ON THE EXPONENTIAL STABILITY OF SOLUTIONS OF $E(u_x)u_{xx} + \lambda u_{xtx} = \rho u_{tt}$

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1. Introduction.

We consider one-dimensional motions $\chi(x,t) = x + u(x,t)$ of a continuum for which the stress σ is related to the strain u_x and the strain rate u_{xt} through the consitutive equation,

$$\sigma(x,t) = \int_0^{u_x(x,t)} E(\xi)\,d\xi + \lambda u_{xt}(x,t).$$

The equation of motion is,

$$E(u_x)u_{xx} + \lambda u_{xtx} = \rho u_{tt}.$$

The function E is the equilibrium Young's modulus, λ the viscosity and ρ the (constant) density of points x in the reference configuration.

In [1] and [2] the authors discussed various properties of solutions of the initial-boundary value problem:

(E) $\quad E(u_x)u_{xx} + \lambda u_{xtx} = \rho u_{tt}, \quad (x,t) \in (0,1) \times (0,\infty),$

(I) $\quad u(x,0) = \varphi(x)$ and $u_t(x,0) = \psi(x) \quad x \in [0,1],$

(B) $\quad u(0,t) = u(1,t) = 0 \quad t > 0.$

In particular, it was shown that (E)-(B) has a unique smooth solution which decays to zero as t tends to $+\infty$, uniformly in x.* What was not obtained was an estimate for the rate of decay. It is this topic which is pursued here.

In [1] and [2] we used energy integrals together with estimates for solutions of the linear heat equation. Here we

*It was also shown that all derivatives through second order tend to zero uniformly in x.

replace the usual energy integrals by ones which are exponentially weighted, an idea which was used by R. J. Duffin [3] for ordinary differential equations. We then make use of the fact that the Green's function for the linear operator,

$$L = E(0)\frac{\partial^2}{\partial x^2} + \lambda\frac{\partial^3}{\partial t\partial x^2} - \rho\frac{\partial^2}{\partial t^2},$$

with (I) and (B), decays exponentially. Our conclusion is that any solution of the nonlinear problem decays exponentially.

2. Statement of Results.

We assume that the function $E : (-\infty,\infty) \to (0,\infty)$ is twice continuously differentiable. The data φ and ψ are in $C^2[0,1]$ and satisfy the compatibility conditions,

$$\varphi(0) = \varphi(1) = \psi(0) = \psi(1) = 0.$$

In addition, $E(\varphi_x)\varphi_{xx} + \lambda\psi_{xx}$ is to vanish at $x = 0$ and 1.

We set

$$J(\varphi,\psi) = \sum_{i=0}^{2} \left(\max_{x\in[0,1]} |\varphi^{(i)}(x)| + \max_{x\in[0,1]} |\psi^{(i)}(x)| \right). \tag{2.1}$$

For any function $U\in C^N([0,1]\times[0,T])$ we set

$$|||U|||_N(t) = \sum_{i=0}^{N} \sum_{k=0}^{i} \max_{x\in[0,1]} \left| \frac{\partial^i U}{\partial x^{i-k}\partial t^k} \right|. \tag{2.2}$$

It was shown in [1] that problem (E)-(B) has a unique solution such that $|||u|||_2(t)$ tends to zero as t tends to infinity. In addition,

$$|||u|||_2(t) \le M_1(J),$$

where M_1 is a smooth function such that $M_1(\xi) \to 0$ as $\xi \to 0^+$.

We shall see that solutions of the linear equation $Lv = 0$ with (I) and (B) satisfy the relation,

$$v = O(e^{-\mu t}),$$

where,

$$\mu = \min_{n \geq 1} \operatorname{Re}\left[\frac{\lambda n^2 \pi^2}{2\rho}\left(1 - \sqrt{1 - \frac{4E(0)\rho}{\lambda^2 n^2}}\right)\right]. \tag{2.3}$$

Note that μ will always be less than or equal to $E(0)/\lambda$ and will equal $E(0)/\lambda$ whenever

$$4E(0)\rho < \lambda^2 \pi^2. \tag{2.4}$$

In order to state our new result, we need some additional notation. For functions $f(x)$ or $f(x,t)$ let

$$|f| \; (|f|(t)) = \max_{x \in [0,1]} |f(x)| \; (\max_{x \in [0,1]} |f(x,t)|), \tag{2.5}$$

and

$$\|f\|^2 \; (\|f\|^2(t)) = \int_0^1 f^2(x)\,dx \; (\int_0^1 f^2(x,t)\,dx). \tag{2.6}$$

Theorem 1. Let μ be as in (2.3). Then there is a constant k, depending on J and satisfying $0 < k < \mu$, and a smooth function $\xi \to M_2(\xi)$, which tends to zero as $\xi \to 0^+$, such that any solution u satisfies:

$$|||u|||_1(t) + |u_{xx}|(t) + \|u_{xt}\|(t) \leq M_2(J)e^{-kt}. \tag{2.7}$$

The constant k approaches μ as J approaches zero. In addition u satisfies, (2.8) $$\int_0^t e^{2k\tau}\{\|u_{tt}\|^2(\tau) + \|u_{xtx}\|^2(\tau)\}d\tau \le M_2(J).$$

3. Energy Integrals.

Before starting our proof, we record a lemma to which we shall appeal throughout the remainder of this paper.

Lemma 1. *Let $v(x,t)$ be C^2 in x and t and satisfy $v(0,t) = v(1,t) = 0$ for $t \ge 0$. Then,*

(3.1) $$\|v\|(t) \le |v|(t) \le \|v_x\|(t) \le |v_x|(t) \le \|v_{xx}\|(t) \le |v_{xx}|(t),$$

where $|\cdot|$ and $\|\cdot\|$ are defined in (2.5) and (2.6) respectively.

Our first step is to obtain two weighted energy integrals replacing formulas (4.2) and (4.4) of [1].

Lemma 2. *Let u be a solution of problem (E),(I),(B) and let k be any positive number. Then the following identities must hold for all $t \ge 0$.*

(3.2) $$e^{2kt}[\rho\|u_t\|^2(t) + 2\int_0^1 \mathcal{E}(u_x)(x,t)dx] + 2\lambda\int_0^t e^{2k\tau}\|u_{xt}\|^2(\tau)d\tau$$

$$= k\int_0^t e^{2k\tau}[\rho\|u_t\|^2(\tau) + 2\int_0^1 \mathcal{E}(u_x)(x,\tau)dx]d\tau + A_1,$$

where

(3.3) $$A_1 \equiv \rho\|\psi\|^2 + 2\int_0^1 \mathcal{E}(\varphi_x)(x)dx, \quad \text{and} \quad \mathcal{E}(\eta) \equiv \int_0^\eta \int_0^\xi E(\gamma)d\gamma\, d\xi;$$

(3.4) $$\lambda e^{2kt}\|u_{xx}\|^2(t) + 2\int_0^t e^{2k\tau}\int_0^1 (E(u_x) - k\lambda)u_{xx}^2(x,\tau)dx\, d\tau$$

$$= 2\rho e^{2kt}\int_0^1 u_{xx}u_t(x,t)dx - 4\rho k\int_0^t e^{2k\tau}\int_0^1 u_{xx}u_t(x,\tau)dx\, d\tau$$

$$+ 2\rho\int_0^t e^{2k\tau}\|u_{xt}\|^2(\tau)d\tau + A_2,$$

where

(3.5) $$A_2 \equiv \lambda\|\varphi_{xx}\|^2 - 2\rho \int_0^1 \varphi_{xx}\psi dx.$$

Equations (3.2) and (3.4) are obtained by multiplying (E) by $e^{2kt}u_t$ and $e^{2kt}u_{xx}$ respectively, integrating over $(0,1)x(0,t)$ and using the fact that the boundary conditions imply that $u_t(0,t) = u_t(1,t) = 0$. (Compare [1]).

We use one result from [1]. This is that any solution satisfies the inequality,

$$|u_x|(t) \leq M_1(J).$$

This implies that there exist positive constants $\underline{E}$ and $\bar{E}$ such that,

(3.6) $$\underline{E} \leq E(u_x) \leq \bar{E}.$$

It follows from (3.6) and the definition of $\mathcal{E}$ that,

(3.7) $$\underline{E}\|u_x\|^2(t) \leq 2\int_0^1 \mathcal{E}(u_x)(x,t)dx \leq \bar{E}\|u_x\|^2(t).$$

A key quantity in our calculations is the weighted norm $\Gamma(t)$ defined by,

(3.8) $$\Gamma(t)^2 = e^{2kt}[\|u_t\|^2(t) + \|u_x\|^2(t)], \ k \geq 0.$$

We observe first that (3.7), when substituted into (3.2), yields the two inequalities,

(3.9) $$\Gamma(t)^2 \leq B_1[A_3 + k\int_0^t \Gamma^2(\tau)d\tau],$$

$$(3.10)\qquad \int_0^t e^{2k\tau}\|u_{xt}\|^2(\tau)\,d\tau \le B_1[A_3 + k\int_0^t \Gamma^2(\tau)\,d\tau],$$

where

$$(3.11)\qquad B_1 = \frac{\max(1,\rho,\bar{E})}{\min(2\lambda,\rho,\underline{E})},$$

and $A_3 = \max(A_1, |A_2|)$.

Our next step is to obtain relations between the quantity $\Omega(t)$, defined by,

$$(3.12)\qquad \Omega(t) = e^{kt}\|u_{xx}\|(t),$$

and $\Gamma(t)$. The results are as follows. For any $k < \underline{E}/\lambda$ we have,

$$(3.13)\qquad \Omega(t) \le B_2[(A_3 + k\int_0^t \Gamma^2(\tau)\,d\tau)^{1/2} + k(\int_0^t \Omega^2(\tau)\,d\tau)^{1/2}],$$

$$(3.14)\qquad \int_0^t \Omega^2(\tau)\,d\tau \le \frac{\bar{B}_2}{(\underline{E}-k\lambda)^2}[A_3 + k\int_0^t \Gamma^2(\tau)\,d\tau],$$

where B_2 and $\bar{B}_2$ are constants depending only on $\underline{E}, \bar{E}, \lambda,$ and ρ.

In the next section, we derive bounds for the integral of Γ in terms of the integral of Ω. These bounds when combined with (3.13) and (3.14) will enable us to show that, for some positive $k < \mu$, both Γ and Ω are bounded.

Consider (3.13) first. Equations (3.4), (3.6) and (3.11), together with Schwarz's inequality, yield the preliminary estimate,

$$(3.15)\qquad \Omega^2(t) + (\underline{E} - k\lambda)\int_0^t \Omega^2(\tau)\,d\tau$$
$$\le B_3[e^{2kt}\|u_t\|(t)\|u_{xx}\|(t) + k(\int_0^t e^{2k\tau}\|u_t\|^2(\tau)\,d\tau)^{1/2}$$
$$\cdot(\int_0^t \Omega^2(\tau)\,d\tau)^{1/2} + \int_0^t e^{2k\tau}\|u_{xt}\|^2(\tau)\,d\tau + A_3],$$

where

$$(3.16) \qquad B_3 = \frac{\max(1,4\rho)}{\min(2,\lambda)} .$$

If we now restrict k to be less than $\underline{E}/\lambda$ and make use of (3.9), (3.10) and (3.1) with $v = u_t$, we see that,

$$(3.17) \qquad \Omega^2(t) \le B_4[N_1(t)\Omega(t) + N_1^2(t) + kN_1(t)(\int_0^t \Omega^2(\tau)d\tau)^{1/2}] .$$

Here

$$B_4 = \max(B_3, B_1 B_3)$$

and $N_1(t)$ is defined by,

$$N_1^2(t) = A_3 + k \int_0^t \Gamma^2(\tau)d\tau .$$

Equation (3.13) now follows easily from (3.12) with

$$B_2 = B_4 + \frac{3}{2} \sqrt{B_4} .$$

To obtain (3.14) we insert the results of (3.13) into (3.15), and make use of (3.9), (3.10) and (3.1) with $v = u_t$, to obtain the inequality,

$$(3.18) \qquad (\underline{E} - k\lambda)\int_0^t \Omega^2(\tau)d\tau \le B_5 N_1(t) k (\int_0^t \Omega^2(\tau)d\tau)^{1/2} + B_5 N_1(t)^2 ,$$

where

$$B_5 = \max(B_4 + 1,\ B_2 B_4 + 1) .$$

This yields the inequality (3.14) with some $\bar{B}_2$, depending only on $\underline{E}, \bar{E}, \rho$ and λ.

4. Additional Estimates and Completion of the Proof of Theorem 1.

In order to proceed, we shall need estimates for the linear inhomogeneous, initial-boundary value problem:

$(E)_L$ $E(0)u_{xx} + \lambda u_{xtx} - \rho u_{tt} = f$, $(x,t) \in (0,1) \times (0,\infty)$,

(I) $u(x,0) = \varphi(x)$ and $u_t(x,0) = \psi(x)$, $x \in [0,1]$,

(B) $u(0,t) = u(1,t) = 0$, $t > 0$.

The observation that (E) may be rewritten as $(E)_L$ with

$$f \equiv (E(0) - E(u_x))u_{xx}, \tag{4.1}$$

the estimates for the linear problem, and inequality (3.14) will ultimately provide, <u>for some</u> k, $0 < k < \mu$, the <u>key inequality</u>:

$$\int_0^t \Gamma^2(\tau)\,d\tau \le \bar{M}(A_3), \quad t \ge 0. \tag{4.2}$$

μ, $\Gamma^2(\tau)$, and A_3 are defined in (2.3), (3.8), and (3.11) respectively, and $\xi \to \bar{M}(\xi)$ is a smooth function which tends to zero as ξ tends to zero.

It is easily verified that the solution of the linear problem $(E)_L$-(B) is given by:

$$u(x,t) = \int_0^1 G^{(1)}(x,\xi,t)\varphi(\xi)\,d\xi + \int_0^t G^{(2)}(x,\xi,t)\psi(\xi)\,d\xi + \int_0^t \int_0^1 G^{(2)}(x,\xi,t-\tau) f(\xi,\tau)\,d\xi\,d\tau \tag{4.2}$$

where

$$G^{(1)}(x,\xi,t) = \sum_{n=1}^{\infty} \left(e^{\mu_n^+ t} + e^{\mu_n^- t}\right) \sin n\pi x \, \sin n\pi\xi, \tag{4.3}$$

$$(4.4)\qquad G^{(2)}(x,\xi,t) = \sum_{n=1}^{\infty} \left(\frac{e^{\mu_n^+ t} - e^{\mu_n^- t}}{\mu_n^+ - \mu_n^-}\right) \sin n\pi x \; \sin n\pi\xi,$$

$$(4.5)\qquad \mu_n^+ = \frac{-\lambda n^2\pi^2}{2\rho}\left[1 + \sqrt{1 - \frac{4E(0)\rho}{\lambda^2 n^2\pi^2}}\right], \text{ and}$$

$$(4.6)\qquad \mu_n^- = \frac{-\lambda n^2\pi^2}{2\rho}\left[1 - \sqrt{1 - \frac{4E(0)\rho}{\lambda^2 n^2\pi^2}}\right].$$

The formulas (4.3) and (4.4) are valid provided $\frac{4E(0)\rho}{\lambda^2 n^2\pi^2} \neq 1$ for all $n \geq 1$. If this condition fails for some n, say N, then the Nth terms in $G^{(1)}$ and $G^{(2)}$ are replaced by $2e^{-\frac{\lambda N^2\pi^2 t}{2}} \sin N\pi x \; \sin N\pi\xi$ and $2te^{-\frac{\lambda N^2\pi^2 t}{2}} \sin N\pi x \; \sin N\pi\xi$ respectively. The resulting analysis of the problem is then appropriately modified and no difficulties arise. In light of this statement, we shall assume throughout that

$$(4.7)\qquad \frac{4E(0)\rho}{\lambda^2 n^2\pi^2} \neq 1, \; n \geq 1.$$

Our main result for the linear problem is the following theorem.

<u>Theorem</u> 2. <u>There exists a constant</u> D, <u>depending on</u> $E(0)$, ρ <u>and</u> λ, <u>and independent of</u> k, <u>such that</u>

$$(4.8)\qquad \int_0^t \Gamma^2(\tau)\,d\tau \leq \frac{D}{\mu - k}\,\Gamma^2(0) + \frac{D}{(\mu - k)^2}\int_0^t e^{2k\tau}\|f\|^2(\tau)\,d\tau,$$

<u>for all</u> $0 \leq k < \mu$. <u>Again</u>,

$$\Gamma^2(t) \equiv e^{2kt}(\|u_t\|^2(t) + \|u_x\|^2(t)),$$

and

$$0 < \mu \equiv \min_{n\geq 1} \frac{\lambda n^2\pi^2}{2\rho}\left[1 - \sqrt{1 - \frac{4E(0)\rho}{\lambda^2 n^2\pi^2}}\right] \leq E(0)/\lambda.$$

Proof: We shall prove the theorem for the u_x term; the calculation for u_t is similar. We first observe that u_x may be written as the sum of three terms F_1, F_2, and F_3; i.e.

$$u_x(x,t) = F_1(x,t) + F_2(x,t) + F_3(x,t)$$

where

$$F_1(x,t) = \int_0^1 G_x^{(1)}(x,\xi,t)\varphi(\xi)\,d\xi,$$

$$F_2(x,t) = \int_0^1 G_x^{(2)}(x,\xi,t)\psi(\xi)\,d\xi, \text{ and}$$

$$F_3(x,t) = \int_0^t \int_0^1 G_x^2(x,\xi,t-\tau)f(\xi,\tau)\,d\xi\,d\tau.$$

Since $\|F_1 + F_2 + F_3\|^2(t) \le 2(\|F_1\|^2(t) + \|F_2\|^2(t) + \|F_3\|^2(t))$, it suffices to look at each term separately. For F_1 we have

$$\|F_1\|^2(t) = \int_0^1 \{\sum_{n=1}^{\infty} (e^{\mu_n^+ t} + e^{\mu_n^- t}) n\pi \cos n\pi x\, \varphi_n\}^2 dx$$

$$= \sum_{n=1}^{\infty} \frac{(e^{\mu_n^+ t} + e^{\mu_n^- t})^2}{2} n^2\pi^2\varphi_n^2$$

where

$$\varphi_n \equiv \int_0^1 \sin n\pi\xi\; \varphi(\xi)\,d\xi$$

is the nth Fourier coefficient of φ. The inequality

$$|e^{\mu_n^+ t} + e^{\mu_n^- t}| \le 2e^{-\mu t}$$

and the identity

$$\|\varphi_x\|^2 = 2 \sum_{n=1}^{\infty} n^2\pi^2\varphi_n^2 \quad ,$$

then yield the estimate

$$\|F_1\|^2(t) \le e^{-2\mu t}\|\varphi_x\|^2.$$

It now follows that

$$\int_0^t e^{2k\tau}\|F_1\|^2(\tau)\,d\tau \le \frac{\|\varphi_x\|^2}{2(\mu-k)}, \quad 0 \le k < \mu.$$

The estimate for F_2 is similar. The result is

$$\|F_2\|^2(t) \le 2Ce^{-2\mu t}\|\psi\|^2$$

and

$$\int_0^t e^{2k\tau}\|F_2\|^2(\tau)\,d\tau \le \frac{C\|\psi\|^2}{2(\mu-k)}, \quad 0 \le k < \mu,$$

$$C \equiv \max_{n\ge 1} \frac{n^2\pi^2}{|\mu_n^+ - \mu_n^-|^2} = \max_{n\ge 1} \frac{\rho}{\lambda^2 n^2\pi^2 \left|1 - \frac{4E(0)\rho}{\lambda^2 n^2\pi^2}\right|}.$$

The constant C is finite because of the assumption (4.7). The F_3 term requires slightly more care. For any $\tau > 0$ we have

$$\|F_3\|^2(\tau) = \int_0^1 \Big\{\sum_{n=1}^{\infty}\Big(\int_0^\tau \frac{e^{\mu_n^+(\tau-\eta)} - e^{\mu_n^-(\tau-\eta)}}{(\mu_n^+ - \mu_n^-)} f_n(\eta)\,d\eta\Big) n\pi \cos n\pi x\Big\}^2 dx$$

$$= \sum_{n=1}^{\infty}\Big\{\int_0^\tau \frac{e^{\mu_n^+(\tau-\eta)} - e^{\mu_n^-(\tau-\eta)}}{\mu_n^+ - \mu_n^-} f_n(\eta)\,d\eta\Big\}^2 \frac{n^2\pi^2}{2}$$

$$\le C \sum_{n=1}^{\infty} 2\Big\{\int_0^\tau e^{-\mu(\tau-\eta)}\,|f_n(\eta)|\,d\eta\Big\}^2$$

where C is as above and

$$f_n(\eta) \equiv \int_0^1 \sin n\pi\xi \; f(\xi,\eta)\,d\xi.$$

Multiplying the last inequality by $e^{2k\tau}$ and integrating the result over $(0,t)$ we obtain,

$$\int_0^t e^{2k\tau}\|F_3\|^2(\tau)\,d\tau$$
$$\leq C\sum_{n=1}^{\infty} 2\int_0^t\int_0^\tau\int_0^\tau e^{2k\tau}e^{-\mu(\tau-\eta_1)}e^{-\mu(\tau-\eta_2)} f_n(\eta_1)f_n(\eta_2)\,d\eta_1 d\eta_2 d\tau.$$

Since

$$\int_0^t\int_0^\tau\int_0^\tau e^{2k\tau}e^{-\mu(\tau-\eta_1)}e^{-\mu(\tau-\eta_2)}|f_n(\eta_1)||f_n(\eta_2)|\,d\eta_1 d\eta_2 d\tau$$
$$= \int_0^t e^{-(\mu-k)\tau_1}e^{-(\mu-k)\tau_2}\int_{\max(\tau_1,\tau_2)}^t e^{k(\eta-\tau_1)}e^{k(\eta-\tau_2)}|f_n(\eta-\tau_1)||f_n(\eta-\tau_2)|\,d\eta\,d\tau_1 d\tau_2$$
$$\leq \left[\int_0^t e^{-(k-\mu)\tau_1}e^{-(k-\mu)\tau_2}d\tau_1 d\tau_2\right]\int_0^t e^{2k\eta}|f_n(\eta)|^2 d\eta$$
$$\leq \frac{1}{(\mu-k)^2}\int_0^t e^{2k\eta}|f_n(\eta)|^2 d\eta,$$

and since

$$\int_0^t e^{2k\eta}\|f\|^2(\eta)\,d\eta = 2\sum_{n=1}^{\infty}\int_0^t e^{2k\eta}|f_n|^2(\eta)\,d\eta,$$

we obtain the inequality:

$$\int_0^t e^{2k\tau}\|F_3\|^2(\tau) \leq \frac{C}{(\mu-k)^2}\int_0^t e^{2k\eta}\|f\|^2(\eta)\,d\eta. \quad \text{q.e.d.}$$

We now complete the proof of Theorem 1. We take f as in (4.1) and make use of (4.8) to obtain:

$$(4.9) \qquad \int_0^t \Gamma^2(\tau)\,d\tau \leq \frac{D}{\mu-k}\Gamma^2(0) + \frac{D(\bar{E}-E(0))}{(\mu-k)^2}\int_0^t \Omega^2(\tau)\,d\tau, \quad 0 \leq k < \mu,$$

where again

$$\Gamma^2(\tau) \equiv e^{2k\tau}(\|u_t\|^2(\tau) + \|u_x\|^2(\tau)),$$
$$\Omega^2(\tau) = e^{2k\tau}\|u_{xx}\|^2(\tau),$$

and $\bar{E}$ is the upper bound for $E(\cdot)$. If we substitute the above result into (3.14) and observe that $\Gamma^2(0) \leq A_3$ (see (3.11)), we obtain the inequality:

$$\int_0^t \Omega^2(\tau)\,d\tau \leq \left[\frac{D}{\mu - k} + \frac{D(\bar{E} - E(0))\bar{B}^2}{(\mu - k)^2(\underline{E} - k\lambda)}\right]A_3$$

$$(4.10) \qquad + \frac{D(\bar{E} - E(0))\bar{B}_2 A_3}{(\mu - k)^2(\underline{E} - k\lambda)}\, k \int_0^t \Omega^2(\tau)\,d\tau,$$

for all $0 \leq k < \min(\mu, \underline{E}/\lambda)$. Inequality (4.2) now follows for any $0 < k < \min(\mu, \underline{E}/\lambda)$ such that

$$(4.11) \qquad \frac{D(\bar{E} - E(0))\bar{B}_2 A_3}{(\mu - k)^2(\underline{E} - k\lambda)}\, k < 1.$$

That k may be chosen arbitrarily close to μ as $A_3 \to 0$ is clear from the form of (4.11).

The remainder of Theorem 1 now follows from the arguments employed in [1] and [2], from equations (3.2) and (3.4), and from the new identities:

$$(4.12) \qquad 2\rho \int_0^t e^{2k\tau}\|u_{tt}\|^2(\tau)\,d\tau + \lambda e^{2kt}\|u_{xt}\|^2(t)$$
$$= 2\int_0^t e^{2k\tau}\int_0^1 E(u_x)u_{xx}u_{tt}(x,\tau)\,dx\,d\tau$$
$$+ 2k\lambda \int_0^t e^{2k\tau}\|u_{xt}\|^2(\tau)\,d\tau + \lambda\|\psi_x\|^2, \text{ and}$$

$$(4.13) \qquad u_{xx}(x,t) = \frac{\rho}{\lambda}u_t - e^{-\int_0^t \frac{E(u_x(x,\eta))\,d\eta}{\lambda}} \cdot \left[\int_0^t \rho u_\tau(x,\tau)\frac{E(u_x(x,\tau))}{\lambda^2}\, e^{\int_0^\tau \frac{E(u_x(x,\eta))\,d\eta}{\lambda}}\,d\tau\right]$$

$$+ e^{-\int_0^t \frac{E(u_x(x,\eta))d\eta}{\lambda}} [\varphi_{xx}(x) - \frac{\rho}{\lambda}\psi(x)].$$

Equation (4.12) is obtained by multiplying (E) by u_{tt}, integrating the result over $(0,1)x(0,t)$ and making use of the fact that the boundary conditions (B) imply that $u_{tt}(0,t) = u_{tt}(1,t) = 0$. To obtain (4.13) we regard (E) as an ordinary differential equation for u_{xx} and solve the initial value problem (E) together with the initial condition $u_{xx}(x,0) = \varphi_{xx}(x)$.

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