section is a lemma which will be of use in the study of homogeneous linear equations. It also indicates the role played by the »obliviator» inequality (A.4). In order to motivate this result let us rewrite equation (1.5) in the form,

$$\ddot{y}(t) = -L(y^{t}) = W(y^{u}) + G(t) \qquad (3.19),$$

where

$$L(y^{r}) = G(O)y(t) + \int_{o}^{t} G'(t - T)y(r)dr$$

$$W(y^{t}) = \int_{t}^{0} G'(T)y(t - T)dr = \int_{o}^{00} G'(t + T)y(-T)dT.$$

If y satisfies the initial condition (1.7) then the quantity $W(y^{t})$ is known, namely,

$$W(y^{t}) = \int_{O}^{oo} G»(t + r)<p(r)dT = \int_{O}^{oo} K(t + r)<p(T)dr \qquad (3.2o).$$

Thus (3.19) assumes the form

$$\dot{y} = L(y_t) + \phi(t) \qquad t > 0$$

where $\phi(t)$ is a known function. This is an equation to which Laplace transform techniques can clearly be applied but it is important to know the behavior of ϕ for large t and our result gives some information about the term $W(y^t)$.

Lemma 3.5. There exists a constant Q, independent of φ, such that

$$W(y^t) \le Q\|\varphi\|_r e^{-\gamma t}.$$

proof: We have by (3.20) and (2.6),

$$|W(y^t)| \le \int_0^\infty k(t+\tau)^{1/p}|K(t+\tau)||\varphi(\tau)|\,d\tau$$

$$= \int_0^\infty \left(\frac{k(t+\tau)}{k(\tau)}\right)^{1/p} k(\tau)^{1/p}|\varphi(\tau)||K(t+\tau)|\,d\tau.$$

But by (2.3) we have

$$\left(\frac{k(t+\tau)}{k(\tau)}\right)^{1/p} \le e^{-\gamma t}.$$

Hence we have

$$|W(y^t)| \le e^{-\gamma t}\int_0^\infty k(\tau)^{1/p}|\varphi(\tau)||K(t+\tau)|\,d\tau$$

$$\le e^{-\gamma t}\left(\int_0^\infty k|\varphi|^p\right)^{1/p}\left(\int_0^\infty |K(t+\tau)|^q\right)^{1/q} = e^{-\gamma t}\|\varphi\|_r Q.$$

4. Laplace Transforms.

In this section, we wish to collect some results concerning Laplace transforms. These are specializations of the results used in [6] and they are obtained from the book of Doetsch [4].

Let $s = \xi + i\eta$ be complex. We write,

$$\mathcal{L}_s(f) = \int_0^{\infty} e^{-st} f(t)\,dt, \quad \lambda_\xi(F) = (2\pi)^{-1} e^{\xi t} \int_{-\infty}^{+\infty} e^{i\eta t} F(\xi + i\eta)\,d\eta.$$

It is known that under suitable conditions these two operators are inverses of each other. We shall need the following result. This is essentially theorem 2, page 266 of [4] and is a consequence of the Riemann-Lebesgue lemma.

Lemma 4.1. *Suppose* $F(s)$ *is analytic in* $\mathrm{Re}\ s > a$ *and continuous in* $\mathrm{Re}\ s \geq a$ *and that*

$$F(s) = \frac{c}{s} + O\left(\frac{1}{s^2}\right) \text{ as } |s| \to \infty, \text{ uniformly in } \mathrm{Re}\ s \geq a \qquad (4.19),$$

where c *is a constant. Then* $\lambda_\xi(F)$ *is independent of* ξ *in* $\xi \geq a$ *and defines a function* $f(t)$ *such that,*

(i) $\mathcal{L}_s(f) = F$ *in* $\mathrm{Re}\ s \geq a$,

(ii) $f(t) = O(e^{at})$ *as* $t \to \infty$.

Let us investigate the Laplace transform of G a little more closely. Since

$$G(\tau) = G(0) + \int_0^{\tau} K(\xi)\,d\xi,$$

we have

$$\mathcal{G}(s) = \mathcal{L}_s(G) = \frac{G(0)}{s} + \frac{\mathcal{L}_s(K)}{s} \qquad (4.2).$$

By (2.6) and (2.4) observe that $\mathcal{L}_s(K)$ is analytic in $\mathrm{Re}\ s > -\gamma$ and, by (B.1), K is differentiable at 0. One can then derive, from theorem 8 page 197 of [4], the following result.

Lemma 4.2. $\mathcal{L}_s(K)$ satisfies the relation

$$\mathcal{L}_s(K) = \frac{K(0)}{s} + O(\frac{1}{s^2}) \text{ as } |s| \rightarrow \infty \qquad (4.3),$$

uniformly in Re $s > -\mu$.

We can now state and prove the fundamental result which guarantees exponential stability for the linear equations. We give the result in a form which is applicable both to (1.5) and (1.10) and also to the equation arising in non-homogeneous viscoelasticity.

Theorem 6. Let $\{\beta_n\}$ be a sequence of positive numbers with $\beta_n \geq \beta > 0$ for all n. Then there exists a $\beta > 0$* such all solutions of the equations,

$$s = -\beta_n \mathcal{G}(s) \qquad (4.4),$$

satisfy the inequality,

$$\text{Re } s \leq -\beta \qquad (4.5).$$

Proof: We write $s = \xi + i\eta$ and $\mathcal{L}_s(K) = A + iB$. Then, by (4.2), (4.4) is equivalent to the two equations,

$$\xi = \frac{-\beta_n G(0)\xi}{\xi^2 + \eta^2} - \frac{\beta_n(A\xi + B\eta)}{\xi^2 + \eta^2} \qquad (4.6)$$

$$\eta = \frac{\beta_n G(0)\eta}{\xi^2 + \eta^2} - \frac{\beta_n(B\xi - \eta A)}{\xi^2 + \eta^2} \qquad (4.7)$$

*β must of course be less than γ in order that the theorem even be meaningful.

(Note that G(0) > 0 since G' < 0 and G(oo) > O) .

We observe first that (4.4) can have no solutions s, with Re s =0 for any n, since Re Q(s) > 0 when Re s = O. A second observation is that there cannot exist a sequence, ST = £, + i*?w °f solutions of (4.4) such that £, —* 0 while the
X K K K
rjv's remain bounded. For the condition Re Q > 0 on Re s = 0 implies that for any M there exists an e > O and an a > O such that Re 5(s) > a in Re s > -e, |lxn s| <, M.

Thus the only way in which the conclusion of the theorem could be violated would be to have a sequence s, K = £, K + irj, K of o. solutions of (4.4), for n = n^, such that £k-*0 while 17^—^0 We show that this cannot occur either.

Suppose such a sequence existed and consider (4.7). From (4.3) we deduce that M^w^) and B^k^k^ tend to zero as k tends to infinity. Hence we derive from (4.7) the result that

$$\lim_{k\to\infty} \frac{\beta_{n_k}}{\xi_k^2 + \eta_k^2} = \frac{1}{G(0)} \qquad (4.8).$$

Also from (4.3)^ we see that

$$A(\xi_k,\eta_k) = \frac{K(0)\xi_k}{\xi_k^2 + \eta_k^2} + O\left(\frac{1}{\xi_k^2 + \eta_k^2}\right)$$

$$B(\xi_k,\eta_k) = \frac{-K(0)\eta_k}{\xi_k^2 + \eta_k^2} + O\left(\frac{1}{\xi_k^2 + \eta_k^2}\right).$$

Thus (4.6) yields,

$$\xi_k = \left(\frac{\beta_{n_k}}{\xi_k^2 + \eta_k^2}\right)(G(0) \,\tilde{*}v + "2\_\_ *T + "2\_\_ h + 0\left(\frac{1}{\xi_k^2 + \eta_k^2}\right)\}.$$

Hence if we pass to the limit in this equation and use (4.8) we obtain,

But this cannot be since K(0) = G'(0) < 0. This contradiction completes the proof of theorem 6.

Let us consider a particular example. Suppose that the influence function of the space R is k(a) = e *' . Then

K(a) = $\tilde{B}e^{-U(J}$ for $v > 2y$ will satisfy (A.5) . *Then* we have

$$G(T) = A + B \int_0^{\tau} e \; da = A - Be$$

where $A = \hat{A}^* + \tilde{B}/V$ and $B = \tilde{B}/v$. This will satisfy (B. 2) and (B.3) if $A > 0$ and $B < 0$. We have then

$$\psi(s) = \frac{-}{s} - \frac{-}{s + \nu}.$$

Let us consider (4.4) for a single $j3_n = 1$, the situation which will arise in the next section when we study (1.5) . Equation (4.4) becomes then,

$$s^3 + us^2 + (A - B)s + AV = 0 \qquad (4.9).$$

Since B is negative, it is not too difficult to verify that all solutions of this equation have negative real parts (as is guaranteed by our theorem).

Note that if $A = G(\infty)$ were zero, then (4.9) would have $s = 0$ as a solution. This will show that we need strict inequality in (B.2). For $A = 0$ the roots are $s_1 = 0$ and

$$s_{2,3} = \frac{-\nu \pm \sqrt{\nu^2 + 4B}}{2} .$$

s_2 and s_3 have negative real parts and the three roots are distinct. Now for A sufficiently small the roots of (4.9) are continuous functions $s_i(A)$ and they remain distinct. Moreover $s_1(A)$, that root such that $s_1(0) = 0$, will remain as the one with largest real part. Thus, by choosing A sufficiently small, we can make (4.9) have a root with real part arbitrarily close to zero. As we shall see in the next section, this means that we can thus have linear equations with arbitrarily small natural decay rates.

5. Stability for Linear Problems.

We write equation (1.5) in the form (3.1) that is,

$$\ddot{y}(t) = -L(y^t) - W(y^t) + g(t) \quad t > 0 \qquad (5.1).$$

Let us formally take the Laplace transform of (5.1). If we let $\hat{y}(s)$, $\hat{w}$ and $\hat{g}$ denote respectively the transforms of y, W and g we have then,

$$s^2\hat{y}(s) - s\varphi(0) - \dot{y}_0(0) = \mathcal{L}_s(L(y^t)) - \hat{w}(s) + \hat{g}(s) \qquad (5.2).$$

Now

$$\mathcal{L}_s(L(y^t)) = G(0)\hat{y}(s) + (sQ(s) - G(0))\hat{y}(s) = sQ(s)\hat{y}(s),$$

and hence (5.2) becomes,

$$(s + \mathcal{G}(s))\hat{y}(s) = (-\hat{w}(s) + \hat{g}(s) + s\varphi(0) + \dot{y}_0)/s \qquad (5.3)$$

Now by lemma 3.5 we know that $\hat{w}$ is analytic in $\text{Re } s > -\gamma$. We know also that $s + \mathcal{G}(s)$ is analytic in $\text{Re } s > -\gamma$ and by theorem 6 we can find a $\beta < \gamma$, such that $s + \mathcal{G}(s)$ is without zeroes in $\text{Re } s \geq -\beta$. Hence the function,

$$\hat{R}(s) = (s^2 + s\mathcal{G}(s))^{-1}$$

is analytic in $\text{Re } s \geq -\beta$ except possibly at $s = 0$. But by lemma (3.3), it follows that,

$$\mathcal{G}(s) - \frac{G(\infty)}{s} \text{ is regular near } s = 0,$$

hence $\hat{R}(s)$ is regular at $s = 0$.

We can now prove theorem (1). Suppose first that g satisfies (i) of theorem (1), that is $g = O(e^{-\mu t})$, $\mu > \beta$. Then g is analytic in $\text{Re } s \geq -\beta$ hence the function $\hat{y}(s)$, defined by

$$\hat{y}(s) = \hat{R}(s)[-\hat{w}(s) + \hat{g}(s) + s\varphi(0) + \dot{y}_0] \qquad (5.4),$$

is also analytic there. From (4.2) and (4.3) we find,

$$\hat{R}(s) = (s^2 + G(0) + \frac{K(0)}{s} + O(\frac{1}{s^2}))^{-1}$$

$$= \frac{1}{s^2}(1 - \frac{G(0)}{s^2} - O(\frac{1}{s^3})).$$

Since $\hat{w}$ and $\hat{g}$ tend to zero as s tends to infinity it follows that

$$\hat{y} = \frac{1}{s^3}\left(1 - \frac{G(0)}{s^2}\right)\left(-\hat{w}(s) + \hat{g}(s) + s\varphi(0) + y_0\right) + O\left(\frac{1}{s^{\wedge}}\right) \tag{5.5}$$

It follows from lemma (4.1) that the function

$$y(t) = \mathcal{L}^{-1}(\hat{y}) \tag{5.6}$$

is independent of ξ in $\xi \geq \xi_0$, satisfies $\mathcal{L}_s(y) = \hat{y}$ and also the condition,

$$y(t) = O(e^{\xi t}) \qquad \text{as} \quad t \to \infty .$$

We propose to show that y, as defined by (5.6), is a solution of (5.1) and that $y(0) = \varphi(0)$, $\dot{y}(0) = \dot{y}_0$. The estimate (5.5) enables us to show that y can be differentiated twice with respect to t. We have for $\xi > \xi_0$,

$$y(t) = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} e^{st}\hat{y}(s)\,ds.$$

Substitution of the estimate (5.5) into this formula leads to a number of terms which can be evaluated explicitly plus a term for which two differentiations with respect to t is justified. For example, the first term is

$$\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} e^{st}\,\frac{\hat{w}(s)}{s^{2}}\,ds = \int_0^t\int_0^{\tau} w(\xi)\,d\xi\,d\tau .$$

The remainder term,

$$\frac{1}{2\pi}\int_{\xi-i\infty}^{\xi+i\infty} e^{st}\,O\left(\frac{1}{s^4}\right)ds$$

can clearly be differentiated twice.

Once we know that y is twice differentiable, it is a tedious but straightforward calculation to show that it satisfies (5.1) as well as $y(0) = \varphi(0)$, $\dot{y}(0) = \dot{y}_0$. We omit the details.

If g satisfies condition (ii) of theorem (1), that is $g(t) = O(e^{-\mu t})$ with $\mu < \beta$ then the proof proceeds in the same way except that now one can infer only that $\hat{y}$ is analytic in $\operatorname{Re} s \geq \mu'$ for any $\mu' < \mu$. Hence lemma (4.1) yields the estimate $y(t) = O(e^{-\mu' t})$ for any $\mu' < \mu$.

It remains only to establish the uniqueness of the solutions. Let y_1 and y_2 be two solutions of (1.5) and (1.7). Then $y = y_1 - y_2$ vanishes in $t \leq 0$ and satisfies,

$$\ddot{y}(t) = -L(y^t).$$

Multiply by $\dot{y}(t)$ and integrate from $-\infty$ to T. This gives by lemma (3.1),

$$\frac{1}{2}\dot{y}(T)^2 = -\int_{-\infty}^{T} L(y^t)\dot{y}(t)\,dt \leq 0.$$

Hence $\dot{y}(T) = 0$ and, since $y(0) = 0$, it follows that $y(t) \equiv 0$.

We want to study the solution (5.5) a little more closely. We can apply lemma (4.1) to $\hat{R}(s)$ and deduce that there exists a function $R(\tau)$ such that $\hat{R}$ is its transform and that,

$$|R(\tau)| \leq R_0 e^{-\beta\tau} \qquad (5.7).$$

Moreover, as in lemma (4.2),

$$\hat{R}(s) = \frac{R(0)}{s} + O(\frac{1}{s^2})$$

On the other hand we say that $R = O(s^{-2})$ as $s \quad \infty$

and hence $R(0) = 0$. Since $sR(s)$ is also analytic in $\operatorname{Re} s \geq -\beta$ it follows that R' exists and satisfies an estimate of the same form as (5.7).

By the convolution theorem, we can write the solution (5.5), (5.4) in the form,

$$y(t) = -\int_0^t R(t-\tau)[W(y^\tau) + g(\tau)]d\tau + \varphi(0)R'(t) + \dot{y}_0 R(t) \qquad (5.8).$$

From lemma (3.5) and the estimate (5.7) we have,

$$\left|\int_0^t R(t-\tau)W(y^\tau)\right| \leq R_0 Q\|\varphi\|_r \int_0^t e^{-\beta(t-\tau)}e^{-\gamma\tau} \leq c_1 e^{-\beta t}\|\varphi\|_r .$$

Thus, we have from (5.8) the estimate,

$$|y(t)| \leq ce^{-\beta t}(\|\varphi\| + |\dot{y}_0|) + R_0 e^{-\beta t}\int_0^t e^{\beta\tau}|g(\tau)|d\tau \qquad (5.9).$$

The proof of theorem 2 is very similar to that of theorem 1; the only complication arises in the calculation of the transform. Hence, we give just an outline of the proof. We rewrite equation (1.10) in the form

$$u_{tt} = L(u^t_{xx}(x,\cdot)) + W(u^t_{xx}(x,\cdot)) + g \qquad (5.10),$$

where

$$W(u^t_{xx}(x,\cdot)) = \int_0^\infty G'(t+\tau)u_{xx}(x,-\tau)d\tau = \int_0^\infty G'(t+\tau)\varphi_{xx}(x,\tau)d\tau .$$

If we formally take transforms we have, as in (5.2),

$$s\hat{u} - \mathcal{G}(s)\hat{u}_{xx} = \left(\hat{w} + \hat{g} + s\varphi(x,0) + u^0(x)\right)/s \qquad (5.11).$$

The transform of conditions (1.11) will be,

$$u(0,s) = \hat{a}(s) \qquad u(L,s) = \hat{b}(s) \qquad (5.12),$$

where $\hat{a}$ and $\hat{b}$ denote the transforms of a and b.

Equations (5.11) and (5.12) constitute a standard two-point boundary value problem. It will have a solution so long as $s/Q(s)$ is not equal to an eigenvalue of $-\frac{d^2}{dx^2}$ with zero boundary conditions, that is to one of the numbers $-n^2$, $n = 1,2,\cdots$. But by theorem 6 we know that for some $\beta > 0$

$$s + n^2 Q(s) \neq 0$$

for $\text{Re } s \geq -\beta$.

If the right side of (5.11) and the functions $\hat{a}$ and $\hat{b}$ are regular in $\text{Re } s \geq -\beta$ then it can be shown that the solution of (5.11) and (5.12) is also regular there. If instead, the right side and $\hat{a}$ and $\hat{b}$ are regular in $\text{Re } s \geq -\mu$, $\mu < 0$ then the solution is regular in $\text{Re } s \geq -\mu$. Thus one has the same two cases as in theorem (1) (see [6] for very similar calculations).

We remark that the case of inhomogeneous viscoelasticity can be treated in a very similar way. The only change is that now Q depends on x as well as s. Hence (5.11) and (5.12) becomes a more general Sturm-Liouville problem. The values of s to be avoided are now the eigen-values of $-\frac{d^2}{dx^2}$ relative to $(Q(x,s))^{-1}$. Once again these form a sequence $\{\beta_n\}$ of negative numbers tending to minus infinity so that theorem 6 is still applicable.

6. Stability for Nonlinear Problems.

In this section we give a proof of theorem 3. Here we are considering the equation (2.12) that is,

$$\ddot{y}(t) = -L(y^t) + F(y^t) \qquad (6.1),$$

where L is exponentially stable and $F(y^t) = o(\|y^t\|)$. We still have the initial conditions,

$$y(t) = \varphi(-t) \text{ in } t < 0 \quad \dot{y}(0) = \dot{y}_0 \qquad (6.2).$$

Suppose we have a solution of (6.1) and (6.2) in some interval $(0,T)$. Then, if we denote the quantity $\|\varphi\| + |\dot{y}_0|$ by r, we have by (5.9),

$$|y(t)| \le c r e^{-\gamma t} + R_0 e^{-\gamma t} \int_0^t e^{\gamma\tau} |F(y^\tau)| d\tau. \qquad (6.3).$$

We know that given any $\varepsilon > 0$ we can find a $\delta(\varepsilon) > 0$ such that,

$$|F(y^t)| \le \varepsilon \|y^t\| \quad \text{for} \quad \|y^t\| < \delta(\varepsilon) \qquad (6.4).$$

Thus so long as $\|y^t\| \le \delta(\varepsilon)$ we have, by (6.3),

$$|y(t)| \le c r e^{-\gamma t} + R_0 \varepsilon e^{-\gamma t} \int_0^t e^{\gamma\tau} \|y^\tau\| d\tau. \qquad (6.5).$$

Next, we observe that

$$\|y^t\| = |y(t)| + \left(\int_0^\infty |y(t - r)|^p k(r) dr\right)^{1/p}$$

$$\le |y(t)| + c' \left\{ \left(\int_t^\infty |\varphi(r - t)|^p k(\tau) d\tau\right)^{1/p} + \left(\int_0^t |y(r)|^p k(t - r) dr\right)^{1/p} \right\}. \qquad (6.6).$$

The second term on the right can be estimated using (2.3) and we can estimate $k(t - \tau)$ in the third term by (2.4). These two estimates yield,

$$\|y^t\| \le |y(t)| + c\|\varphi\|_r e^{-\gamma t} + c_\varepsilon e^{-\gamma t} \left(\int_0^t |y(\tau)|^p e^{p\gamma\tau} d\tau\right)^{1/p} \qquad (6.7).$$

Let v be a positive number such that $v < /J$ and $v < y - fi$ and set

$$A(t) = |y(t)|e\wedge-\wedge S_B(t) = H y \wedge e \wedge * .$$

Then (6.5) can be written as.

$$A(t) \leq cre^{-\nu t} + R_o \epsilon e^{-\nu t} \int_0^t B(T)e^{UT}dr \qquad (6.8).$$

By Hölder's inequality we have

$$\left(\int_0^t B(\tau)e^{\nu\tau}d\tau\right)^p \leq c_1 e \wedge \int_0^t B(T)^p dr .$$

Hence (6.8) yields,

$$A(t)^p \leq c_2[r^p e^{-p\nu t} + \epsilon^p \int_0^t B(r)^p dr] \qquad (6.9)$$

From (6.7) we obtain,

$$B(t) \leq A(t) + c'\|\varphi\|_r e^{-(\gamma-\beta+\nu)t} + c''e^{-(\gamma-\beta+\nu)t}\left(\int_0^t A(T)^p e^{p(\gamma-\beta+\nu)\tau}d\tau\right)^{1/p},$$

or

$$B(t)^p \leq c_3[A(t)^p + \|\varphi\|_r^p e^{-p(\gamma-\beta+\nu)t}$$

$$+ e^{-p(\gamma-\beta+\nu)t}\int_0^t A(\tau)^p e^{p(\gamma-\beta+\nu)\tau}d\tau] \qquad (6.10)$$

We substitute (6.10) into (6.9) and obtain,

$$A(t)^p \leq c\wedge r^p e^{-P\wedge} + \epsilon^p \int_0^t A(r)^p dr + c_4\epsilon^p\|\varphi\|_r^p$$

$$+ \epsilon^p \int_0^t e^{-P\wedge\wedge+\wedge T}\left(\int_0^\tau A(\mu)^p e^{p(\gamma-\beta+\nu)\mu}d\mu\right)d\tau] \qquad (6.11)$$

If we interchange the order of integration in the last term we find that it has the form,

$$\epsilon^p \int_0^t A(\mu)^p e^{p(\gamma-\beta+\nu)\mu} \left(\int_\mu^t e^{-p(\gamma-\beta+\nu)\tau} d\tau \right) d\mu \leq c_5 \epsilon^p \int_0^t A(\mu)^p d\mu.$$

Thus (6.11) becomes,

$$A(t)^p \leq c_6 [r^p e^{-p\nu t} + \epsilon^p \|\varphi\|_r^p + \epsilon^p \int_0^t A(\tau)^p d\tau]. \qquad (6.12).$$

We set

$$\Gamma = c_6 (r^p + \epsilon^p \|\varphi\|_r^p), \ \bar{\epsilon} = c_6 \epsilon^p. \qquad (6.13).$$

Then if we apply Gronwall's inequality to (6.12) we have,

$$A(t)^p \leq \Gamma + \epsilon\Gamma \int_0^t e^{\epsilon(t-s)} ds \leq \Gamma(1 + \epsilon e^{\epsilon t}).$$

Thus

$$|y(t)| = A(t) e^{-(\beta-\nu)t} \leq \Gamma(1 + \epsilon e^{\epsilon t}) e^{-(\beta-\nu)t}. \qquad (6.14).$$

If we choose $\epsilon < 1$ and $\epsilon < \beta - \nu$ it follows from (6.14) that

$$|y(t)| \leq 2\Gamma \qquad (6.15),$$

and then from (6.7) we obtain

$$\|y^t\| \leq (2 + c'')\Gamma + c' \|\varphi\|_r \leq (2 + c'')\Gamma + c'r. \qquad (6.16).$$

Note also that $\Gamma \leq 2c_6 r$.

Now we are ready to choose the δ of theorem 3. We choose δ so that $\delta < \delta(\epsilon)$ and

$$(2 + c'') 2c_6 \delta + c' \delta < \delta(\epsilon),$$

where $\delta(\epsilon)$ is as in (6.4). Then if $\|\varphi\| + |\dot{y}_0| < r$ we have, by (6.16),

$$\|y^t\| \leq \delta(\epsilon). \qquad (6.17)$$

We have (6.17) initially since 6 < 6(€) and we have shown that (6.17) continues to hold so long as the solution exists. It follows that F on the right side of (6.1) remains bounded. But differentiation of (5.8) leads to an estimate like (5.9) for $\dot{y}(t)$. Then the boundedness of F implies that of $\dot{y}$. Standard arguments (see [3]) show that we can continue the solution of (6.1) so long as $y(t), \dot{y}(t)$ and $\|y_t\|$ remain bounded; hence we can continue for all time. Equation (6.14) yields the exponential decay and the proof is complete.

Notice that, since v is arbitrary, we have proved that the rate of decay of the nonlinear equation is arbitrarily close to the natural decay rate of the linearized equation.

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