EQUIPARTITION OF ENERGY

IN WAVE MOTION

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Abstract

Of concern are solutions of the classical wave equation in three-dimensions. It is shown that if a solution has compact support then after a finite time the kinetic energy of the wave is constant and equals the potential energy. The proof employs the Paley-Wiener theorem of Fourier analysis.

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Equipartition of Energy in Wave Motion

1. Introduction.

In a recent paper [1] Jerome Goldstein gives an elegant analysis of abstract wave equations. The central result of his paper is an equipartition theorem stating that the difference of the kinetic energy and the potential energy vanishes as the time approaches infinity. Goldstein points out that the genesis of this equipartition property is simply the Riemann-Lebesgue lemma of Fourier analysis.

This observation of Goldstein suggests that other theorems of Fourier analysis should be connected with wave motion problems. To this end the present note develops a connection with the Paley-Wiener theorem. This theorem characterizes Fourier transforms of functions having compact support. Thus attention is here confined to a wave having compact support. Then it is shown to be a consequence of the Paley-Wiener theorem that after a finite amount of time has elapsed the kinetic energy is constant and equals the potential energy. This proof applies to waves in space of odd dimension.

After being informed of this result, Goldstein was able to show that the kinetic energy also becomes constant for certain abstract wave equations. His results are given in an accompanying paper [2].

2. The Paley-Wiener Theory.

The following is a well known property of the Fourier transform

Paley-Wiener Theorem. A function $F(z)$ *is a Fourier transform of the type*

$$F(z) = (2\pi)^{-\frac{1}{2}} \int_{-c}^{c} e^{izt} f(t)\,dt \ , \ c > 0$$

where $f(t)$ *is of class* $L_2(-c,c)$ *if and only if* $F(z)$ *is of class* $L_2(-\infty,\infty)$ *and is an entire function of exponential type* c. *That is*

$$|F(z)| = Oe^{|z|(c+\epsilon)} \quad \text{for all} \quad \epsilon > 0.$$

The following consequence of the Paley-Wiener theorem is needed.

Corollary. Let $H(z)$ *be an entire function of exponential type* c *and let* H *be absolutely integrable on the real axis. Then the Fourier transform of* H *is zero at points outside the interval* $[-c,c]$.

Proof. The function $K(z) = \int_0^z H(z)\,dz$ is clearly an entire function of exponential type c. Moreover on the real axis

$$|K(x)| \leq \int_{-\infty}^{\infty} |H(x)|dx$$

Then by a suitable application of the Phragmen-Lindelof theorem it follows that if $z = x+iy$

$$|K(z)| \leq Ae^{c|y|}$$

for a constant A. The Cauchy integral formula applied to a circle of unit radius and centered on the x-axis gives

$$|K'(x)| \leq Ae^{c}.$$

Thus $H(x) = K'(x)$ is uniformly bounded. Consequently if $H(x)$ is of class $L_1(-\infty,\infty)$ it also is of class $L_2(-\infty,\infty)$. Then the proof is completed by first using the Paley-Wiener theorem and then using the L_2 Fourier inversion theorem.

3. The classical wave equation.

Of concern is the wave equation

(1) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2} .$$

The kinetic energy K of a solution u is defined to be

(2) $$K(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_t{}^2 dxdydz = \|u_t\|^2$$

where $u_t = \partial u/\partial t$. The potential energy P is defined to be

(3) $$P(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (u_x{}^2 + u_y{}^2 + u_z{}^2)\, dxdydz.$$

Theorem: Let u(x,y,z,t) be a solution of the wave equation and be of compact support in space. Then the kinetic energy is constant and is equal to the potential energy for $t \geq b$ where b satisfies

(4) $$u(x,y,z,0) = u_t(x,y,z,0) = 0 \quad \text{for} \quad x^2 + y^2 + z^2 > b^2.$$

Proof: The Fourier transform of the function u is denoted by U and is defined as

(5) $$U(\xi,\eta,\zeta,t) = (2\pi)^{-3/2} \iiint_{-\infty}^{\infty} e^{i(\xi x+\eta y+\zeta z)} u(x,y,z,t)\, dxdydz.$$

For short this will be denoted as $U = Tu$. We assume, of course, that u has continuous derivatives of the second order. Then taking the Fourier transform of this wave equation we find that

(6) $$(\xi^2+\eta^2+\zeta^2)U(\xi,\eta,\zeta,t) = \partial^2 U(\xi,\eta,\zeta,t)/\partial t^2.$$

This step is readily justified by the smoothness and the compact support assumed. Solving (6) gives

(7) $$U(\xi,\eta,\zeta,t) = F_1(\xi,\eta,\zeta)\cos\rho t + F_2(\xi,\eta,\zeta)\frac{\sin\rho t}{\rho}$$

where $\rho = (\xi^2+\eta^2+\zeta^2)^{1/2}$. Here $F = Tf$ where

(8) $$f_1(x,y,z) = u(x,y,z,0), \quad f_2(x,y,z) = u_t(x,y,z,0).$$

Since $U_t = Tu_t$ the Parseval theorem gives

(9) $$\|U_t\|^2 = \|u_t\|^2.$$

Thus the kinetic energy can be expressed in the form

(10) $$K = \|\rho F_1\sin\rho t - F_2\cos\rho t\|^2.$$

Carrying out the square operation and making use of the identity $2\sin\rho t\cos\rho t = \sin 2\rho t$ gives

(11) $$2K = \|\rho F_1\|^2 + \|F_2\|^2 + \iiint_{-\infty}^{\infty}[(\rho^2|F_1|^2 - |F_2|^2)\cos 2\rho t - \rho(F_1F_2^* + F_1^*F_2)\sin 2\rho t]\,d\xi d\eta d\zeta.$$

It is now to be shown the integral terms in (11) vanish for $t \geq b$ introduce spherical coordinates so

(12) $$\xi = \rho\sin\theta\cos\varphi, \quad \eta = \rho\sin\theta\sin\varphi, \quad \zeta = \rho\cos\theta,$$

$$(13)\qquad F(\rho,\Theta,\varphi) = (2\pi)^{-3/2}\iiint e^{i\rho r\cos(\rho,r)} f(x,y,z,0)\,dxdydz.$$

Here $r^2 = x^2+y^2+z^2$ and $\cos(\rho,r) = (\xi x+\eta y+\zeta z)/\rho r$. Since f has compact support we may assume $r \leq b$. Thus if Θ and φ are fixed (13) converges for all values of ρ regarded as a complex variable. Moreover it is seen that F is an entire function of ρ of exponential type b. In other words

$$(14)\qquad |F| \leq Ae^{b|\rho|}$$ where

$$(15)\qquad A = (2\pi)^{-3/2}\iiint |f(x,y,z)|\,dxdydz.$$

Also note (12) and (13) yield the identity

$$(16)\qquad F(-\rho,\Theta,\varphi) \equiv F(\rho,\pi-\Theta,\varphi+\pi).$$

One of the integrals in (11) is

$$(17)\qquad I_2(t) = \iiint_{-\infty}^{\infty} \cos(2\rho t)\,|F_2|^2 d\xi d\eta d\zeta.$$

It follows by the hypotheses that this integral is absolutely convergent.

Define the function G as

$$(18)\qquad G_2(\rho,\varphi,\Theta) = F_2(\rho,\Theta,\varphi)\,F_2^*(\rho,\Theta,\varphi).$$

Since both F_2 and F_2^* are entire functions of ρ it follows that G is an entire function and

$$(19)\qquad |G_2(\rho,\varphi,\Theta)| \leq A^2 e^{2b|\rho|}.$$

Now express the integral (17) as an iterated integral in spherical coordinates. Thus

(20) $$I_2(t) = \int_o^{\infty} \cos(2\rho t) H_2(\rho)\, d\rho \qquad \text{where}$$

(21) $$H_2(\rho) = \rho^2 \int_o^{2\pi} d\varphi \int_o^{\pi} G_2(\rho,\varphi,\Theta) \sin\Theta d\Theta .$$

It follows that H_2 is an entire function of ρ and by (19)

(22) $$|H_2(\rho)| \leq 8\pi A^2 \rho^2 e^{2b|\rho|} .$$

Thus H_2 is an entire function of exponential type $2b$.

From (16) and (18) it follows that

(23) $$G_2(-\rho,\Theta,\varphi) = G_2(\rho,\pi-\Theta,\varphi+\pi) .$$

Hence

(24) $$H_2(-\rho) = \rho^2 \int_o^{2\pi} d\varphi \int_o^{\pi} G_2(-\rho,\pi-\Theta,\varphi+\pi) \sin\Theta d\Theta .$$

Let $\Theta' = \pi - \Theta$ and $\phi' = \phi + \pi$ in this integral and it is seen that $H_2(-\rho) = H_2(\rho)$ so H_2 is an even function of ρ.

The function F_2 is of class L_2 in space so G_2 is of class L_1. Consequently the function H_2 is of class L_1 on the real axis. But H_2 is of exponential type $2b$ so according to the corollary of the Paley-Wiener theorem the Fourier transform of H_2 vanishes outside the interval $[-2b, 2b]$. Since $H_2(\rho)$ is an even function the Fourier transform may be written as a cosine transform as in (20).

This shows that $I_2(t) = 0$ for $t > b$. By continuity $I_2(b) = 0$ also.

The integral

(25) $$I_1(t) = \iiint_{-\infty}^{\infty} \rho^2 |F_1|^2 \cos(2\rho t)\, d\xi d\eta d\zeta$$

is analyzed by the same argument. The extra factor ρ^2 does change the parity or the exponential type so $I_1(t) = 0$ for $t \geq b$ also.

The integral

(26) $$I_3(t) = \iiint_{-\infty}^{\infty} \rho(F_1^* F_2 + F_1 F_2^*) \sin(2\rho t)\, d\xi d\eta d\zeta$$

is analyzed by a similar argument. It may be written in iterated form as

(27) $$I_3(t) = \int_0^{\infty} \sin(2\rho t) H_3(\rho)\, d\rho .$$

Here $H_3(\rho)$ is of odd parity because of the factor ρ in (26). Thus the Fourier transform of $H_3(\rho)$ over the whole real axis can be expressed in the form (27) and the Paley-Wiener theorem gives $I_3(t) = 0$ for $t \geq b$. Thus it has been shown that if $t \geq b$ then (11) becomes

(28) $$2K = \|\rho F_1\|^2 + \|F_2\|^2 .$$

Consequently the kinetic energy is constant.

The total energy E is defined to be

$$E = K + P \tag{29}$$

Green's theorem is seen to give

$$\frac{dE}{2dt} = \int (u_t u_{tt} + \nabla u \cdot \nabla u_t)\, dv = \int u_t (u_{tt} - \Delta u)\, dv = 0. \tag{30}$$

Thus the total energy is constant for all time. Parseval's theorem applied to (2), (3), and (29) at time $t = 0$ gives

$$E = \|\rho F_1\|^2 + \|F_2\|^2. \tag{31}$$

Comparing (28) and (31) gives $K = E/2$. Thus $K = P$ and the proof of Theorem 1 is complete.

The same proof is seen to be valid for one-dimensional waves but not for two dimensional waves.

Theorem 2. The statement of Theorem 1 is false for two-dimensional wave motion.

Proof. Then the corresponding function $H_2(\rho)$ has odd parity. Thus I_2 is not the Fourier transform of an entire function unless I_2 vanishes identically. Similar statements apply to I_2 and I_3. Thus by the Paley-Wiener theorem the kinetic energy can be constant for $t \geq b$ only if it is constant for all t. But this is not true in general.

References.

[1] J. A. Goldstein, "An asymptotic property of solutions of wave equations," Proc. Amer. Math. Soc. 23 ____(1969).

[2] J. A. Goldstein, "An asymptotic property of solutions of wave equations.II," Jour. Math. Analysis and Applications ______ (1970).