PROPERTIES OF STRESS FUNCTIONALS FOR MATERIALS WITH ELASTIC RANGE, WITH APPLICATIONS TO CLASSICAL THEORIES OF ELASTIC-PLASTIC MATERIALS

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Contents

Correction to Section 9: The analysis and conclusions given in Section 9 for rate-independent materials are in error and should be replaced by the following discussion. (The underlined sentence on page 1.2 should accordingly be deleted.)

If π is rate-independent, then $\dot{\pi}(f)$ is given by

$$\dot{\pi}(f) = \gamma\bar{\pi}(f)$$

where $\bar{\pi}$ is rate-independent. The expression for $\dot{\pi}(f)$ obtained for simple shear will be of this form if and only if there exists functions $\bar{\eta}$, $\bar{\phi}$ and $\bar{\nu}$ such that

$$\eta(\gamma^2, t-t_o) = \bar{\eta}(K^2)$$

$$\phi(\gamma^2, t-t_o) = (t-t_o)\bar{\phi}(K^2)$$

$$\nu(\gamma^2, t-t_o) = (t-t_o)\bar{\nu}(K^2)$$

where $K = \gamma(t-t_o)$. The expression for $\dot{\pi}(f)$ then becomes

$$\begin{aligned}\dot{\pi}(f) = {} & (\bar{\eta}(K^2) + \lambda_1 + 2\lambda_2)\ A_1 - \frac{1}{2}\ (t-t_o)\bar{\nu}(K^2)A_2 \\ & - \frac{1}{2}\ (t-t_o)\bar{\phi}(K^2)(1(\mathrm{tr}\ A_1^2) - 2A_1^2) \\ & + \frac{1}{6}\ (t-t_o)(\bar{\phi}(K^2)+\bar{\nu}(K^2))\ 1\ \mathrm{tr}\ A_1^2\ .\end{aligned}$$

The stress at time t is obtained by integrating this expression. One obtains:

$$\pi(f) = [(\lambda_1+2\lambda_2)\ K + \int_0^K \bar{\eta}(s^2)ds]\ (\underset{\sim}{e_1} \otimes \underset{\sim}{e_2} + \underset{\sim}{e_2} \otimes \underset{\sim}{e_1})$$

$$- \frac{1}{6} \int_0^{K^2} [\bar{\phi}(s) + \bar{\nu}(s)]ds\ \underset{\sim}{e_1} \otimes \underset{\sim}{e_1}$$

$$- \frac{1}{6} \int_0^{K^2} [\bar{\phi}(s) - 2\bar{\nu}(s)]ds\ \underset{\sim}{e_2} \otimes \underset{\sim}{e_2}$$

$$- \frac{1}{6} \int_0^{K^2} [-2\bar{\phi}(s) + \bar{\nu}(s)]ds\ \underset{\sim}{e_3} \otimes \underset{\sim}{e_3} + \pi_o\ .$$

Here, π_o is the Cauchy stress at time $t = t_o$ which necessarily is a hydrostatic pressure. Note that the functions $\bar{\eta}$, $\bar{\nu}$ and $\bar{\phi}$, which determine the flow rule also determine the dependence of the stresses upon the total amount of shear. The shear stress π_{12} is given by

$$\pi_{12}(f) = (\lambda_1+2\lambda_2)K + \int_0^K \bar{\eta}\ (s^2)ds$$

and the normal stress differences by

$$\pi_{11}(f) - \pi_{22}(f) = -\frac{1}{2} \int_0^{K^2} \bar{\nu}(s)ds$$

$$\pi_{11}(f) - \pi_{33}(f) = -\frac{1}{2} \int_0^{K^2} \bar{\phi}(s)ds\ .$$

1. Introduction.

The purpose of this paper is to indicate how certain features of the classical theories of elastic-plastic materials arise naturally in recent theories of materials with elastic range. The recent theories take the point of view that knowledge of the stress, work, or free energy as a functional of deformation (and possibly temperature) history should determine specific characteristics of elastic-plastic materials. For example, PIPKIN and RIVLIN [1] show that a condition on a derivative of the work functional determines whether or not the work done in closed cycles of deformation is positive, and this condition on the work functional is shown to imply convexity of the yield surface in a theory with infinitesimal deformations. In [2], I have shown that a derivative of the free energy functional measures the dissipation arising from inelastic deformations, and this derivative reduces to the plastic power when deformations are small.

In the present paper I show how two derivatives of the stress functional, namely, the global and local history gradients, determine, respectively, the permanent stretching tensor and, when "elastic deformations" are small, a decomposition of the total stretching tensor into permanent and recoverable parts. The permanent stretching tensor is given as a multiple of the global history gradient, and this relation is a general form of the "flow rules" given in classical theories. For the case of small " elastic deformations," the global history gradient is replaced by the local history gradient in the expression for the permanent stretching tensor. Moreover, in this case I derive the classical decomposition of the total stretching tensor (" strain-rates" relative to the present configuration) into a term linear in the stress rates and a second term which is the permanent stretching tensor. In the case of arbitrary elastic deformations, the flow rules take their simplest form when the permanent stretching entails no volume change. The case where volume changes associated with permanent deformation are included is treated without undue complications. It should be noted that, since rate effects may influence the stress during " loading" , i.e., when permanent stretching occurs, the theory given here has been constructed so as to include such influences.

An example of the use of the general form for the flow rule is given in Section 9 where I discuss a simple shearing motion of an isotropic material with elastic range. For the case of small elastic deformations, I show that <u>normal stress effects can occur if and only if the material is rate-dependent</u>. The case of arbitrary elastic deformations can be treated using the results in Section 7, but this case leads to a more involved analysis which I have chosen not to include here.

The flow rule for the permanent stretching tensor can be used to obtain results about the behavior of the permanent stretching tensor under superposed rotations and changes of reference configuration. In fact, if the stress-functional has certain transformation properties under such influences, then the permanent stretching tensor will have the same transformation properties. I give another result of this type which implies that <u>a permanent deformation history transforms as an ordinary deformation history</u> (under superposed rotations) <u>if and only if the "elastic response", i.e. the response relative to the permanent configuration, is an isotropic function</u>. Thus, the physical prejudice that permanent deformations are possible deformations of the material is intimately connected with the nature of the response of the material measured relative to a permanent configuration.

2. Histories, Functionals.

Let R denote the set of all real numbers, V a three-dimensional inner product space, $\underline{\mathcal{L}}$ the set of all (second order) tensors on V, and $\underline{\mathcal{L}}^+$ the elements of $\underline{\mathcal{L}}$ with positive determinant. If $\xi: R \to \underline{\mathcal{L}}^+$ and if $\underline{t} \in R$, the function $\xi^t: [0,\infty) \to \underline{\mathcal{L}}^+$ defined by

$$\xi^t(s) = \xi(t-s), \quad s \in [0,\infty)$$

is called the *history of* ξ *up to time* t. Henceforth, I use the term history to denote any function $\underline{f}: [0,\infty) \to \underline{\mathcal{L}}^+$. The values $\underline{f}(\underline{s})$ of such a function have a physical interpretation as values of the deformation gradient (with respect to a fixed reference configuration) for a material point in a motion of some body.

I consider only those histories which are continuous and piecewise continuously differentiable. A superposed dot above a history denotes differentiation with respect to $\underline{s}$, and it follows that

$$\dot{\xi}^t(s) = -\frac{d}{d\tau}\,\xi(\tau)\big|_{\tau=t-s} \; .$$

If f is a history and if $\sigma \in [0,\infty)$, the *σ-section of* $\underline{f}$ is the history $\underline{f}_\sigma$ defined by

$$f_\sigma(s) = f(s+\sigma), \quad s \in [0,\infty).$$

A history $\underline{g}$ is said to be a *continuation* of a history $\underline{f}$ if there exists $\sigma \in [0,\infty)$ such that $\underline{g}_\sigma = \underline{f}$. In addition, if $\underline{a} \in \underline{\mathcal{L}}^+$ and if $\underline{g}(0) = \underline{a}$, then $\underline{g}$ is said to be a *continuation of* $\underline{f}$ *to* $\underline{a}$.

A functional π is a mapping from the collection of all histories into some finite dimensional vector space W. In this paper, the vector space W will be the space $\underline{\mathcal{L}}$ introduced above. In this case the value $\pi(f)$ represents the Cauchy stress corresponding to the history $\underline{f}$. If $\mathcal{S}$ is some non-empty collection of histories, π is said to be **path-independent** on $\mathcal{S}$ if $\pi(\underline{g}) = \pi(\underline{h})$ whenever $\underline{g}(0) = \underline{h}(0)$ and $\underline{g}$ and $\underline{h}$ are elements of $\mathcal{S}$.

3. Elastic Range and Related Concepts

Let $\underset{\sim}{E}$ denote a subset of $\underline{\ell}^+$. If $\underline{f}$ is any history the symbol $C(\underline{f},\underset{\sim}{E})$ denotes all continuations $\underline{g}$ of $\underline{f}$ such that for some $\sigma>0$ $\underline{g}_\sigma = \underline{f}$, and $\underline{g}(\underline{s}) \in \underset{\sim}{E}$ for every $\underline{s} \in [0,\sigma)$. It may happen that a functional π is such that given any history $\underline{f}$, there is a set $\underset{\sim}{E}(\underline{f}) \subset \underline{\ell}^+$ such that π is path-independent on $C(\underline{f},\underset{\sim}{E}(\underline{f}))$. PIPKIN and RIVLIN [1] and OWEN [2] have used such functionals to discuss the mechanics and thermodynamics of a class of materials which includes elastic-plastic materials. The purpose of this section is to present a precise definition of the term elastic range using the ideas presented in the papers [1], [2].

Let π be a functional and $\underline{f}$ a given history.

Definition. *A non-empty set $\underset{\sim}{E} \subset \ell^+$ is said to be an elastic region of π corresponding to f if the following conditions are satisfied:*

E1. *$\underset{\sim}{E}$ is an open, connected set;*

E2. *For some $\sigma \in (0,\infty)$, the set $\{f(s)\,|\,s \in (0,\sigma)\}$ is a subset of $\underset{\sim}{E}$;*

E3. *π is path-independent on $C(f,\underset{\sim}{E})$;*

E4. *Let $\pi^*(\cdot,\underline{f})$: $\underset{\sim}{E} \to \ell$ be defined by*

$$\pi^*(a,f) = \pi(g),\ a \in \underset{\sim}{E}$$

where g is any continuation of f to a such that $g \in C(f,\underset{\sim}{E})$. With $\pi^*(\cdot,f)$ so defined, then

$$\text{a) } \pi^*(\cdot,f) \in C^1(\underset{\sim}{E})$$

and b) *the limit*

$$D\pi(f) = \lim_{a \to f(0)} \nabla\pi^*(a,f)$$

exists and satisfies

$$\pi^*(a,f) - \pi(f) = D\pi(f)(a-f(0)) + o|a-f(0)|$$

for $a \in \underset{\sim}{E}$.

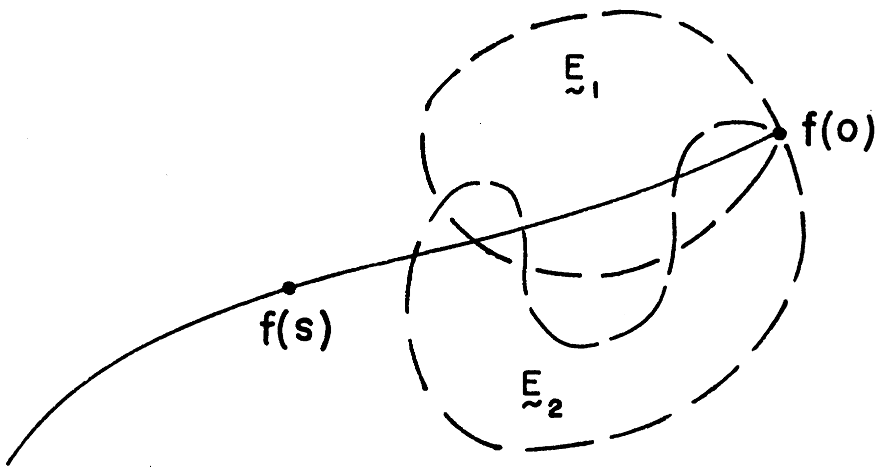

FIGURE 1

Figure 1 shows two elastic regions of π corresponding to f. It cannot be expected that the union of two such regions will be an elastic region corresponding to f. In order to be able to consider a distinquished elastic region, an additional assumption is needed: there exists an elastic region $\underset{\sim}{E}(f)$ of π corresponding to f which contains all other elastic regions of π corresponding to f. $\underset{\sim}{E}(f)$ is called the elastic range of π corresponding to f.

The requirement that $\underset{\sim}{E}(\underset{\sim}{f})$ be open and connected is made for the sake of mathematical clarity and does not severly restrict the class of sets to be considered. The condition E2 is essential in what follows; specifically, it insures that certain operators on the functional π are well defined. This condition specifies that recently encountered points $\underline{f}(\underline{s})$ be accessible from the present one, $\underline{f}(0)$, through continuations of f which remain in $\underset{\sim}{E}(\underset{\sim}{f})$. (In classical plasticity, this condition would imply that, during loading, a reversal in the direction of the strain-rate would cause the material to unload. If the yield surface is smooth and does not shrink during loading, then this condition is satisfied in the classical theories.) Condition E3 is simply the condition of path-independence mentioned above. E4 is a smoothness assumption on π, restricted to the set $C(\underline{f},\underset{\sim}{E}(\underline{f}))$. The functions $D\pi(\underline{f})$ and $\nabla\pi^*(\underline{a},\underline{f})$ are linear functions from $\underline{\mathcal{L}}$ into $\underline{\mathcal{L}}$. If $\underline{f}(0) \in \underset{\sim}{E}(\underline{f})$, then $\nabla\pi^*(\underline{f}(0),\underline{f}) = D\pi(\underline{f})$. The function $D\pi(\underline{f})$ is called the *elastic gradient of* π *at* $\underline{f}$ (c.f. [2]). It is worth noting that E4b implies that $\lim_{\underline{a}\to\underline{f}(0)} \pi^*(\underline{a},\underline{f}) = \pi(\underline{f})$. It is convenient to define $\pi^*(\underline{f}(0),\underline{f})$ to be equal to $\pi(\underline{f})$. If $\underline{f}(0)$ is not in $\underset{\sim}{E}(\underline{f})$, this definition gives a continuous extension of $\pi^*(\cdot,\underline{f})$ to $\underset{\sim}{E}(\underline{f}) \cup \{\underline{f}(0)\}$.

It is easy to show that if $\underline{f}$ is a rest history, i.e., for some $\underline{a} \in \underline{\mathcal{L}}^+$, $\underline{f}$ is equal to the history $\underline{a}^+$ defined by

$$a^+(s) = a, \quad s \in [0,\infty),$$

then $\underline{f}(0) \in \underset{\sim}{E}(\underline{f})$. (This result follows from property E2).

4. The Local and Global History Gradients

Given a functional π it is possible to define functionals $\delta\pi$ and π' using the fact that π has an elastic range $\underset{\sim}{E}(\underline{f})$ corresponding to each history $\underline{f}$. The functional π' was introduced by PIPKIN and RIVLIN [1] and the functional $\delta\pi$ was first discussed by this author in [2]. Here, I shall given definitions for the two functionals under consideration as well as two useful properties of these functionals.

Let $\underline{f}$ be a given history and let $\sigma_o \in [0,\infty)$ be such that $\{\underline{f}(\underline{s}) | \underline{s} \in (0,\sigma_o)\} \subset \underset{\sim}{E}(\underline{f})$.

Definition. The local history gradient $\delta\pi(\underline{f})$ of π at f is defined by

$$\delta\pi(f) = \lim_{\sigma \downarrow 0} \frac{1}{\sigma}\{\pi^*(f(\sigma),f) - \pi^*(f(\sigma),f_\sigma)\}$$

whenever the limit exists.

Henceforth I assume that $\delta\pi(\underline{f})$ exists for every history $\underline{f}$.
Of course, $\delta\pi(\underline{f})$ is an element of $\underline{\ell}$.

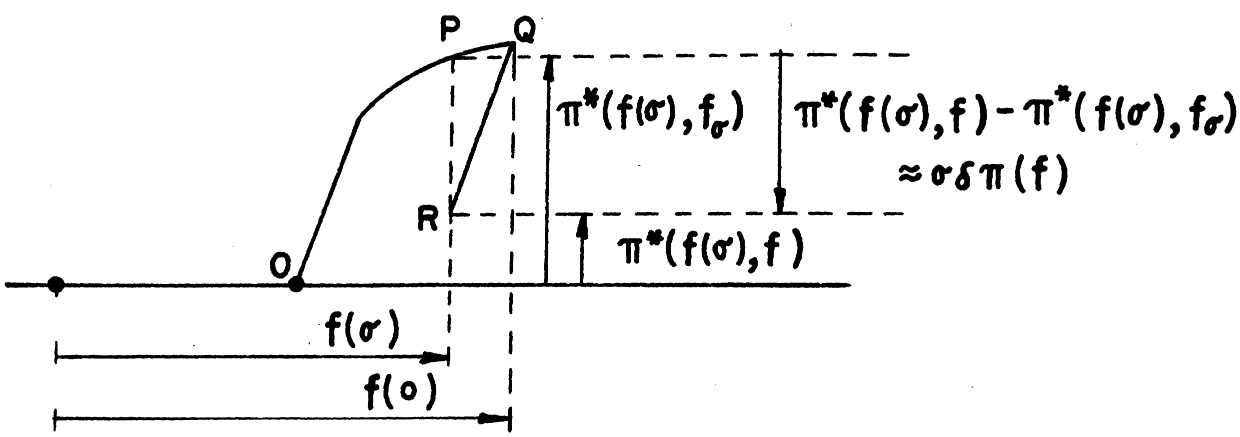

FIGURE 2

Figure 2 illustrates the computation of $\delta\pi(\underline{f})$ in terms of a one-dimensional stress-strain curve. If the material is rate-independent, then $-\delta\pi(\underline{f})$ represents the slope of the curve RQ minus the slope of the curve OPQ at Q (provided $|\dot{\underline{f}}(0)| = 1$). This example and the definition above indicate that $\delta\pi$ measures changes in stress due to short, closed cycles of deformation gradient. Such a cycle would begin with $\underline{f}(\sigma)$, proceed to $\underline{f}(0)$ through the points $\underline{f}(\underline{s})$, $\underline{s} \in (0,\sigma)$, and then either retrace this path in the opposite direction, or proceed to $\underline{f}(\sigma)$ through any path entirely within $\underset{\sim}{E}(\underline{f})$. Such a cycle and the corresponding stress response are illustrated in Figure 3.

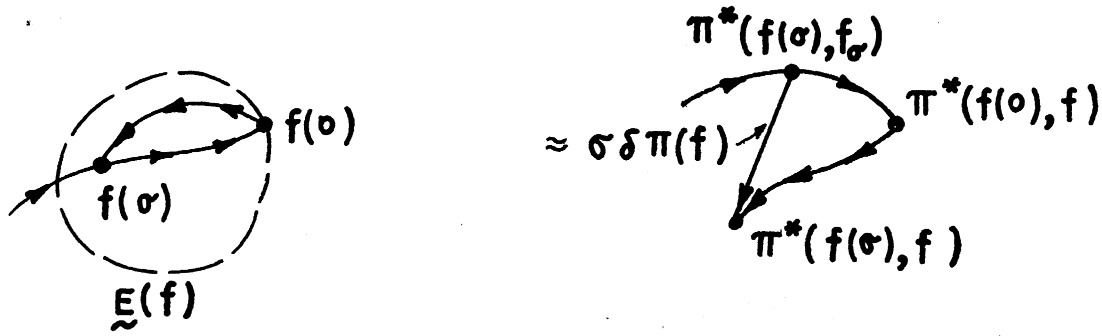

FIGURE 3

Let $\underline{f}$ be a given history and let $\sigma_1 \in (0,\infty)$ and $\underline{a} \in \underline{\ell}^+$ be such that $\underline{a} \in \underset{\sim}{E}(\underline{f}_\sigma)$ for every $\sigma \in [0,\sigma_1]$.

Definition. The global history gradient $\pi'(a,f)$ of π at a and f is defined by

$$\pi'(a,f) = \lim_{\sigma \downarrow 0} \frac{1}{\sigma}\{\pi^*(a,f) - \pi^*(a,f_\sigma)\}$$

whenever the limit exists.

Henceforth, I assume that $\pi'(\underline{a},\underline{f})$ exists whenever $\underline{a}$ and $\underline{f}$ satisfy the conditions stated above.

Figure 4 illustrates the computation of $\pi'(\underline{a},\underline{f})$ in terms of a one-dimensional stress-strain curve.

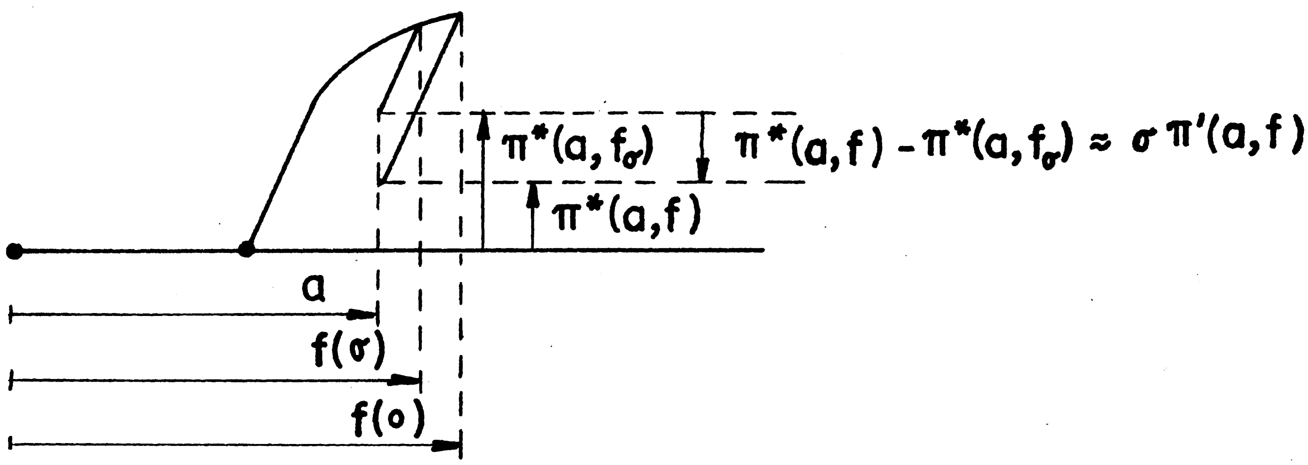

FIGURE 4

Here, $\sigma\pi'(\underline{a},\underline{f})$ measures the approximate effect of the path $\{\underline{f}(\underline{s})\,|\,\underline{s} \in [0,\sigma)\}$ on the value $\pi^*(\underline{a},\underline{f}) - \pi^*(\underline{a},\underline{f}_\sigma)$. In a more general situation, the significance of $\pi'(\underline{a},\underline{f})$ is given in Figure 5.

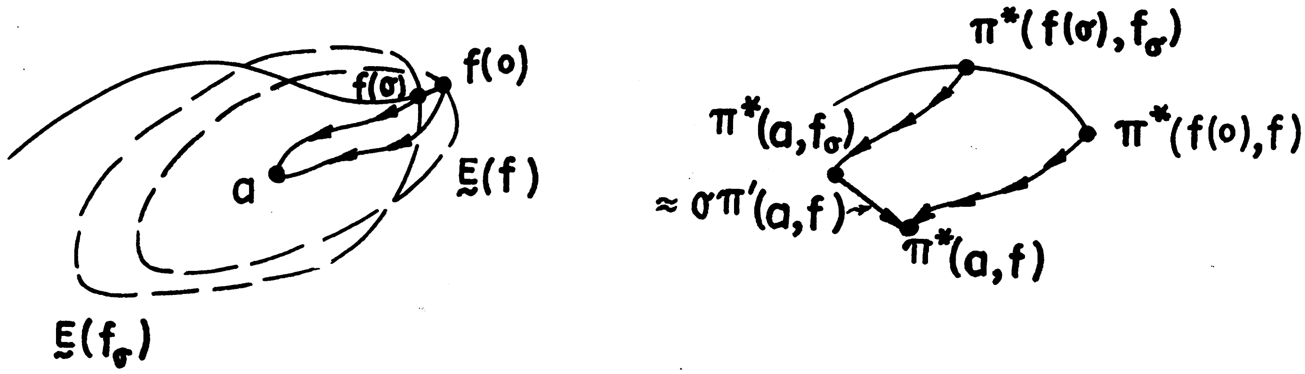

FIGURE 5

If $\underline{f}(0) \in \underset{\sim}{E}(\underline{f})$ and if the sets $\underset{\sim}{E}(\underline{f}_\sigma)$ coincide for σ in some interval $[0,\hat{\sigma}]$, $\hat{\sigma} > 0$, then it follows that

$$\delta\pi(f) = \pi'(a,f) = 0$$

for every $\underline{a} \in \underset{\sim}{E}(\underline{f})$. In fact, for some $\tilde{\sigma} \in [0,\hat{\sigma}]$,

$$\begin{aligned}\delta\pi(f) &= \lim_{\sigma\downarrow 0}\frac{1}{\sigma}\{\pi^*(f(\sigma),f) - \pi^*(f(\sigma),f_\sigma)\}\\ &= \lim_{\sigma\downarrow 0}\frac{1}{\sigma}\{\pi^*(f(\sigma),f_{\tilde{\sigma}}) - \pi^*(f(\sigma),f_{\tilde{\sigma}})\}\\ &= 0.\end{aligned}$$

A similar argument shows that under the above hypotheses, the global history gradient vanishes.

A second result, which will be useful in Section 8, is the following: _If_ f _is any history, then_

$$\dot{\pi}(f) \overset{\text{def}}{=} \lim_{\sigma \downarrow 0} \frac{1}{\sigma}\{\pi(f) - \pi(f_\sigma)\}$$

exists and is given by

$$\dot{\pi}(f) = -D\pi(f)\dot{f}(0) + \delta\pi(f).$$

(See [2] for a proof of this result.) Hence, _the local history gradient and the elastic gradient determine_ $\dot{\pi}$, _the rate of change of_ π.

5. Permanent Deformations

The notion of "permanent deformation" rests on the assumption that small deformations measured relative to a distinguished point in the elastic range of a given history produce a response which is independent of the given history. In other words, there is no difference between the "states" of the material in the distinguished configurations corresponding to any two histories. This assumption now will be incorporated in the present theory.

Assumption: There is a neighborhood $\underset{\sim}{N}$ of the identity tensor and a function $\pi_o: \underset{\sim}{N} \to \mathcal{L}$ such that, given any history f, there exists a history p_f satisfying

P1. $p_f(\sigma)$ is an element of $\underset{\sim}{E}(f_\sigma)$ for every $\sigma \in [0,\infty)$,

P2. $\liminf_{\sigma \downarrow 0} d(p_f(\sigma), \partial\underset{\sim}{E}(f_\sigma)) > 0$,

P3. $\pi^*(bp_f(\sigma), f_\sigma) = \pi_o(b)$ for every $b \in \underset{\sim}{N}$ such that $bp_f(\sigma) \in \underset{\sim}{E}(f_\sigma)$.

Here, $d(\underline{p}_{\underline{f}}(\sigma), \partial\underset{\sim}{E}(\underline{f}_\sigma))$ is the (closest) distance between $\underline{p}_{\underline{f}}(\sigma)$ and the boundary of $\underset{\sim}{E}(\underline{f}_\sigma)$. $\underline{p}_{\underline{f}}$ is called a *permanent deformation history corresponding to* $\underline{f}$.

The stress for a continuation of $\underline{f}_\sigma$ within $\underset{\sim}{E}(\underline{f}_\sigma)$ to $\underline{p}_{\underline{f}}(\sigma)$ is independent of σ and is given by $\pi_o(1)$, where 1 denotes the identity tensor. The variable $\underline{b}$ appearing in P3 plays the role of the "elastic part" of the deformation gradient. An assumption similar to P3 appears in the paper of LEE and LIU [3]. The present assumption is weaker in that the neighborhood

$\underset{\sim}{N}\underline{p}_{\underline{f}}(\sigma) = \{\underline{a} = \underline{b}\underline{p}_{\underline{f}}(\sigma) | \underline{b} \in \underset{\sim}{N}\}$, of $\underline{p}_{\underline{f}}(\sigma)$ need not include $\underline{f}(\sigma)$. Hence, the stress $\pi(\underline{f}_\sigma)$ is not necessarily given by $\pi_o(\underline{f}(\sigma)\underline{p}_{\underline{f}}(\sigma)^{-1})$. If $\underset{\sim}{E}(\underline{f}_\sigma) \subseteq \underset{\sim}{N}\underline{p}_{\underline{f}}(\sigma)$ then

$$\pi(f_\sigma) = \pi_o(f(\sigma)p_f(\sigma)^{-1}).$$

This case will be discussed in Section 8.

Let us agree to write $\underset{\sim}{E}(\sigma)$ and $\underline{p}(\sigma)$ for $\underset{\sim}{E}(\underline{f}_\sigma)$ and $\underline{p}_{\underline{f}}(\sigma)$ when $\underline{f}$ and $\underline{p}$ are fixed in a particular discussion. The symbols $L_{\underline{p}}(\sigma)$ and $D_{\underline{p}}(\sigma)$ denote the tensors

$$L_p(\sigma) = \mathring{p}(\sigma)p(\sigma)^{-1}$$

and

$$D_p(\sigma) = \frac{1}{2}(L_p(\sigma) + L_p(\sigma)^T), \quad \sigma \in [0,\infty).$$

The tensor $-D_{\underline{p}}(\sigma)$ is called the *permanent stretching tensor at time* σ. In Section 7, a relation between the permanent stretching tensor and the global history gradient will be derived.

6. Restrictions Imposed by Frame-Indifference and by Material Symmetry.

If π is frame-indifferent or satisfies invariance properties which correspond to material symmetries, then elastic ranges and the local and global history gradients have similar properties. Since, in Section 7, the global history gradients are shown to determine the permanent stretching tensor, the properties deduced in this section for the global history gradient will determine invariance properties of the permanent stretching tensor.

It will be convenient to assume that π satisfies a weak principle of frame indifference of the form:

$$\pi(Q_o f) = Q_o\, \pi(f) Q_o^T$$

for every $\underset{\sim}{f}$ and for every orthogonal tensor Q_o. Here, $Q_o \underset{\sim}{f}$ is the history

$$(Q_o f)(s) = Q_o f(s), \quad s \in [0,\infty).$$

Use of this principle instead of the stronger form ([4],p.60) circumvents some technical difficulties associated with the possible lack of smoothness of the boundaries of elastic ranges. The weaker form is strong enough to obtain useful restrictions on quantities derived from the stress functional.

I 1. *For every orthogonal* Q_o,

$$\underset{\sim}{E}(Q_o f) = Q_o\, \underset{\sim}{E}(f).$$

Proof. It suffices to show that $Q_o\, \underset{\sim}{E}(\underset{\sim}{f}) \subseteq \underset{\sim}{E}(Q_o \underset{\sim}{f})$. In fact, if this inequality holds for every Q_o and $\underset{\sim}{f}$, then

$$\underset{\sim}{E}(Q_o f) = Q_o Q_o^T \underset{\sim}{E}(Q_o f)$$
$$\subseteq Q_o\ \underset{\sim}{E}(Q_o^T Q_o f) = Q_o\ \underset{\sim}{E}(f)$$

so that $Q_o \underset{\sim}{E}(f) = \underset{\sim}{E}(Q_o f)$. In order to verify the inequality $Q_o\ \underset{\sim}{E}(\underline{f}) \subseteq \underset{\sim}{E}(Q_o \underline{f})$, it suffices to verify that $Q_o\ \underset{\sim}{E}(\underline{f})$ is an elastic region for $Q_o \underline{f}$. E1 is satisfied by $Q_o\ \underset{\sim}{E}(\underline{f})$, and E2 follows since, if $\underline{f}(s) \in \underset{\sim}{E}(\underline{f})$ for s near zero, then $(Q_o \underline{f})(s) = Q_o \underline{f}(s) \in Q_o\ \underset{\sim}{E}(\underline{f})$ for s near zero. In order to show that π is path-independent on $\mathcal{C}(Q_o \underline{f}, Q_o\ \underset{\sim}{E}(\underline{f}))$, note that if $\underline{g}$, $\underline{g}'$ are in $\mathcal{C}(Q_o \underline{f}, Q_o\ \underset{\sim}{E}(\underline{f}))$ with $\underline{g}(0) = \underline{g}'(0) = a$, then $Q_o^T \underline{g}$, $Q_o^T \underline{g}'$ are in the set $\mathcal{C}(\underline{f}, \underset{\sim}{E}(\underline{f}))$ with $Q_o^T \underline{g}(0) = Q_o^T \underline{g}'(0) = Q_o^T \underline{a}$. Hence, using frame-indifference one obtains

$$\pi(\underline{g}) = Q_o\ \pi(Q_o^T \underline{g}) Q_o^T$$
$$= Q_o\ \pi(Q_o^T \underline{g}') Q_o^T$$
$$= \pi(\underline{g}'),$$

since π is path-independent on $\mathcal{C}(\underline{f}, \underset{\sim}{E}(\underline{f}))$. Therefore, E3 is satisfied by $Q_o\ \underset{\sim}{E}(\underline{f})$. Moreover, one finds that for $\underline{a} \in Q_o\ \underset{\sim}{E}(f)$,

$$\pi^*(a, Q_o f) = \pi^*(Q_o Q_o^T a,\ Q_o f)$$
$$= Q_o\ \pi^*(Q_o^T a, f) Q_o^T,$$

so that $\pi^*(\cdot, Q_o \underline{f})$ is C^1 on $Q_o \underset{\sim}{E}(\underline{f})$. Moreover,

$$\nabla\ \pi^*(a, Q_o f) b = Q_o [\nabla \pi^*(Q_o^T a, f) Q_o^T b] Q_o^T,$$

so that $D\pi^*(Q_o f)$ exists and

$$\pi^*(a,Q_of) - \pi(Q_of) =$$

$$= Q_o[\pi^*(Q_o^T a,f) - \pi(f)]Q_o^T$$

$$= Q_o[D\pi(f)(Q_o^T[a-Q_of(0)])]Q_o^T$$

$$+o|Q_o^T[a-Q_of(0)]|$$

$$= D\pi(Q_of)[a-(Q_of)(0)] +$$

$$+ o|a - (Q_of)(0)|.$$

These considerations show that E4 is satisfied, which, together with the previous arguments, imply that $Q_o \underset{\sim}{E}(\underline{f})$ is an elastic region for $Q_o\underline{f}$. Therefore, $Q_o \underset{\sim}{E}(\underline{f}) \subseteq \underset{\sim}{E}(Q_o\underline{f})$ since the elastic range $\underset{\sim}{E}(Q_o\underline{f})$ is a maximal elastic region.

I 2. <u>For every orthogonal</u> Q_o, <u>the history gradients satisfy</u>

$$\delta\pi(Q_of) = Q_o\ \delta\pi(f)Q_o^T$$

$$\pi'(a,Q_of) = Q_o\pi'(Q_o^T a,f)Q_o^T.$$

Proof. One has:

$$\delta\pi(Q_of) = \lim_{\sigma\downarrow 0} \frac{1}{\sigma}\{\pi^*(Q_of(\sigma),Q_of) - \pi^*(Q_of(\sigma),Q_of_\sigma)\}$$

$$= \lim_{\sigma\downarrow 0} \frac{1}{\sigma}\ Q_o\{\pi^*(f(\sigma),f) - \pi^*(f(\sigma),f_\sigma)\}Q_o^T$$

$$= Q_o\ \delta\pi(f)Q_o^T,$$

and

$$\pi'(a,Q_of) = \lim_{\sigma\downarrow 0} \frac{1}{\sigma}\{\pi^*(a,Q_of) - \pi^*(a,Q_of_\sigma)\}$$

$$= \lim_{\sigma\downarrow 0} \frac{1}{\sigma}\{\pi^*(Q_oQ_o^T a,Q_of) - \pi^*(Q_oQ_o^T a,Q_of_\sigma)\}$$

$$= Q_o\ \pi'(Q_o^T a,f)Q_o^T.$$

I 3. *If p_f is a permanent deformation history for f, then $Q_o p_f$ is a permanent deformation history for $Q_o f$ for every orthogonal Q_o if and only if*

$$\pi_o(Q_o b Q_o^T) = Q_o \pi_o(b) Q_o^T$$

for every orthogonal Q_o

Note. I 3 asserts that permanent deformation histories transform in the same way as ordinary histories if and only if π_o is an isotropic function.

Proof. One must show that P1, P2 and P3 are satisfied by $Q_o \underline{p}_{\underline{f}}$ (with respect to the history $Q_o \underline{f}$). First, since $Q_o \underset{\sim}{E}(\underline{f}_\sigma) = \underset{\sim}{E}(Q_o \underline{f}_\sigma) = \underset{\sim}{E}((Q_o \underline{f})_\sigma)$, and since $\underline{p}_{\underline{f}}(\sigma) \in \underset{\sim}{E}(\underline{f}_\sigma)$, it follows that $(Q_o \underline{p}_{\underline{f}})(\sigma) \in \underset{\sim}{E}(Q_o \underline{f}_\sigma)$. Moreover,

$$\begin{aligned} d(Q_o p_f(\sigma), \partial \underset{\sim}{E}((Q_o f_\sigma)) &= \\ &= d(Q_o p_f(\sigma), \partial Q_o \underset{\sim}{E}(f_\sigma)) \\ &= d(p_f(\sigma), \partial \underset{\sim}{E}(f_\sigma)), \end{aligned}$$

so that P1 and P2 are satisfied. Next, suppose that $\underline{b} \in \underset{\sim}{N}$ and $\underline{b} Q_o \underline{p}_{\underline{f}}(\sigma)$ is in the set $\underset{\sim}{E}(Q_o \underline{f}_\sigma)$. It follows that

$$\begin{aligned} \pi^*(bQ_o p_f(\sigma), Q_o f_\sigma) &= \\ &= \pi^*(Q_o(Q_o^T b Q_o) p_f(\sigma), Q_o f_\sigma) \\ &= Q_o \pi^*(Q_o^T b Q_o p_f(\sigma), f_\sigma) Q_o^T \\ &= Q_o \pi_o(Q_o^T b Q_o) Q_o^T. \end{aligned}$$

Note that $Q_o^T b Q_o \in \underset{\sim}{N}$ and $Q_o^T b Q_o \underline{p_f}(\sigma)$ is in the set $\underset{\sim}{E}(\sigma)$. Hence, P3 will be satisfied for every orthogonal Q_o if and only if

$$\pi_o(b) = \pi^*(bQ_o p_f(\sigma), Q_o f_\sigma)$$
$$= Q_o \pi_o(Q_o^T b Q_o) Q_o^T,$$

i.e. π_o must be in isotropic function. Thus, the proof of I.3 is complete

I 4. <u>If for every orthogonal</u> Q_o <u>and some permanent deformation history</u> p_f <u>one has that</u> $Q_o p_f$ <u>is a permanent deformation history for</u> $Q_o f$, <u>then</u>

$$\pi_o(bQ_o) = \pi_o(b)$$

<u>for all orthogonal</u> Q_o <u>if and only if</u>

$$\pi_o(Q_o b) = Q_o \pi_o(b) Q_o^T$$

<u>for all othogonal</u> Q_o.

Proof. The result follows immediately from I 3.

S 1. <u>If for some unimodular tensor</u> $H \in \ell$ <u>one has</u>

$$\pi(fH) = \pi(f)$$

<u>for every</u> f, <u>then</u>

$$\delta\pi(fH) = \delta\pi(f)$$

$$\pi'(aH, fH) = \pi'(a, f)$$

<u>for every</u> f.

Proof. These results are immediate consequences of the symmetry property of π and the definitions of the history gradients.

If the stress functional satisfies the strong form of the principle of frame indifference, stronger results analogous to I1-I4 can be obtained despite the technical difficulty mentioned at the beginning of this section. This difficulty arises because superposition of non-constant rigid rotation histories may cause E2 to be violated. In turn, the existence of the history gradients at histories modified by the rotations cannot be obtained from the existence at the original histories using the arguments given in the proof of I 2. Nevertheless, the history gradients can be defined on the troublesome modified histories through the principle of material frame indifference, and this definition can be shown to be consistent with the values obtained on modified histories which are not pathological, i.e. for which E2 is valid. In Section 9, the strong form of frame indifference is used to obtain a representation for the local history gradient for simple shearing motions, and it is hoped that these few remarks will justify this procedure to the reader. Elsewhere, the results I 1 -I 4, as stated and proved in this section for the weaker form of frame indifference, suffice in the development that follows.

7. Flow Rules.

In this section it will be shown that the permanent stretching tensor is determined explicitly by the stress functional through the global history gradient and the function π_o. The simple form of the relation arises from the assumption that π_o is an isotropic function. In the last section, it was shown that this assumption of isotropy is equivalent to the assumption that permanent deformation histories transform in the same way as deformation histories under superposed constant rigid motions.

The following result will be needed in deriving the relation for the permanent stretching tensor: *If* π_o *is isotropic and satisfies either of the identities in* I 4, *then there exist* $\lambda_o, \lambda_1, \lambda_2 \in R$ *such that*

$$\nabla\pi_o(1)u = \lambda_o \text{tr } u \; 1 + 2(\lambda_1 + 2\lambda_2)\hat{u}$$

for every $u \in \mathcal{L}$. *Here* $\hat{u}$ *denotes the symmetric part of* u *and* tr *is the trace operation*. Note that when $\lambda_2 = 0$, this relation has the same form as the classical constitutive equation for infinitesimal elastic materials, with λ_o and λ_1 corresponding to Lame's constants. The form for $\nabla\pi_o(1)$ is derived by explicit calculation using representations for isotropic functions (see [4], p. 142, equation (47.20)).

The main result of this section will now be stated and proved: *Let* f *be a history and* p_f *a permanent deformation history corresponding to* f. *If* π_o *is isotropic and satisfies either of the identities in* I 4, *and if* $\lambda_1 + 2\lambda_2 \neq 0$, *then*

$$D_p(0) + \frac{\lambda_o}{2(\lambda_1+2\lambda_2)} (\mathrm{tr}\, D_p(0))1 = \frac{\pi'(p(0),f)}{2(\lambda_1+2\lambda_2)} .$$

If the permanent stretching is isochoric, i.e. $\mathrm{tr}\, D_p(0) = 0$, then

$$D_p(0) = \frac{\pi'(p(0),f)}{2(\lambda_1+2\lambda_2)} .$$

Both of these relations will be referred to as flow rules, in keeping with the terminology used in theories of plasticity. The condition that the permanent stretching be isochoric is generally used in the classical theories.

The identity in the theorem will be proved by using the definition of $\pi'(\underline{p}(0),\underline{f})$. This procedure necessitates a verification that $\pi^*(\underline{p}(0),\underline{f}_\sigma)$ is defined for sufficiently small values of σ. It suffices to show that $\underline{p}(0) \in \underset{\sim}{E}(\sigma)$ for small σ. This statement follows from the continuity of $\underline{p}$ and conditions P1 and P2. In fact, suppose there exists a sequence $\{\sigma_n\}$ with limit zero such that $\underline{p}(0) \notin \underset{\sim}{E}(\sigma_n)$ for every $\underline{n}$. This condition implies that some boundary point $\underline{a}_n$ of $\underset{\sim}{E}(\sigma_n)$ lies on the (closed) line segment joining $\underline{p}(0)$ and $\underline{p}(\sigma_n)$. Consequently, one has

$$\begin{aligned} d(p(\sigma_n), \partial E(\sigma_n)) &\leq d(p(\sigma_n), a_n) = \\ &= d(p(\sigma_n), p(0)) - d(p(0), a_n) \\ &\leq d(p(\sigma_n), p(0)) \end{aligned}$$

where d measures the distance between points in $\underline{\mathcal{L}}^+$,

$$d(a_1,a_2) = \mathrm{tr}[(a_1-a_2)(a_1-a_2)^T].$$

However, the last inequality and the continuity of $\underset{\sim}{p}$ contradict P2, and hence $\underset{\sim}{p}(0) \in \underset{\sim}{E}(\sigma)$ for σ sufficiently small. It follows that the computation shown below is meaningful:

$$\pi'(p(0),f) = \lim_{\sigma \downarrow 0} \frac{1}{\sigma}\{\pi^*(p(0),f) - \pi^*(p(0),f_\sigma)\}$$

$$= \lim_{\sigma \downarrow 0} \frac{1}{\sigma}\{\pi_o(1) - \pi_o(p(0)p(\sigma)^{-1})\}$$

$$= \lim_{\sigma \downarrow 0} [\nabla\pi_o(1)\frac{1}{\sigma}\{p(\sigma)-p(0)\}p(\sigma)^{-1}]$$

$$= \nabla\pi_o(1)L_p(0).$$

Note that the continuity of $\underset{\sim}{p}$ guarantees that $\underset{\sim}{p}(0)\underset{\sim}{p}(\sigma)^{-1}$ is in $\underset{\sim}{N}$ for σ sufficiently small. The flow rules follow from the formulas for $\nabla\pi_o(1)\underline{u}$.

8. A Decomposition for the Stretching Tensor.

In this section attention is restricted to the case where $\underset{\sim}{N}\underline{p}(\sigma) \supseteq \underset{\sim}{E}(\sigma)$ for every $\sigma \in [0,\infty)$. This condition implies that for every σ,

$$\pi^*(a,f_\sigma) = \pi_o(ap(\sigma)^{-1})$$

for every $\underline{a} \in \underset{\sim}{E}(\sigma)$. In this case the local history gradient reduces to

$$\begin{aligned}
\delta\pi(f) &= \lim_{\sigma\downarrow 0} \frac{1}{\sigma}\{\pi^*(f(\sigma),f) - \pi^*(f(\sigma),f_\sigma)\} \\
&= \lim_{\sigma\downarrow 0} \frac{1}{\sigma}\{\pi_o(f(\sigma)p(0)^{-1}) - \pi_o(f(\sigma)p(\sigma)^{-1})\} \\
&= -\lim_{\sigma\downarrow 0} \frac{1}{\sigma}\{\nabla\pi_o(f(\sigma)p(0)^{-1})f(\sigma)[p(\sigma)^{-1}-p(0)^{-1}]\} \\
&= \nabla\pi_o(f(0)p(0)^{-1})[f(0)p(0)^{-1}L_p(0)].
\end{aligned}$$

The elastic gradient $D\pi(\underline{f})$ takes the simple form

$$D\pi(f)u = \nabla\pi_o(f(0)p(0)^{-1})[up(0)^{-1}]$$

for every $\underline{u} \in \underline{\mathcal{L}}$. The formulae for $\delta\pi$ and $D\pi$ along with the expression for $\dot{\pi}$ given at the end of Section 4 combine to give

$$\dot{\pi}(f) = -\nabla\pi_o(e)[L(0)e - eL_p(0)]$$

where

$$L(0) = \dot{f}(0)f(0)^{-1}$$

and

$$e = f(0)p(0)^{-1}.$$

If the magnitude of $\underline{e}$-1 is small compared with unity, the expression for $\dot{\pi}(\underline{f})$ takes the approximate form

$$\dot{\pi}(f) = -\nabla\pi_o(1)[L(0)-L_p(0)].$$

The expression for $\nabla\pi_o(1)\underline{u}$ given in section 7 yields, with $D(0) = \frac{1}{2}[L(0)+L(0)^T]$,

$$\dot{\pi}(f) = -\lambda_o tr(D(0)-D_p(0))1 = 2(\lambda_1+2\lambda_2)(D(0)-D_p(0)).$$

For isochoric plastic stretching one finds that

$$\dot{\pi}(f) = -\lambda_o \; tr \; D(0)1 - 2(\lambda_1+2\lambda_2)(D(0)-D_p(0))$$

from which it follows that

$$tr\; \dot{\pi}(f) = -[3\lambda_o + 2(\lambda_1+2\lambda_2)]tr\; D(0).$$

One then obtains the <u>decomposition for the deviatoric stretching tensor</u>:

$$-D(0)^{\#} = \frac{\dot{\pi}(f)^{\#}}{2(\lambda_1+2\lambda_2)} - D_p(0)^{\#}$$

where, for any tensor $\underline{u}$,

$$u^{\#} = u - \tfrac{1}{3}tr\; u\; 1.$$

If $\lambda_2 = 0$ it follows that

$$-D(0)^{\#} = \frac{\dot{\pi}(f)^{\#}}{2\lambda_1} - D_p(0)^{\#},$$

which is the <u>classical decomposition for the deviatoric stretching tensor</u> (c.f. [5]). (This relation is not frame-indifferent, due to the presence of the term in $\mathring{\pi}$. This situation arises because of the approximation $f(0)p(0)^{-1} \cong 1$. It is interesting to note that this relation is frame-indifferent in the weak sense defined in Section 6.)

The calculation given above suggests the following remark: the present theory is a generalization of the classical theories of elastic-plastic materials, and it reduces to the classical theories when the "elastic deformation" $(\underline{f}(0)\underline{p}(0)^{-1} - 1)$ is small. Actually, the present theory allows for rate-dependent behavior, so that a further restriction on the response functional π would be needed to obtain classical formulae which depend upon the rate-independence of the material.

Note that in the case of small elastic deformations, the local and global history gradients agree, i.e.

$$\delta\pi(f) = \pi'(p(0), f) = \nabla\pi_o(1) L_p(0).$$

Hence, in this approximation if one assumes further that the permanent stretching is isochoric, then the permanent stretching is determined by the local history gradient and the function π_o.

9. Simple Shear for an Isotropic Material with Elastic Range.

Let us consider a material with elastic range which is isotropic with respect to a homogeneous configuration $\underset{\sim}{\varkappa}$, i.e.

$$\pi(fQ_o) = \pi(f)$$

for every orthogonal tensor Q_o. It follows from S1, Section 6, that the history gradients π' and $\delta\pi$ satisfy similar identities. Moreover, the flow rules derived in Section 7 give the result: the permanent stretchings (at $s = 0$) for the histories f and fQ_o are the same for every orthogonal Q_o.

If the elastic deformations are small, the last remark in Section 7 gives (when the permanent stretching is isochoric),

$$D_p(0) = \frac{1}{2(\lambda_1+2\lambda_2)}\,\delta\pi(f).$$

Suppose now that the material undergoes a simple shearing motion with respect to $\underset{\sim}{\varkappa}$, i.e. the description of the motion relative to $\underset{\sim}{\varkappa}$ has the form

$$\begin{aligned} \underset{\sim}{\chi}_{\underset{\sim}{\varkappa}}(\underset{\sim}{X},t) &= \underset{\sim}{X}_o + F(t)(\underset{\sim}{X}-\underset{\sim}{X}_o), & t \geq t_o \\ &= \underset{\sim}{X} &, \quad t < t_o. \end{aligned}$$

Here, $\underset{\sim}{X}$ and $\underset{\sim}{X}_o$ are points in $\underset{\sim}{\varkappa}$ and

$$F(t) = 1 + \gamma(t-t_o)\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_2, \quad t \geq t_o,$$

where $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ is an orthonormal basis of V and γ is any real number. Let us fix $t \in R$ and define

$$f(s) = \begin{cases} F(t-s), & 0 \leq s \leq t - t_o \\ 1 \quad , & t - t_o < s < \infty \end{cases} .$$

From the isotropy and frame indifference of $\delta\pi$ it can be shown ([4],p. 78) that

$$\delta\pi(f) = \pi_1(A_1,A_2,B(t),t-t_o)$$

where

$$B(t) = F(t)F(t)^T$$

and π_1 is an isotropic function. A_1 and A_2 are the first two Rivlin-Eriksen tensors which, along with $B(t)$, are given by

$$A_1 = -2D(0) = \gamma(\underset{\sim}{e}_1\otimes\underset{\sim}{e}_2+\underset{\sim}{e}_2\otimes\underset{\sim}{e}_1)$$

$$A_2 = \quad = 2\gamma^2\underset{\sim}{e}_2\otimes\underset{\sim}{e}_2$$

$$B(t) = 1 + (t-t_o)A_1 + (t-t_o)^2[A_1^2-\tfrac{1}{2}A_2], \quad t \geq t_o.$$

Since $B(t)$ is a function of $t - t_o$, A_1 and A_2 it follows that

$$\delta\pi(f) = \pi_2(A_1,A_2,t-t_o)$$

where π_2 is an isotropic function. Arguments used in deriving the form of the stress in steady viscometric flows (e.g.[6]) yield the representation for π_2:

$$\begin{aligned}\pi_2(A_1,A_2,t-t_o) &= \eta\, A_1 - \tfrac{1}{2}\nu A_2 \\ &\quad - \tfrac{1}{2}\phi[1(\text{tr } A_1^2) - 2A_1^2] \\ &\quad + \tfrac{1}{6}(\nu+\phi)1(\text{tr } A_1^2),\end{aligned}$$

where η,ν and ϕ are scalar-valued functions of γ^2 and $t - t_o$. The last term was chosen so as to guarantee that π_2 and, hence, $\delta\pi(f)$ and $D_p(0)$ are trace free. The decomposition of Section 8,

$$\dot{\pi}(f)^{\#} = 2(\lambda_1+2\lambda_2)[-D(0)^{\#}+D_p(0)^{\#}],$$

takes the form

$$\dot{\pi}(f) = \dot{\pi}(f)^{\#} = 2(\lambda_1+2\lambda_2)[\tfrac{1}{2}A_1 + \frac{1}{2(\lambda_1+2\lambda_2)}\pi_2(A_1,A_2,t-t_o)]$$

$$= (\eta + \lambda_1 + 2\lambda_2)\ A_1 - \tfrac{1}{2}\nu A_2$$

$$- \tfrac{1}{2}\phi[1(\mathrm{tr}\ A_1^2)-2A_1^2] + \tfrac{1}{6}(\nu+\phi)1(\mathrm{tr}\ A_1^2).$$

If π is a <u>rate-independent</u> functional, then $\dot{\pi}(f)$ must be of the form:

$$\dot{\pi}(f) = \gamma\tilde{\pi}(f)$$

where $\tilde{\pi}$ is rate-independent. Since A_1^2, A_2, ν and ϕ are even functions of γ, $\dot{\pi}(f)$ takes the last form if and only if, together, the terms involving these quantities vanish, and η is a function of $\gamma^2(t-t_o)^2$. Thus, if π is rate-independent one obtains

$$\dot{\pi}(f) = [(\lambda_1+2\lambda_2) + \eta(\gamma^2(t-t_o)^2)]A_1.$$

Whether or not the material is rate-independent, the expression for $\dot{\pi}$ can be integrated to give the value $\pi(f)$ in terms of the functions shown above and the value of stress at time t_o. (From the isotropy and the special form of $\underset{\sim}{\chi}_{\underset{\sim}{\kappa}}$, the stress at time t_o reduces to a hydrostatic pressure.) One concludes that

1. Normal stress effects can occur if and only if the material is rate-dependent.

and 2. For a rate-independent material, the stresses are determined by λ_1, λ_2 and by a single function η of the total shear.

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