NETWORK MODELS

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Abstract

The steady flow of electrical current through a network of conductors has served as a suggestive model for a variety of mathematical theories. This paper describes electrical models related to the following theories: series-parallel graphs, parallel addition of matrices, lattice theory, generalized inverses, Grassmann algebra, Wang algebra, matroids, extremal length, Rayleigh's reciprocal relation and the width-length inequality.

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NETWORK MODELS

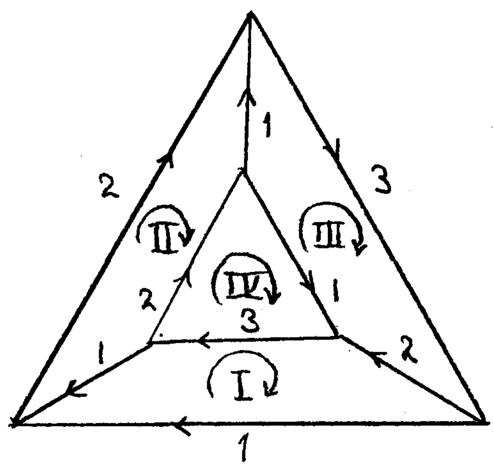
The history of science shows that the development of mathematics has been accelerated by the use of models. Thus geometric diagrams have served as models for algebraic relations. Gambling has served as a model for probability theory. Gravitation has served as a model for harmonic functions. Such models have accelerated mathematical development for three main reasons: (i) Attention is focused on significant problems. (ii) Models aid the intuition in perceiving complex relations. (iii) New concepts are suggested.

Since the days of Ohm and Kirchhoff, the study of electrical networks has stimulated developments in practically every branch of mathematics. For example network models have contributed to topology, nonlinear differential equations, function theory, boolean algebra and information theory.

Networks are still an abundant source of mathematical problems. This paper describes several such problems which have interested me. These problems are varied but will all involve the steady flow of electrical current through a network of resistors obeying Ohm's law. This is the classical Kirchhoff network and of main concern here will be the joint resistance of such a network.

1. <u>The four color problem</u>.

One of the first network models which attracted my interest was an electrical correspondence of map coloration. This can be illustrated by the simple map shown in Fig. 1 consisting of a triangular region surrounded by four quadrilateral regions. This map is colored by four "colors" I, II, III and IV. This coloration has been chosen so that the color IV does not appear on the boundary regions.



A map coloration
Figure 1

If a region has the color number X let a cyclic current of X amperes flow clockwise around the boundary of the region. Thus IV amperes flows around the boundary of the triangle. Now superimpose all of these cyclic flows. The strength of the resultant flow in the boundary edges is given in arabic numerals and the direction is indicated by arrows. Thus one edge of the triangle separates regions colored IV and I so the strength of the current through this edge is IV - I = 3 amperes. Clearly then the edge currents have strengths 1, 2, or 3. Moreover, the total current flow satisfies Kirchhoff's current conservation law. This comes about because the total current flow is a superposition of cyclic flows.

Now consider an arbitrary map colored in four colors I, II, III and IV such that IV does not appear on the boundary. Then by the same procedure given in the above example, it is seen that there is a conservative flow in the edges such that the current strength in each edge is 1, 2, or 3 amperes.

Conversely, suppose that there is a conservative flow through the edges of a map such that the current strength in each edge is 1, 2, or 3 amperes. Now it is an elementary theorem of network theory that any conservative flow can be achieved by a superposition of cyclic currents flowing clockwise around the edges of the regions. Then the boundary regions have cyclic values ± 1, ± 2, ± 3 because the cyclic current must be equal to the boundary edge current. Next, observe that neighboring region must have cyclic currents differing by an integer. Thus by moving from region to region it is seen that all cyclic currents are integer valued.

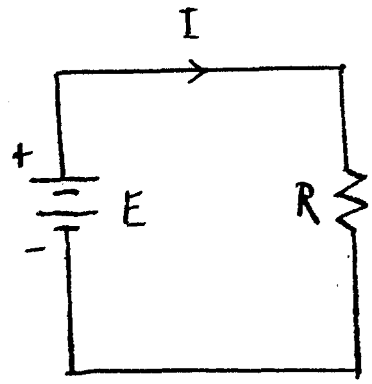
Next reduce the cyclic currents mod 4. Then we obtain a coloration of the map in colors I, II, III and IV. Since the edge currents are not congruent to zero mod 4 it follows that adjacent regions have different color. Moreover, the boundary regions are not colored IV. This proves the following theorem.

<u>A planar map can be colored in four colors if and only if there is a conservative flow through the edge network such that the current through an edge has strength</u> 1, 2 <u>or</u> 3 <u>amperes</u>.

To extend these ideas to networks which are not planar, I made the following conjecture. Consider a network such that every branch (edge) is on some circuit. Then there is a conservative flow such that the current through a branch has strength 1, 2, 3 or 4 amperes. Similar ideas were independently developed by W. T. Tutte.

2. Series-parallel networks.

Shown in Fig. 2 is a simple circuit containing a battery of voltage E and a resistor of resistance R ohms. Then the current I flowing in the circuit is determined by the relation

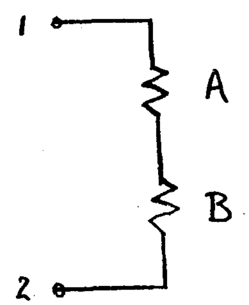


A simple circuit
Figure 2

(1) $$\frac{E}{I} = R > 0.$$

This is Ohm's law.

Shown in Fig. 3 are two resistors connected in series. One resistor has resistance A ohms and the other has resistance B ohms. Then the joint resistance R between terminals 1 and 2 is given by the formula



Reistors in series
Figure 3

(2) $$R = A + B.$$

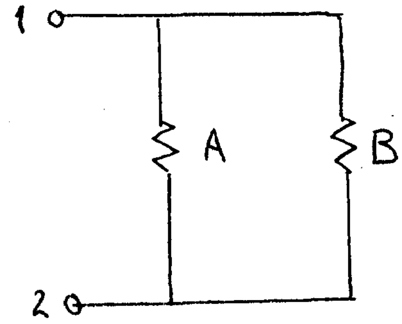
On the other hand, the two resistors could be connected in parallel as shown in Fig. 4. Conductance is the reciprocal of of resistance and conductances add in the parallel connection so

$$R^{-1} = A^{-1} + B^{-1}.$$

Solving for R gives

$$R = \frac{AB}{A + B} \tag{3}$$

and this is the formula for the joint resistance R of two resistors in parallel.



Resistors in parallel
Figure 4

To have a convenient short notation for the joint resistance of resistors connected in parallel let

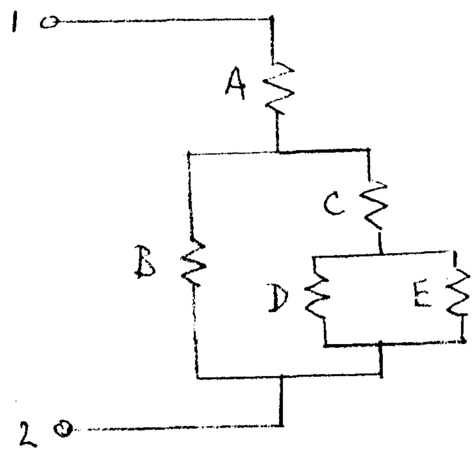
$$A : B = \frac{AB}{A + B} . \tag{4}$$

Then $A : B$ may be regarded as a new operation termed <u>parallel addition</u> [17]. Parallel addition is defined for any non-negative numbers. The network model shows that parallel addition is commutative and associative. Moreover, multiplication is distributive over this operation.

Consider now an algebriac expression in the operations (+) and (:) operating on positive numbers A, B, C, etc. An example is

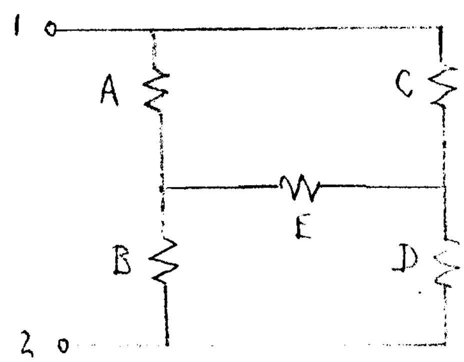
(5) $$R = A + B : (C + D : E).$$

To give a network interpretation of such a polynomial read A + B as "A series B" and A : B as "A parallel B" then it is clear that the expression (5) is the joint resistance of the network shown in Figure 5. Networks obtained from such polynomials are termed <u>series-parallel</u> <u>connections</u>.



A series-parallel connection
Figure 5

Not every network is a series-parallel connection. In particular it can be checked that the Wheatstone bridge connection of Fig. 6 is not a series-parallel connection. In fact it follows from an analysis given in reference [12] that a network is a series-parallel connection if and only if there is no embedded network having the Wheatstone bridge configuration. Another simple characterization of series-parallel connection has been given by Riordan and Shannon [23].



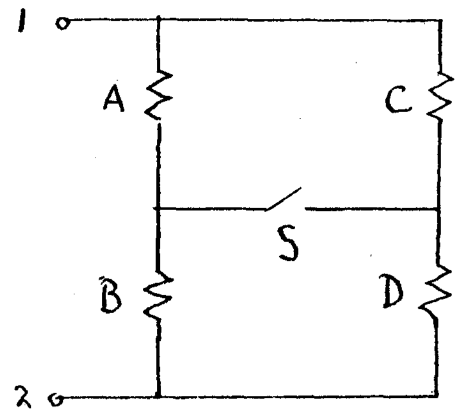
The Wheatstone bridge connection
Figure 6

According to a principle stated by Rayleigh [22] the current flow through a network may be described as taking the paths of least resistance. Alfred Lehman [18] used Rayleigh's principle to derive an interesting inequality termed <u>the series-parallel inequality</u>. He considered a network such as shown in Fig. 7. Then the joint resistance when the switch S is open is

$$R_{oo} = (A + B) : (C + D).$$

On the other hand, when the switch S is closed the joint resistance is

$$R_o = A : C + B : D.$$



Lehman's connection
Figure 7

But the current takes the paths of least resistance and there is less constraint with the switch closed so

$$A : C + B : D \leq (A + B) : (C + D). \tag{6}$$

This is the series-parallel inequality and in ordinary algebra it is expressed as

$$\frac{AC}{A + C} + \frac{BD}{B + D} \leq \frac{(A + B)(C + D)}{A + B + C + D}.$$

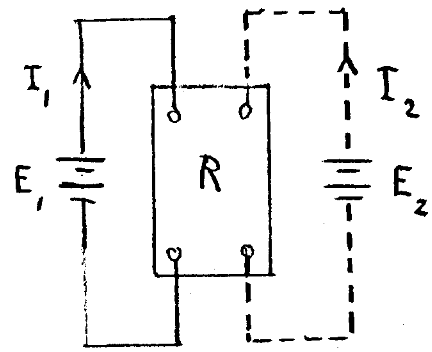
It is worth noting that Lehman's connection corresponds to replacing the resistor E in the Wheatstone bridge convection with a switch. Let R_E denote the joint resistance of Wheatstone's bridge. Then the following is a generalization of Lehman's inequality

$$R_O \leq R_E \leq R_\infty . \tag{7}$$

This also is a consequence of Rayleigh's principle. The inequality on the right side of (7) is obtained by putting the resistor E in series with the switch. The inequality on the left is obtained by putting the resistor E in parallel with the switch.

3. The parallel addition of matrices.

The various relationships just described become more interesting and suggestive when the scalar formulation of Ohm's law is replaced by a vector formulation. For example, Fig. 8 depicts a resistor box with two pairs of terminals. The first pair is in circuit 1, denoted by a solid line, and the second pair is in circuit 2, denoted by a dashed line. Then the currents and voltages in these circuits satisfy equations of the form



A resistor box
Figure 8

$$\text{(8a)} \qquad \begin{aligned} E_1 &= R_{11}I_1 + R_{12}I_2 \\ E_2 &= R_{21}I_1 + R_{22}I_2 \end{aligned}$$

In vector form, these equations can be written as

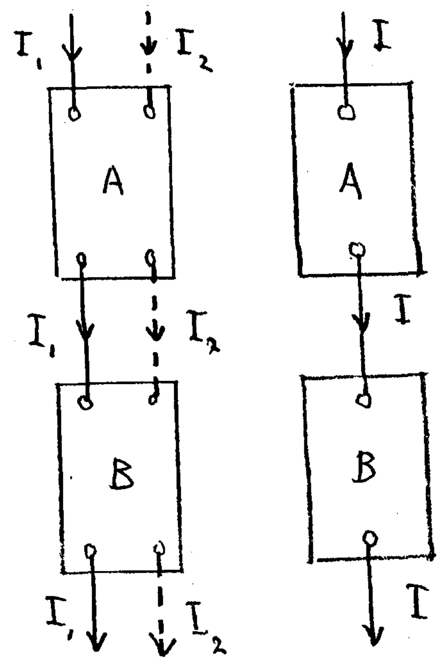
$$\text{(8b)} \qquad \underset{\sim}{E} = R\underset{\sim}{I}.$$

If the box just contains interconnected resistors, then it results that R is a symmetric matrix. Moreover, by the conservation of energy it follows that R is positive semidefinite. Therefore, in what follows, an arbitrary symmetric positive semidefinite matrix R shall be termed a <u>resistance matrix</u>. For an appropriate generalization where R is not symmetric, see [13].

Resistance boxes may be added in series as is shown on the left side of Fig. 9. If A and B are the resistance matrices of the two boxes, then the joint resistance matrix R is given by

$$R = A + B. \tag{9}$$

In other words series connection of resistance boxes corresponds to addition of their matrices. (This assumes that the current I_1 in the first circuit of box A is the same as the current in the first circuit of box B. This can be achieved by use of isolation transformers.) The right side of Fig.9 gives an abbreviated symbolism for the series addition of resistor boxes.



Series addition of resistor boxes

Figure 9

Note that any current vector $\underset{\sim}{I}$ can be the input to a resistor box. However, not every voltage vector $\underset{\sim}{E}$ can be an output if R is a singular matrix. In any case, the following well-known theorem relates the range spaces of semi-definite matrices.

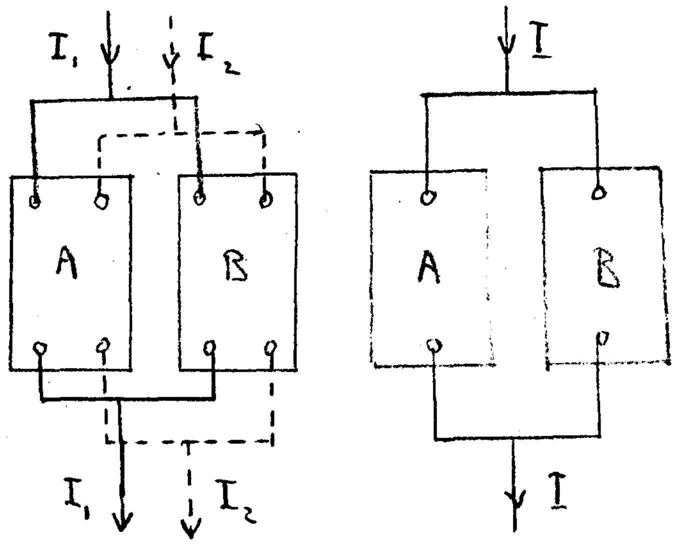
(10) $$\text{Range}(A + B) = \text{Range } A + \text{Range } B.$$

It is equally possible to connect the resistor boxes in parallel as shown on the left side of Fig. 10. The right side gives the symbolic diagram. First, suppose A and B are non-singular then

$$R^{-1} = A^{-1} + B^{-1}$$

where R is the joint resistance matrix of the parallel connection. Solving for R gives

(11) $$R = A(A + B)^{-1}B.$$



Parallel connection of resistor boxes

Figure 10

Again it is convenient to have a short notation for the operation on the right side of (11) so let

(12) $$A : B = A(A + B)^{-1}B$$

define <u>parallel</u> <u>addition</u> of matrices A and B. Various properties of this new operation were developed in a joint paper with William N. Anderson [2]. Some of these properties will now be described.

First, note that relation (10) shows that Range $(A + B) \supset$ Range B for semidefinite matrices. Hence $(A + B)^{-1}B$ is then well defined. This shows that the operation $A : B$ is defined for any pair of positive semidefinite matrices.

By virtue of the network model, we expect that the parallel sum is both commutative and associative. A direct proof of commutativity follows from the following manipulations:

$$A : B = (A + B - B)(A + B)^{-1}B = B - B(A + B)^{-1}B.$$
$$B : A = B(A + B)^{-1}(A + B - B) = B - B(A + B)^{-1}B.$$

This proves that

(13) $$A : B = B(A + B)^{-1}A.$$

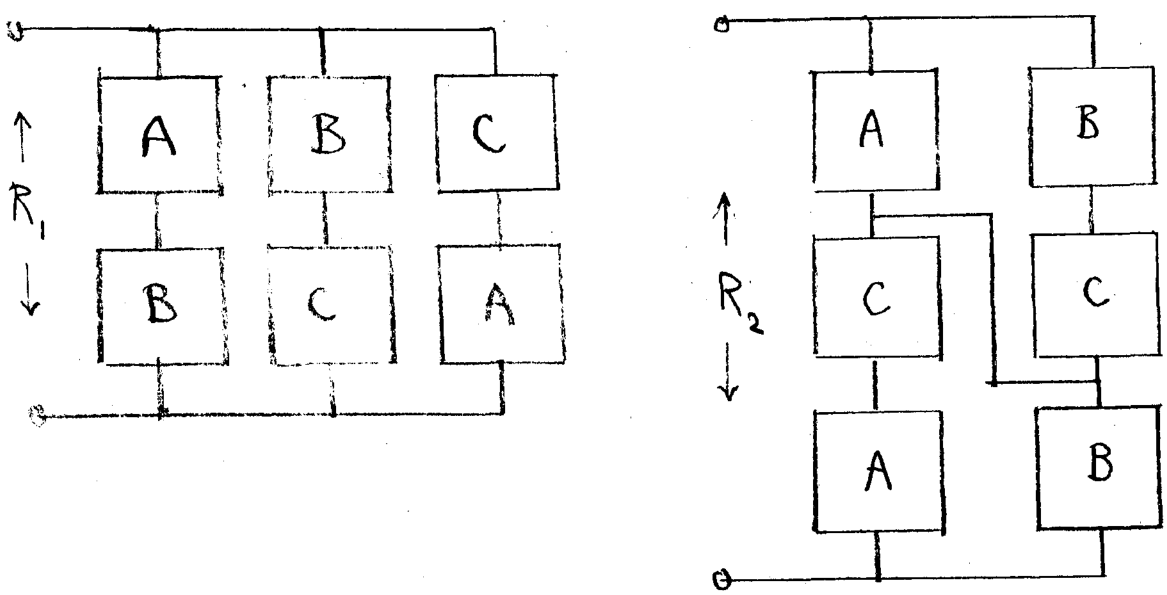
It is obvious from relation (12) that Range $(A : B) \subset$ Range A. Likewise, relation (13) shows that Range $(A : B) \subset$

Range B. Further analysis gives that actually

(14) $$\text{Range}(A : B) = \text{Range } A \cap \text{Range } B.$$

Relations (10) and (14) reveal a remarkable duality between series addition and parallel addition of matrices.

To give an application of the above duality principle the networks shown in Figure 11 are now analyzed.



Matrices with the same range

Figure 11

Clearly the joint resistance matrix of the first network is

$$R_1 = (A + B) : (B + C) : (C + A).$$

If α, β and γ are the range spaces of A, B, and C respectively it follows from (10) and (14) that

$$\text{Range } R_1 = (\alpha + \beta) \cap (\beta + \gamma) \cap (\gamma + \alpha).$$

On the other hand the joint resistance matrix of the second network is

$$R_2 = A : (B + C) + B : (A + C).$$

Therefore the range of this matrix is

$$\text{Range } R_2 = \alpha \cap (\beta + \gamma) + \beta \cap (\gamma + \alpha).$$

The subspaces of a vector space form a modular lattice. There are various identities which hold on a modular lattice. In particular

(15) $$(\alpha + \beta) \cap (\beta + \gamma) \cap (\gamma + \alpha) \equiv \alpha \cap (\beta + \gamma) + \beta \cap (\alpha + \gamma)$$

as is shown by Birkhoff [3]. This proves

(16) $$\text{Range } R_1 = \text{Range } R_2.$$

The reader will see that there are various analogous procedures for constructing networks with the same range.

The parallel sum operation is found to satisfy various inequalities. Thus the norm, trace, and determinant satisfy the following inequalities:

$$\|A : B\| \leq \|A\| : \|B\|, \tag{17}$$

$$\mathrm{tr}(A : B) \leq (\mathrm{tr}\, A) : (\mathrm{tr}\, B), \tag{18}$$

$$\det(A : B) \leq (\det A) : (\det B). \tag{19}$$

Here the notation (:) on the right side of these inequaliites denotes the scalar parallel operation. These inequalities give further manifestations of the duality between series and parallel addition.

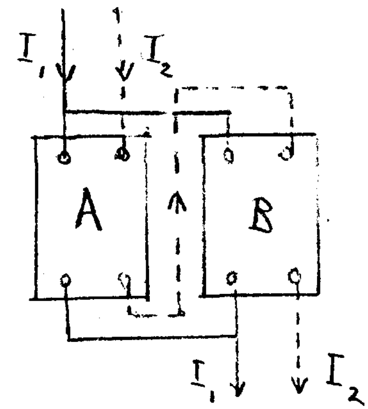
The network connections used by Lehman to obtain the series-parallel inequality can be extended to resistor boxes. It is then found that

$$A : C + B : D \leq (A + B) : (C + D) \tag{20}$$

for positive semi-definite matrices A, B, C and D. Here $A \leq B$ means that $B - A$ is positive semidefinite.

The scalar inequality (7) refers to the Wheatstone bridge and is a generalization of the series-parallel inequality. Presumably (7) can be extended to matrices; however, this is an open question.

Another type of connection of resistor boxes is termed the hybrid connection. In the hybrid connection some circuits are put in series and some circuits are put in parallel. Such a connection is shown in Fig. 11b. By use of the hybrid connection an elegant solution of the network synthesis problem was found [25].



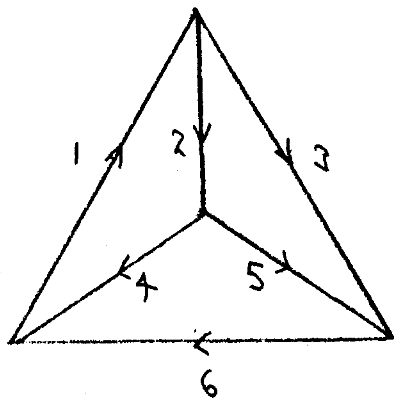
Hybrid connection of resistor boxes

Figure 11b

The joint resistance matrix R of the hybrid connection may be termed <u>hybrid addition</u> of matrices A and B. Some recent studies by George Trapp and the writer have revealed several properties of hybrid addition. In particular, the series-parallel inequality (20) is valid if A : B now denotes hybrid addition of A and B.

4. The Bott-Duffin duality analysis.

It has long been known that many theorems about electrical networks have companion theorems obtained by interchanging current and voltage variables and replacing resistance by conductance. A theory of this electrical duality was developed in collaboration with Raoul Bott [4]. To explain this approach it suffices to consider a simple directed network such as shown in Fig. 12. This network has six branches and so let $I_1, I_2, \ldots, I_6$ be the currents flowing in the directed branches. Then Kirchhoff's current law states that the total current entering a node vanishes. For example at the node where branches (1,2,3) meet



A directed network
Figure 12

$$I_1 - I_2 - I_3 = 0 \quad \text{etc.} \tag{21}$$

Let $V_1, V_2, \ldots, V_6$ be voltage drops across the branches. Kirchhoff's voltage low states that the voltage drops around any circuit vanishes. For example branches (1,2,4) form a circuit so

$$V_1 + V_2 + V_4 = 0 \quad \text{etc.} \tag{22}$$

It was observed by Herman Weyl [24] that the two laws of Kirchhoff cause current and voltage to be orthogonal i.e.

$$\sum_{j=1}^{6} I_j V_j = 0. \tag{23}$$

This holds for any current flow I satisfying Kirchhoff's first law and any voltage V satisfying Kirchhoff's second law. For an application of (23) see [8].

Weyl's theorem (23) was the inspiration for our general duality analysis. But we changed (23) from a theorem to a postulate. Thus consider a six-dimensional Euclidean space E. Let K be the subspace of E corresponding to vectors $V = (V_1, V_2, \ldots, V_6)$ which satisfy Kirchhoff's voltage law. Then let P be the perpendicular projection operator from E into K. Thus P is a symmetric matrix such that $P^2 = P$. Let $P' = I - P$ then P' is also a perpendicular projection matrix and $PP' = 0$. Moreover P' projects E into K' the orthogonal complementary subspace of K. By Weyl's theorem it follows that K' is the space corresponding to vectors satisfying Kirchhoff's current law.

To relate the perpendicular projection matrix P to electrical properites let G be a diagonal matrix whose diagonal elements $g_1, g_2, \ldots, g_6$ are positive numbers giving the conductances of the six branches. Then the discriminant D is defined as

$$D = \det(GP + P'). \tag{24a}$$

Thus D is a multilinear form in $g_1, \ldots, g_6$

$$D = g_1 g_2 g_3 + \ldots + g_2 g_4 g_5. \tag{24b}$$

Each term of D is a product of branches which form a tree of the network.

The transfer matrix T is defined as

$$T = P(GP + P')^{-1}. \tag{25}$$

This matrix has the following physical significance. Suppose that a current source of strength I is inserted in branch 6. Then the voltage across branch 2 is $V_2 = T_{26} I$. In fact all electrical properties of the network are given by the transfer matrix T.

Electrical duality comes about by defining G' to be a diagonal matrix whose diagonal elements $g_1', g_2', \ldots, g_6'$ are the resistances of the branches. Then the dual discriminant is defined as

$$D' = \det(G'P' + P), \tag{26a}$$

Thus D' has the form

(26b) $$D' = g_4' g_5' g_6' + \ldots + g_1' g_3' g_6'.$$

Each term of D' is a product of branches whose complement is a tree. The <u>dual transfer matrix</u> is

(27) $$T' = P'(G'P' + P)^{-1}.$$

This matrix has the following physical signficance. Suppose that a battery of voltage E is inserted in branch 6 then the current in branch 2 is $I_2 = T'_{26}D$.

We term the correspondence between primed and unprimed symbols <u>electrical duality</u>. Moreover, there are various relations between the primal and dual symbols. For example

(28) $$G'T' + TG = 1.$$

A beautiful algebraic structure develops from the above concepts. Moreover, two far reaching generalizations are feasible: (i) P can be chosen to be an arbitrary perpendicular projection matrix. (ii) G can be taken to be an arbitrary matrix. In this generalization T is termed the <u>constrained inverse</u> of G. The constrained inverse exists if and only if the discriminant $D \neq 0$. These generalizations also have elctrical and mechanical interpretations.

5. <u>How to use Wang algebra</u>.

K. T. Wang managed an electrical power plant and in his spare time he sought simple rules for solving the network equations. These rules appear to have a connection with the Bott-Duffin analysis. To make this connection Bott and I restated his rules as three postulates for an algebra:

(i) $\quad xy = yx.$

(ii) $\quad x + x = 0.$

(iii) $\quad xx = 0.$

Here x and y are arbitrary elements of the algebra [5,10].

To see how to apply the Wang algebra consider the network shown in Fig. 13. Let the branch conductances a,b,c,d and e be regarded as independent generators of the Wang algebra. A star element of the algebra is defined to be the sum of branches meeting at a node. Thus the star element at node 3 is $(a + b + c)$. To find the discriminant, the rule is to carry out the Wang product of all stars except one. Then omitting the star at node 2 gives

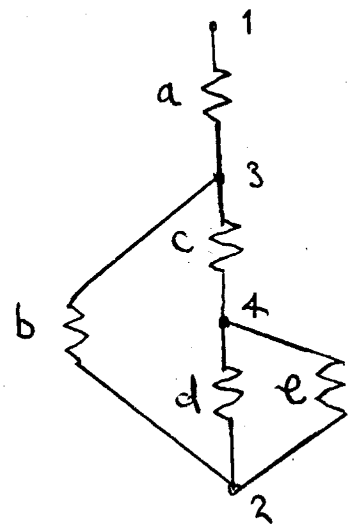


Figure 13

$$D = a(a + b + c)(d + e + c) = (ab + ac)(d + e + c).$$

Thus the discriminant of the network is

$$D = abd + abe + acd + ace + abc. \tag{29}$$

It can be readily checked that the terms of (29) give all the trees of the network of Fig. 13.

To find the joint resistance R between nodes 1 and 2 the rule is to write R as a fraction. The denominator of the fraction is the discriminant D and the numerator N is the product of all stars except those at 1 and 2. Thus

$$R = \frac{N}{D} = \frac{ad + ae + bd + be + cd + ce + ac + bc}{abd + abe + acd + ace + abc}. \tag{30}$$

There are also simple rules for calculating the transfer matrices T and T'.

The network shown in Fig. 13 is a series-parallel connection; in fact, it is the same connection as is shown in Fig. 5. Thus the joint resistance R could also be calculated by the series-parallel formula (5). However, the Wang rules apply even if the network is not of series parallel type.

A proof of the Wang rules was made by first observing that the Wang algebra is the Grassmann algebra when the coefficient field is the integers mod 2. However, the Grassmann algebra gave a more general system of calculation and which could be directly related to the Bott-Duffin analysis.

For example, let the symbols a, b, c and d now be regarded as independent vectors of a real vector space E_4. Then the vectors $(a + d)$, $(a + b + c)$, $(d + c - a)$ form a basis of a subspace S but they are not stars (or circuits) of any network. The Grassman algebra consists of the vectors of E_+ together with outer products formed by the law $xy = -yx$ for any two vectors x and y. Then the outer product π associated with S is

$$\begin{aligned}\pi &= (a + d)(a + b + c)(d + c - a)\\ &= abd + acd + abc + dac + dbc - dba - dca\\ &= 2abd + 3acd + abc + dbc.\end{aligned}$$

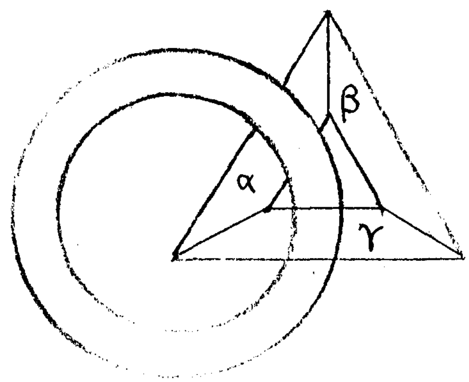
Then according to the Bott-Duffin rule the coefficients of the discriminant are the square of the coefficients of the outer product, thus

$$D = 4abd + 9acd + abc + dbc.$$

Now a, b, c and d denote real numbers.

6. <u>What is a regular matroid</u>?

There are ideal electrical networks which do not obey Kirchhoff's law. For example, Fig. 14 shows a double triangle network linked with a magnetic ring of zero magnetic resistance. Then by Ampere's law, no electric current may link the ring. This imposes a constraint in addition to Kirchhoff's current law. Nevertheless, the Bott-Duffin analysis applies without change. More remarkable is the fact that the Wang algebra also works. For example, the dual discriminant is given by the Wang product $D' = \alpha\beta\gamma$ where α, β and γ are the three square circuits indicated in Fig. 14.



A linked network
Figure 14

This raises the following question. What characterizes subspaces such that the discriminant can be calculated by Wang's short cut method? Let us term such subspaces <u>quasi-Kirchhoffian</u>. Then an answer to the question is given by the following statement [5,10].

<u>Theorem</u>: <u>Let</u> S <u>be an</u> m-<u>dimensional subspace of</u> n-<u>dimensional real vector space</u> E_n. <u>Let</u> C <u>be the set of those vectors of</u> S <u>whose components are</u> $+1$, -1 <u>or</u> 0. <u>Then</u> S <u>is quasi</u>-<u>Kirchhoffian if and only if</u> C <u>is an</u> m-<u>dimensional vector space under addition mod</u> 2.

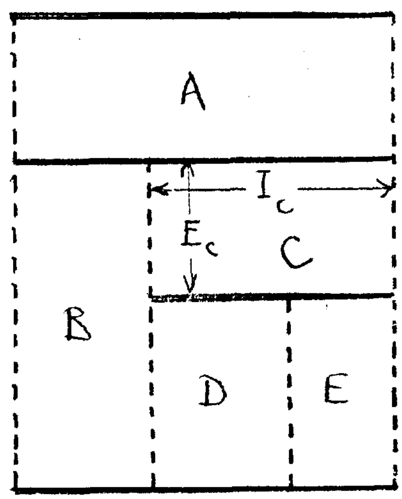
For example, if S is the subspace defined by Kirchhoff's current law then C is the set of flows of unit strength. In other words a vector of C is composed of one or more non-overlapping circuits.

If S is quasi-Kirchhoffian then C may be thought of as being composed of non-overlapping quasi-circuits. A matroid is a set of vectors which behave like circuits under addition mod 2. Thus C is a matroid. In fact the above theorem gives a necessary and sufficient characterization of <u>regular matroids</u>. For a postulational treatment of matroids reference is made to the elegant development given by Minty [20].

7. <u>Squaring the square</u>.

Is it possible to divide a square into squares no two of which are equal? This puzzle resisted attack for years but fell before a massive assault by Brooks, Smith, Stone and Tutte. Making use of an electric network model these authors developed an example in which a square was divided into 26 smaller squares [6]. The minimum number is not known.

The network they employed may be described as a lumped network equivalent to a distributed network. To understand this correspondence consider a situation when a rectangle is divided into smaller rectangles such as shown in Fig. 15. Suppose that the rectangle is constructed out of a thin conducting plate such that a unit square has a resistance of one ohm between opposite sides. Since resistance is proportional to length and inversely proportional to width it follows that a square of any size has resistance of one ohm. Thus if E is the height of the rectangle and I its width, then the resistance from top to bottom is $R = E/I$.



A rectangular network
Figure 15

The streamlines of current flow in the rectangle are vertical lines. Thus, the flow will not be changed if cuts are made along the vertical dashed lines separating the rectangles marked B, C, D, E. The equipotentials are horizontal lines. Thus the flow would not be changed if perfectly conducting bus bars are placed along the solid lines forming upper and lower sides of the rectangles. By virtue of these observations the plate may be regarded as a lumped network having lumped resistors of value A, B, C, D and E ohms. These resistances are determined by the dimensions. Thus if I_C is the width and E_C is the height of the rectangle marked C then $C = E_C/I_C$ etc.

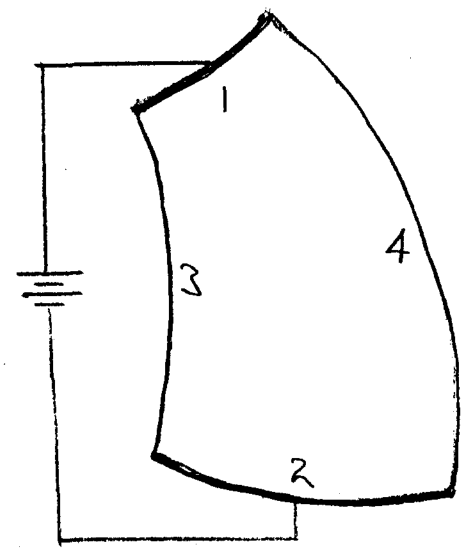
Clearly the lumped network is the same as the series parallel connection shown in Fig. 5. Thus the resistance of the plate is

$$R = A + B : (C + D : E) \tag{31}$$

because the cuts and bus bars do not change resistance. This example of a rectangular network suggests the following conjecture. <u>Every division of a rectangle into rectangles has an equivalent series-parallel connection and every series-parallel connection has an equivalent division of a rectangle into rectangles</u>.

8. Rayleigh's reciprocal relation.

Again consider a conducting plate having resistance of 1 ohm between opposite sides of a unit square. In Fig. 16 is shown a curvilinear quadrilateral plate with sides 1, 2, 3 and 4. The sides 3 and 4 are insulated but sides 1 and 2 are connected to perfectly conducting bus bars (denoted by heavy lines). Let R_{12} be the joint resistance between sides 1 and 2. Next let sides 1 and 2 be insulated and let bus bars be connected to sides 3 and 4. If R_{34} is the joint resistance in in this dual situation then



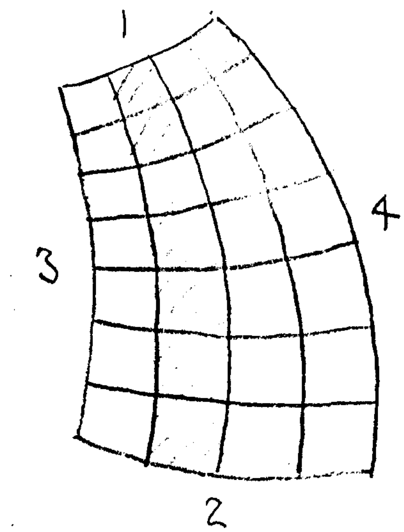
A curvilinear quadrilateral
Figure 16

$$R_{12}R_{34} = 1. \tag{32}$$

This is Rayleigh's reciprocal relation [21].

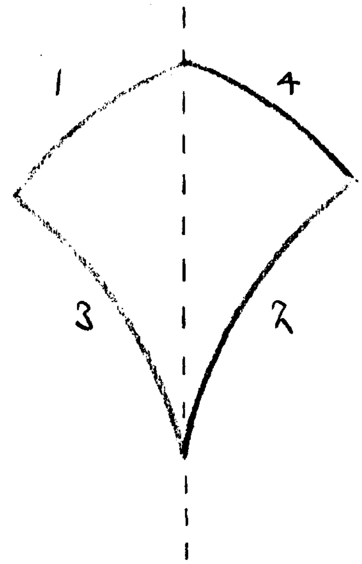
To prove (32) draw the equipotential lines and the streamlines. Since the potential $u(x,y)$ is harmonic the streamlines are the equipotential lines of the conjugate harmonic function

$v(x,y)$. It follows from the Cauchy-Riemann equations that the two sets of equipotentials are orthogonal and divide the region into curvilinear squares such as is shown in Fig. 17. This breaks the flow up into channels. One of these channels is denoted by cross hatching in Fig. 17. This channel is a series of 7 squares so the total resistance is 7 ohms. It is seen that there are 4 channels in parallel so this gives $R_{12} = 7/4$ ohms. Now consider the conjugate problem then Fig. 17 again applies, but the equipotentials and streamlines interchange roles. Thus a channel from side 3 to side 4 is a series of 4 squares and so the channel resistance is 4 ohms. There are 7 channels so the total resistance is $R_{34} = 4/7$ ohms. This is the proof of (32) given by Rayleigh.



Conjugate functions
Figure 17

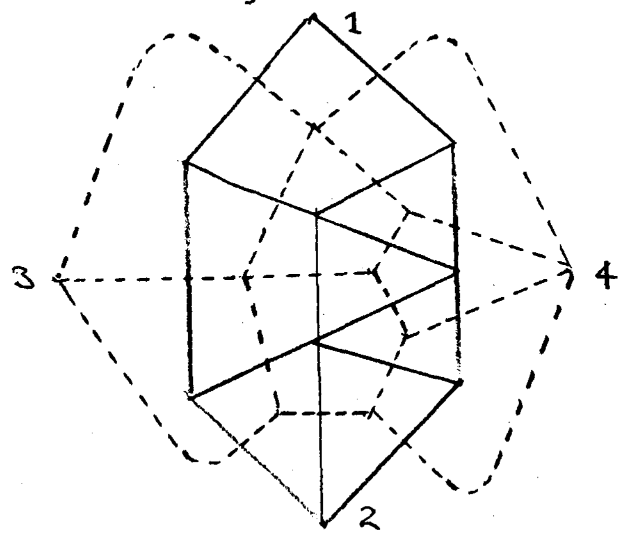
A surprising consequence of the Rayleigh relation arises when the quadrilateral region has bilateral symmetry as in



Self dual conductor
Figure 18

Fig. 18. By symmetry $R_{12} = R_{34}$. Hence by Rayleigh's reciprocal relation $R_{12} = 1$ ohm.

These considerations raise the question of a lumped network analog of Rayleigh's reciprocal relation [9,11]. To answer this question consider Fig. 19. A planar network is shown in solid lines with two distinguished nodes, 1 and 2. Another planar network is shown in dashed lines, and it has two distinguished nodes, 3 and 4. These networks are termed <u>dual</u> because of the following properties: (i) Crossing branches give a one-to-one correspondence between the networks. (ii) A region of one of the network contains one and only one of the nodes of the other network. (iii) The



Dual networks
Figure 19

distinguished nodes are on the boundary and are not in a region. (iv) If branches cross the resistances r and r^* are required to satisfy

$$rr^* = 1. \tag{33}$$

Under these hypotheses it follows that the reciprocal relation (32) holds for the joint resistance of the two networks.

A proof of (32) can be given by noting that Kirchhoff's current law for the primal network defines the same constraint as Kirchhoff's voltage law for the dual network. It follows immediately from the Bott-Duffin analysis that the resistance of the primal network is equal to the conductance of the dual network.

9. Upper bound and lower bound networks.

The potential $u(x,y)$ of a conducting plate satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

To solve potential problems on a computer Laplace's equation is approximated by the difference equation

$$(34) \quad u(x,y) = \frac{u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h)}{4}.$$

Physically this corresponds to replacing the conducting plate by a wire screen having square meshes of side h. Then equation (34) states that the potential at a node is the mean of the potentials at the four neighboring nodes. This is clearly a consequence of the laws of Ohm and Kirchhoff.

It is important to know the nature of the error in replacing the plate by the wire screen. As an approach to this problem consider the rectangular plate shown in Fig. 20. The plate is 4 cms. high and 3 cms. wide so the

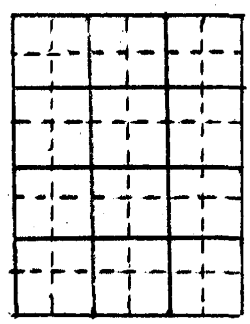
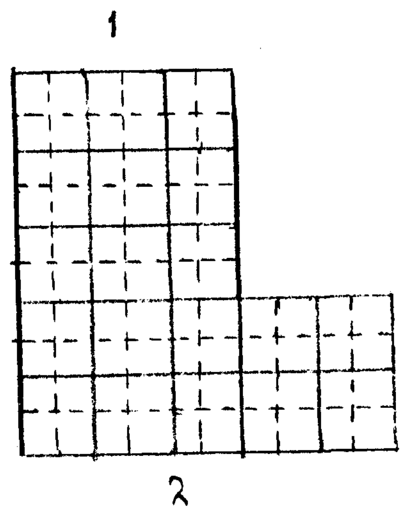


Plate and screen
Figure 20

resistance between the top edge and bottom edge is 4/3 ohms. This suggests that the screen wire should have a resistance of one ohm per centimeter. First suppose that the screen is placed as shown by the dashed lines. Then the horizontal wires carry no current but the current flows through three vertical wires, each having a resistance of 4. The total resistance is 4/3 ohms so there is no error in using the dashed network. Next consider the network indicated by solid lines. Now there are 4 vertical wires and this poses a problem. However, if it is supposed that the wires on the boundary have resistance of 2 ohms per cm. then the correct joint resistance of 4/3 ohms is again obtained.

Next consider an arbitrary region made up of squares. Such a plate is shown in Fig. 21. Let R be the joint resistance from edge 1 to edge 2 when the other edges are insulated. Two screen networks are shown in Fig. 21. The resistance of the square sides are to be 1 ohm



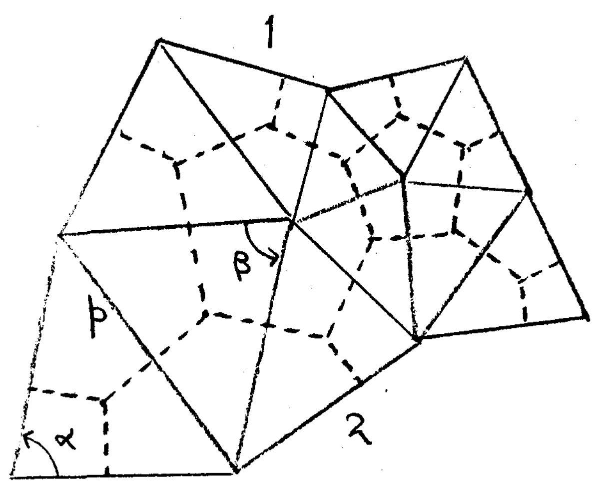
An L shaped plate
Figure 21

inside and 2 ohms on the boundary. We term the solid lines the upper network. Let R^u be the joint resistance of the upper network between edges 1 and 2. We term the dashed lines the lower network. Let R^L be the joint resistance of the lower network. Then the following inequality maintains

$$R^u \leq R \leq R^L. \tag{35}$$

Thus the upper network furnishes an upper bound to the conductance and the lower network furnishes a lower bound to conductance [9].

The proof of (35) is obtained as a special case of a more general theorem in which a polygonal region is triangulated in an arbitrary rashion. Thus consider the polygonal plate shown in Fig. 22. The triangulation of this polygon is denoted by solid lines. The solid lines are termed the upper network. An interior line p of this network is given resistance



A polygonal plate
Figure 22

(36) $$r = \frac{2}{\cot\alpha + \cot\beta}$$

where α and β are the angles opposite the line p as shown in Fig. 22. If p is a boundary line then the term $\cot\alpha$ would be omitted in this formula.

Let $u(x,y)$ be the potential function for the plate problem. Let $w(x,y)$ be a function which is linear in each triangle but which is continuous over the plate region. Then by Dirichlet's principle

(36) $$\iint \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 dx\ dy \le \iint \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 dx\ dy$$

provided $u = w$ on both boundary segment 1 and boundary segment 2. Then by making certain transformations, this inequality is found to be equivalent to

(37) $$1/R_{12} \le 1/R^u_{12}$$

where R_{12} is the joint resistance of the plate and where R^u_{12} is the joint resistance of the upper network. Thus the upper network furnishes an upper bound to conductance.

To obtain a lower bound for conductance we may use Rayleigh's reciprocal relation $R_{12} = 1/R_{34}$. Here 3 and 4 denote boundary segments complementary to segments 1 and 2. But by formula (37) it follows that $1/R_{34} \le 1/R^u_{34}$. Thus

(38) $$R_{12}^{u} \leq R_{12} \leq 1/R_{34}^{u}$$

gives upper and lower bounds to R_{12}.

The lower network is defined to be the dual of the upper network. The lower network is shown by dashed lines in Fig. 22. The resistance of a dashed branch is the reciprocal of the resistance of a branch it crosses. It follows that the joint resistance R_{12}^{L} of the lower network satisfies the reciprocal relation $R_{12}^{L} = 1/R_{34}^{u}$. Substituting this in (38) gives

(39) $$R_{12}^{u} \leq R_{12} \leq R_{12}^{L}.$$

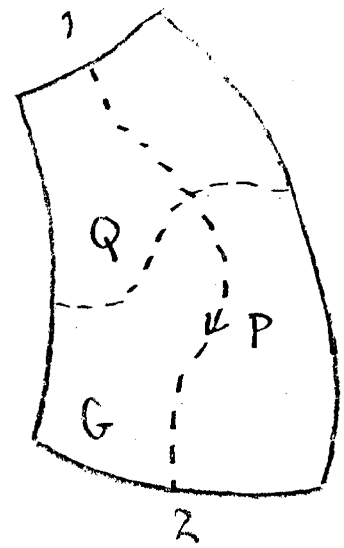
This is the desired bounding relation. For further generalizations see [14] and [16].

If a region is covered by a square lattice then it can be triangulated by inserting one diagonal in each square. It then results that (39) gives the bounding relation (35) stated for square lattices.

10. The extremal length of a network.

Ahlfors and Beurling have introduced the concept of extremel length of a curvilinear quadrilateral [1]. The relation of this geometric concept to complex function theory has been developed at extreme length in the literature so we shall only give the definition. Consider the curvilinear quadrilateral shown in Fig. 23. Then the extremal length is denoted by EL and is defined as

$$\text{(40)} \qquad EL = \sup_{w} \inf_{P} \frac{\left(\int_P w \, dx\right)^2}{\iint_G w^2 dx\, dy}.$$



A region G
Figure 23

Here P is any path from side 1 to side 2 and w(x,y) is any continuous non-negative function defined over the region G.

To give an electrical interpretation of extremal length we imagine the quadrilateral G is a conducting plate of unit resistivity. Let R be the joint resistance between sides 1 and 2. Then $E = IR$ where E is the battery voltage and I is the current flow. The power input is $EI =$

$= E^2/R = I^2R$. On the other hand if $w(x,y)$ is the strength of the current density, the power dissipated in heat is w^2 per square centimeter. Thus by the conservation of energy

(41) $$E^2/R = \iint_G w^2 dx\, dy.$$

If P is a streamline from 1 to 2

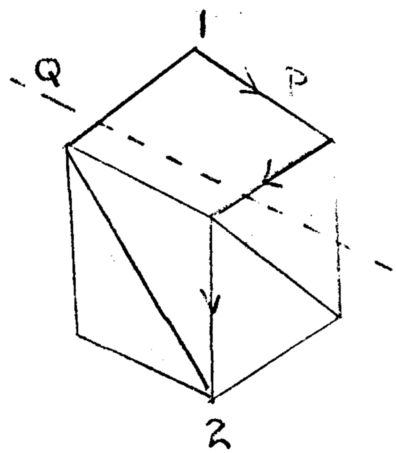
(42) $$E = \int_P w\, ds.$$

Combining (41) and (42) gives

(43) $$R = \frac{(\int_P w\, ds)^2}{\iint_G w^2 dx\, dy}.$$

It is then possible to show that this choice of w and P is optimal. Thus the extremal length is simply the joint resistance.

The concept of extremal length can be extended to networks in the following way [11]. Consider a network G with two distinguished nodes 1 and 2. A <u>path</u> P connecting node 1 and 2 is designated in Fig. 24 by arrows. Let r_j denote the resistance of branch j. Then the extremal length is defined to be



Paths and cuts
Figure 24

$$\text{(45)} \qquad EL = \max_{w} \min_{P} \frac{(\Sigma_P \, r_j w_j)^2}{\Sigma_G r_j w_j^2} \, .$$

Hence w_j is an arbitrary non-negative function defined on the branches. If w_j is actually the strength of the current in branch j then $\Sigma_G r_j w_j^2$ is the power dissipated in heat. If the path P follows the direction of current flow from 1 to 2 then $\Sigma r_j w_j = E$, the battery voltage. Thus by the conservation of energy, the joint resistance satisfies

$$\text{(46)} \qquad R = \frac{(\Sigma_P r_j w_j)^2}{\Sigma_G r_j w_j^2} \, .$$

Again it can be shown that this choice of P and w is optimal. Hence the extremal length of a network is equal to the joint resistance.

The network shown in Fig. 24 is planar, however, the definition holds for general networks. Moreover, the formulation of the network concept of extremal length suggests a related concept termed <u>extremal width</u>. The extremal width is denoted by EW and is defined as

$$\text{(47)} \qquad EW = \max_{w} \min_{Q} \frac{(\Sigma_Q w_j)^2}{\Sigma_G r_j w_D^2} \, .$$

Here Q denotes a cut and is defined as a set of branches which separate node 1 from node 2. A cut is indicated in Fig. 24. If w_j is actually the current strength in branch j then $\min \Sigma_Q w_j = I$ the total current. Also $\Sigma_G r_j w_j^2 = I^2 R$ hence

$$\text{(48)} \qquad \frac{1}{R} = \frac{(\Sigma_Q w_j)^2}{\Sigma_G r_j w_j^2} .$$

This is actually the optimal solution. Thus the following identity holds

$$\text{(49)} \qquad (EL)(EW) = 1.$$

It is worth noting that the definition of joint resistance by means of extremal length makes no explicit appeal to either of Kirchhoff's laws.

If in relation (49) the maximization operations are omitted, then an inequality results. Writing v_j for $r_j w_j$ gives the network inequality

$$\text{(50)} \qquad \Sigma_G v_j w_j \geq (\min_P \Sigma_P v_j)(\min_Q \Sigma_Q w_j) .$$

This is termed the <u>width-length inequality</u> [11]. It holds for arbitrary $w_j \geq 0$ and $v_j \geq 0$. Here v_j is arbitrary because r_j is arbitrary. A special case of (50) had previously been found by Shannon and a generalization to matroids has been given by Lehman [19].

The fact that the width-length inequality holds for arbitrary networks suggests that an analogous relation holds for conducting bodies. Thus consider a topological image G of the cylinder. For example Figure 23 can serve to illustrate this situation but G is now considered to be a solid body rather than a plate. Let the top surface of the "cylinder" be denoted by 1 and let the bottom surface be denoted by 2. Thus P denotes a path in G from 1 to 2. Likewise Q represents a surface cutting the body in two parts such that 1 and 2 are not in the same part. Then the following width-length inequality was conjectured in [11],

$$\iiint_G VW \, dx \, dy \, dz \geq \left(\inf_P \int_P V ds\right)\left(\inf_Q \iint_Q W dA\right) \tag{51}$$

where $V \geq 0$ and $W \geq 0$ are arbitrary continuous functions. Recently this inequality has been proved by W. R. Derrick [7].

In this paper we have brought to light analogies between lumped and distributed networks. In particular relations (50) and (51) give a nice illustration of an analogy between discrete and continuous systems. Another interesting analogy is given in reference [15] which treats a problem of optimum design from both the discrete and continuous point of view.

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