ON DISCONJUGACY AND INTERPOLATION

IN THE COMPLEX DOMAIN

Meira Lavie

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1. Introduction.

In the present paper we are concerned with linear differential equations of the form

$$(1.1) \qquad y^{(n)}(z) + q_{n-1}(z)y^{(n-1)}(z) + \ldots + q_o(z)y(z) = 0,$$

where the coefficients $q_j(z)$, $j = 0,\ldots,n-1$, are assumed to be analytic in a given region R. In particular, we shall be interested in the disconjugacy of such equations. Equation (1.1) is said to be disconjugate in R, if the only solution of (1.1) which vanishes at n (not necessarily distinct) points of R is the trivial solution $y(z) \equiv 0$.

In order to obtain sufficient conditions for disconjugacy of equation (1.1) in a region R, we consider the implications of the hypothesis that there exists a non-trivial solution $y(z)$ which vanishes at n points of R. In doing so, we shall be led to the following question: Let $f(z)$ be analytic in the region R and vanish at n points of R. If $|f^{(n)}(z)| \leq M_n$, $z \in R$, find upper bounds for $|f^{(j)}(z)|$, $j = 0,\ldots,n-1$, $z \in R$.

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*On leave from Technion - Israel Institute of Technology, Haifa.

The same question occurs in interpolation theory: Let $F(z)$ be analytic in the region R, and satisfy $|F^{(n)}(z)| \leq M_n$, $z \in R$. Let $p(z)$ be a polynomial of degree at most $n - 1$, which coincides with $F(z)$ at n points of R, and denote by $r(z) = F(z) - p(z)$ the remainder. Clearly, $r(z)$ vanishes at n points of R, and $|r^{(n)}(z)| = |F^{(n)}(z)| \leq M_n$. The problem of finding upper bounds for $|r^{(j)}(z)|$, $j = 0, \ldots, n-1$, thus reduces to the question stated above.

In section 2 we show that if R is a finite closed convex region, and $f(z)$ is analytic in R and vanishes at $a_i \in R$, $i = 1, \ldots, n$, then upper bounds for $|f^{(j)}(z)|$, $j = 0, \ldots, n-1$, $z \in R$, can be given in terms of $M_n = \text{Max } |f^{(n)}(\zeta)|$, $\zeta \in R$, and the distances $|z - a_i|$, $i = 1, \ldots, n$.

In section 3 we obtain disconjugacy criteria for equation (1.1) in a convex region, and upper bounds for the derivatives to order $(n - 1)$ of the remainder $r(z)$ in the Lagrange interpolation.

In section 4 we generalize the results of section 3. Finally, in section 5, we consider the special case where the convex region R is the unit disk.

2. Bounds for the derivatives of a function with n zeros.

Theorem 1.

Let $f(z)$ be analytic in a finite closed convex region R, and assume that $f(z)$ vanishes at n (not necessarily distinct) points $a_1,\ldots,a_n$ of R. If

$$(2.1)\qquad M_j = \max_{\zeta \in R} |f^{(j)}(\zeta)|, \qquad j = 0,\ldots,n,$$

then, for $z \in R$,

$$(2.2)\qquad |f^{(j)}(z)| \leq \frac{M_n j!}{n!} \sum_{s \in Q_{n-j,n}} \prod_{t=1}^{n-j} |z - a_{i_t}|, \quad j = 0,\ldots,n-1,$$

where the summation in (2.2) is taken over the set $Q_{n-j,n}$ of all the increasing sequences $s = (i_1,\ldots,i_{n-j})$ of $n-j$ integers $1 \leq i_1 < i_2 < \ldots < i_{n-j} \leq n$. Equality holds in (2.2), if

$$(2.3)\qquad f(z) = c\prod_{i=1}^{n}(z - a_i), \quad c = \textit{constant},$$

and $\arg(z - a_i) = \theta$, $i = 1,\ldots,n$, $0 \leq \theta < 2\pi$.

In particular for $j = 0$, (2.2) yields

$$(2.2)_0\qquad |f(z)| \leq \frac{M_n}{n!}\prod_{i=1}^{n}|z - a_i|, \qquad z \in R,$$

and equality holds if $f(z)$ is given by (2.3), while z and a_i, $i = 1,\ldots,n$, are arbitrary.

(If some of the a's coincide, say $a_1 = a_2 = \ldots = a_m = a^*$, $m \leq n$, we assume that $f(z)$ has a zero of multiplicity m at least at $a^* \in R$; i.e. $f(a^*) = f'(a^*) = \ldots = f^{(m-1)}(a^*) = 0$.)

For the proof of Theorem 1, we require the following lemma. (Cf. [5, Lemma 1])

Lemma 1.

If $f(z)$ *is analytic in a region* R *and* $f(\alpha) = 0$, $\alpha \in R$, *then*

$$(2.4) \quad \left(\frac{f(z)}{\alpha - z}\right)^{(k)} = \frac{1}{(\alpha-z)^{k+1}} \int_{\alpha}^{z} (\alpha-\zeta)^k f^{(k+1)}(\zeta)\,d\zeta, \quad k = 0,1,\ldots .$$

The integration in (2.4) *may be carried out along any curve in* R *joining the two points* α *and* z.

In particular, if R *is convex, then*

$$(2.5) \quad \left|\left(\frac{f(z)}{\alpha-z}\right)^{(k)}\right| \leq \frac{1}{k+1} \operatorname*{Max}_{\zeta \in [\alpha,z]} \left|f^{(k+1)}(\zeta)\right|, \quad z \in R, \quad k = 0,1,\ldots,$$

where $[\alpha,z]$ *is the linear segment joining* α *and* z.

(2.4) follows from the identity

$$\left(\frac{f(z)}{\alpha-z}\right)^{(k)} = \frac{N_k(z)}{(\alpha-z)^{k+1}}, \qquad N_k(z) = k! \sum_{j=0}^{k} \frac{(\alpha-z)^j f^{(j)}(z)}{j!} .$$

$f(\alpha) = 0$ implies that $N_k(\alpha) = 0$, and since $N_k'(z) = (\alpha-z)^k f^{(k+1)}(z)$, it follows that

$$N_k(z) = \int_{\alpha}^{z} (\alpha-\zeta)^k f^{(k+1)}(\zeta)\,d\zeta .$$

If R is a convex region, the integration on the right-hand side of (2.4) may be carried out along the linear segment $[\alpha,z]$, and (2.5) follows in an obvious manner.

Proof of Theorem 1.

We prove the theorem by induction on n. If $f(z)$ is analytic in R and vanishes at $a_1 \in R$, then clearly

$$(2.6) \qquad |f(z)| \leq |z - a_1| \max_{\zeta \in R} |f'(\zeta)|, \qquad z \in R.$$

(This is the case $k = 0$, $\alpha = a_1$ of (2.5).) For $n = 2$, we assume that $f(z)$ vanishes at a_1 and a_2 and we set

$$(2.7) \qquad f(z) = (z - a_2)g_1(z) .$$

It follows that $g_1(z)$ is analytic in R and $g_1(a_1) = 0$. (Note that this holds also if $a_1 = a_2$.) Therefore, according to (2.6)

$$(2.6)' \qquad |g_1(z)| \leq |z - a_1| \max_{\zeta \in R} |g_1'(\zeta)| , \qquad z \in R.$$

Setting $k = 1$ and $\alpha = a_2$ in (2.5), we obtain

$$(2.8) \qquad |g_1'(z)| \leq \frac{1}{2} \max_{\zeta \in R} |f''(\zeta)| = \frac{M_2}{2} , \qquad z \in R.$$

(2.7), (2.6)' and (2.8) now imply that

$$|f(z)| \leq \frac{M_2}{2} \prod_{i=1}^{2} |z - a_i|, \qquad z \in R,$$

and

$$|f'(z)| = |(z-a_2)g_1'(z) + g_1(z)| \leq \frac{M_2}{2} \sum_{i=1}^{2} |z-a_i| , \qquad z \in R.$$

Thus, (2.2) is established for $n = 2$.

Assume now that (2.2) has been established for $n \leq m$, and let $f(z)$ vanish at the points $a_1, \ldots, a_{m+1}$ of R. If

$$(2.7)' \qquad f(z) = (z-a_{m+1})g_m(z),$$

then $g_m(z)$ is analytic in R and vanishes at $a_1, \ldots, a_m$. Hence, by our induction hypothesis

$$(2.2)' \quad |g_m^{(j)}(z)| \le \frac{j!}{m!} \max_{\zeta\in R}|g_m^{(m)}(\zeta)| \sum_{s\in Q_{m-j,m}} \prod_{t=1}^{m-j} |z-a_{i_t}|,$$

$$j=0,\dots,m-1, \quad z\in R.$$

In view of (2.7)', it follows by (2.5) that

$$(2.8)' \quad |g_m^{(m)}(z)| \le \frac{1}{m+1} \max_{\zeta\in R}|f^{(m+1)}(\zeta)| = \frac{M_{m+1}}{m+1}, \qquad z \in R.$$

By (2.7)', (2.2)' and (2.8)', we obtain

$$|f^{(j)}(z)| = |(z-a_{m+1})g_m^{(j)}(z) + jg_m^{(j-1)}(z)| \le$$

$$\le \max_{\zeta\in R}|g_m^{(m)}(\zeta)| \Big[\frac{j!}{m!}|z-a_{m+1}| \sum_{s\in Q_{m-j,m}} \prod_{t=1}^{m-j} |z-a_{i_t}| +$$

$$+ \frac{j(j-1)!}{m!} \sum_{s\in Q_{m+1-j,m}} \prod_{t=1}^{m+1-j} |z-a_{i_t}|\Big] \le$$

$$\le \frac{M_{m+1}\, j!}{(m+1)!} \sum_{s\in Q_{m+1-j,m+1}} \prod_{t=1}^{m+1-j} |z-a_{i_t}|, \qquad j=0,\dots,m, \quad z \in R.$$

In the last step we made use of the following statement: If $s = (i_1,\dots,i_{m+1-j}) \in Q_{m+1-j,m+1}$ then, either $i_{m+1-j} = m+1$ and $(i_1,\dots,i_{m-j}) \in Q_{m-j,m}$, or $i_{m+1-j} < m+1$ and then $(i_1,\dots,i_{m+1-j}) \in Q_{m+1-j,m}$.

As the equality assertion is easily verified, this completes the proof of Theorem 1.

We remark that inequality $(2.2)_0$ is known. It can be obtained from Hermite's formula for divided differences (Cf. [9,p. 329], [3,Theorem 2]), or from the representation

$$f(z) = (a_n - z)\dots(a_2 - z)\int_{a_1}^{z}\frac{1}{(a_2-z_1)^2}\int_{a_2}^{z_1}\frac{(a_2-z_2)}{(a_3-z_2)^3}$$

(2.9)

$$\dots\int_{a_{n-1}}^{z_{n-2}}\frac{(a_{n-1}-z_{n-1})^{n-2}}{(a_n-z_{n-1})^n}\int_{a_n}^{z_{n-1}}(a_n-z_n)^{n-1}f^{(n)}(z_n)\,dz_n\dots dz_1,$$

(Cf. [1],[6,Theorem 3.2]), which is valid for a function $f(z)$ which is analytic in a region R and vanishes at $a_i \in R$, $i = 1,\dots,n$. All the integrations in (2.9) are carried out along curves in R. (It is easily confirmed that (2.9) follows from (2.4) by induction.)

We shall also require the following consequence of Lemma 1.

Lemma 2.

Let $f(z)$ be analytic in a convex region R. If the image of R given by $f^{(k)}(z)$, $k \geq 1$, is included in a half plane, (i.e., $\operatorname{Re}\{e^{i\gamma}f^{(k)}(z)\} > 0$, $z \in R$, *for some* γ, $0 \leq \gamma < 2\pi$,) *then $f(z)$ is at most k-valent in R.* (Cf. [13]).

Proof.

Without loss of generality we may assume that $\operatorname{Re}\{f^{(k)}(z)\} > 0$, $z \in R$. We note that if $f(z)$ takes the value b m times in R, then $f(z)-b$ has m zeros in R, while $[f(z)-b]^{(k)} = f^{(k)}(z)$. Therefore, it is sufficient to show that $f(z)$ does not have more than k zeros in R. Assume now that $f(a_i) = 0$, $a_i \in R$, $i = 1,\dots,k$, and set

(2.10) $$f(z) = f_k(z) = f_j(z)\prod_{i=j+1}^{k}(z-a_i), \quad j = 0,1,\dots,k-1.$$

We have to show that $f_o(z) \neq 0$, $z \in R$. We first prove that $\mathrm{Re}\{f^{(k)}(z)\} > 0$, $z \in R$, implies that $\mathrm{Re}\{f_{k-1}^{(k-1)}(z)\} > 0$, $z \in R$. Indeed, by (2.4) and (2.10)

$$(2.11) \quad f_{k-1}^{(k-1)}(z) = \left(\frac{f(z)}{z-a_k}\right)^{(k-1)} = \frac{1}{|z-a_k|^k} \int_o^{|z-a_k|} \rho^{k-1} f^{(k)}(a_k+\rho e^{i\theta})\, d\rho,$$

where $\theta = \arg(z-a_k)$. It now follows from (2.11) that if $\mathrm{Re}\{f^{(k)}(z)\} > 0$, $z \in R$, then $\mathrm{Re}\{f_{k-1}^{(k-1)}(z)\} > 0$, $z \in R$. Since, by (2.10), $f_{j+1}(z) = (z-a_{j+1})f_j(z)$, $j = 0,\ldots,k-1$, it follows similarly that $\mathrm{Re}\{f_j^{(j)}(z)\} > 0$ for $z \in R$. Thus, we finally conclude that $\mathrm{Re}\{f_o(z)\} > 0$, and hence $f_o(z) \neq 0$ for $z \in R$.

We note that this result can also be derived from Hermite's formula for divided differences.

3. Disconjugacy and interpolation in a convex region.

In this section we apply Theorem 1 in order to obtain disconjugacy criteria for equation (1.1) in a convex region R. Estimates for the remainder $r(z)$ (and its derivatives) in Lagrange polynomial interpolation are also obtained. We start with

Lemma 3.

Let $f(z)$ *be analytic in a finite closed convex region* R *of diameter* d. *Assume that* $f(z)$ *vanishes at* n *(not necessarily distinct) points* $a_1,\ldots,a_n$ *of* R. *Then,*

$$(3.1) \qquad |f^{(j)}(z)| \le M_j \le \frac{M_n d^{n-j}}{(n-j)!}, \quad z \in R, \; j = 0,\ldots,n,$$

where M_j, $j = 0,\ldots,n$, *is defined by* (2.1).

Since there are $\binom{k}{r}$ elements in the set $Q_{r,k}$, $1 \le r \le k$, (3.1) follows from (2.2).

Theorem 2.

Let the coefficients $q_j(z)$, $j = 0,\ldots,n-1$, *of the differential equation*

$$(1.1) \qquad y^{(n)}(z) + q_{n-1}(z)\, y^{(n-1)}(z) + \ldots + q_0(z)\, y(z) = 0,$$

be analytic in a finite closed convex region R *of diameter* d. *If*

$$(3.2) \qquad \sum_{j=0}^{n-1} \frac{|q_j(z)|\, d^{n-j}}{(n-j)!} < 1, \quad z \in R,$$

then equation (1.1) *is disconjugate in* R.

Proof.

Suppose that equation (1.1) has a non-trivial solution $y(z)$ which vanishes at n points of R. Then, by Lemma 3,

$$(3.1)' \qquad |y^{(j)}(z)| \le \frac{M_n d^{n-j}}{(n-j)!}, \quad j = 0,\dots,n-1, \quad z \in R,$$

where $M_n = \operatorname{Max}|y^{(n)}(\zeta)|$ for $\zeta \in R$. Since the maximum of $|y^{(n)}(\zeta)|$ is attained at a point $z^* \in R$, we have $M_n = |y^{(n)}(z^*)|$. (1.1) and (3.1)' now imply

$$(3.3) \quad M_n = |y^{(n)}(z^*)| \le \sum_{j=0}^{n-1} |q_j(z^*)|\,|y^{(j)}(z^*)| \le M_n \sum_{j=0}^{n-1} \frac{|q_j(z^*)|d^{n-j}}{(n-j)!}.$$

The number M_n must be positive. Indeed, by (3.1)', $M_n = 0$ would imply that $y(z) \equiv 0$, $z \in R$, which contradicts our hypothesis that $y(z)$ is a non-trivial solution. Hence, it follows from (3.3) that

$$\sum_{j=0}^{n-1} \frac{|q_j(z^*)|d^{n-j}}{(n-j)!} \ge 1 .$$

Inequality (3.2) is thus incompatible with the existence of a non-trivial solution of (1.1) with n zeros in R.

Remarks. (i). A special case of Theorem 2 was obtained by Kim [6,Theorem 3.2]. Indeed, it was this result of Kim which drew our attention to the problem of estimating $|y^{(j)}(z)|$, $j = 0,\dots,n-1$, when $y(z)$ is assumed to have n zeros in a convex region.

(ii). If the strict inequality in (3.2) is replaced by a non-strict inequality we obtain a sufficient condition for disconjugacy of equation (1.1) in an open convex domain of diameter d.

In the next theorem we improve a disconjugacy criterion given by Nehari [9].

Theorem 3.

Let $q_j(z)$, $j = 0,\ldots,n-1$, *be analytic in a finite closed convex region* R, *whose boundary is a piecewise smooth curve* C, *and whose diameter is* d. *If*

$$(3.4)\qquad \sum_{j=0}^{n-1} \frac{d^{n-j-1}}{(n-j-1)!}\int_C |q_j(w)\,dw| < 2,$$

then the differential equation (1.1) *is disconjugate in* R.

Proof.

Suppose that equation (1.1) has a non-trivial solution $y(z)$ which vanishes at n points of R. Then, by Lemma 3,

$$(3.5)\qquad |y^{(j)}(z)| \le \frac{M_{n-1}d^{n-j-1}}{(n-j-1)!}, \quad z \in R,\ j = 0,\ldots,n-1,$$

where $M_{n-1} = \operatorname{Max}|y^{(n-1)}(\zeta)|$ for $\zeta \in R$. (We have used only that $y(z)$ vanishes at $n-1$ points.)

We next prove that if $y(z)$ is analytic in R and vanishes at n points of R, then

$$(3.6)\qquad M_{n-1} \le \frac{1}{2}\int_C |y^{(n)}(w)\,dw|,$$

[9,p. 330]. To establish (3.6) we first show that for every $z \in R$ there exists a point $\zeta \in R$ (ζ depends on z) such that

$$(3.7)\qquad |y^{(n-1)}(z)| \le |y^{(n-1)}(z) - y^{n-1}(\zeta)|.$$

Indeed, suppose there exists $z_0 \in R$, such that for every $\zeta \in R$

(3.8) $$|y^{(n-1)}(zo) - y^{(n-1)}(\zeta)| < |y^{(n-1)}(zo)|$$

holds. (3.8) would then imply that $\operatorname{Re}\{e^{i\gamma}y^{(n-1)}(\zeta)\} > 0$ for $\zeta \in R$ and some γ, $0 \leq \gamma < 2\pi$. By Lemma 2, it would follow that $y(z)$ does not have more than $n-1$ zeros in R. Hence, if $y(z)$ vanishes at n points of R then (3.7) holds. If $z^* \in R$ is such that $|y^{(n-1)}(z^*)| = M_{n-1}$, it follows from (3.7) that there exists $\zeta^* \in R$ such that

(3.7)' $$M_{n-1} = |y^{(n-1)}(z^*)| \leq |y^{(n-1)}(z^*) - y^{(n-1)}(\zeta^*)|.$$

Combining (3.7)' with the inequality

$$|y^{(n-1)}(z^*) - y^{(n-1)}(\zeta^*)| \leq \frac{1}{2}\int_C |y^{(n)}(w)dw|$$

[9, p. 329], we obtain (3.6).

Hence, if $y(z)$ is a non-trivial solution of (1.1) which vanishes at n points of R, it follows from (3.5) and (3.6) that

$$2M_{n-1} \leq \int_C |y^{(n)}(w)dw| = \int_C \left|\sum_{j=0}^{n-1} q_j(w)y^{(j)}(w)\right||dw| \leq$$

$$\leq \sum_{j=0}^{n-1}\int_C |q_j(w)||y^{(j)}(w)||dw| \leq M_{n-1}\sum_{j=0}^{n-1}\frac{d^{n-j-1}}{(n-j-1)!}\int_C |q_j(w)dw|.$$

Since $M_{n-1} > 0$, the last inequality implies that

$$\sum_{j=0}^{n-1}\frac{d^{n-j-1}}{(n-j-1)!}\int_C |q_j(w)dw| \geq 2,$$

which contradicts (3.4).

Remark. As pointed out by Nehari [9], the assumption of analyticity on the boundary C, as well as the assumption that the boundary C is piecewise smooth can be relaxed.

We state now the analogous results for Lagrange polynomial interpolation. We denote by P_{n-1} the set of polynomials of degree at most $n-1$.

Theorem 4.

Let $F(z)$ be analytic in a finite closed convex region R of diameter d. Let $p(z)$ be the unique polynomial in the set P_{n-1} which satisfies

$$(3.9) \qquad p(a_i) = F(a_i), \quad a_i \in R, \quad i = 1,\dots,n.$$

If

$$r(z) = F(z) - p(z)$$

then

$$(3.10) \qquad |r^{(j)}(z)| \le \frac{M_n d^{n-j}}{(n-j)!}, \quad z \in R, \quad j = 0,\dots,n-1,$$

where $M_n = \underset{\zeta \in R}{\mathrm{Max}}|F^{(n)}(\zeta)|$.

If the boundary of R is a piecewise smooth curve C, then

$$(3.11) \quad |r^{(j)}(z)| \le \frac{d^{n-j-1}}{2(n-j-1)!} \int_C |F^{(n)}(w)\,dw|, \quad z \in R, \quad j = 0,\dots,n-1.$$

In view of (3.9), $r(a_i) = 0$, $a_i \in R$, $i = 1,\dots,n$, and since $p^{(n)}(z) \equiv 0$, it follows that $\mathrm{Max}|r^{(n)}(\zeta)| = \mathrm{Max}|F^{(n)}(\zeta)| = M_n$, $\zeta \in R$. The results follow now from (3.1) and (3.6).

4. Generalizations of disconjugacy and interpolation in a convex region.

In this section we generalize the results of the previous section. Our main concern is to obtain conditions which will guarantee the strong disconjugacy of equation (1.1). Strong disconjugacy, which is a more restringent property than disconjugacy, has been recently introduced by London and Schwarz [8]. Equation (1.1) is said to be strongly disconjugate in a given region R, if for every choice of n (not necessarily distinct) points $a_1,\ldots,a_n$ of R, and every sequence of positive integers $k_1,\ldots,k_\ell$ such that $k_1+\ldots+k_\ell = n$, the only solution of (1.1) which satisfies

$$(4.1)\quad \begin{aligned} &y(a_1) = \ldots = y(a_{k_1}) = y^{(k_1)}(a_{k_1+1}) = \ldots = y^{(k_1)}(a_{k_1+k_2}) = \\ &\ldots = y^{(k_1+\ldots+k_{\ell-1})}(a_{k_1+\ldots+k_{\ell-1}+1}) = \ldots = y^{(k_1+\ldots+k_{\ell-1})}(a_n) = 0, \end{aligned}$$

is the trivial solution $y(z) \equiv 0$, [8, p. 495].

Strong disconjugacy implies both disconjugacy ($\ell=1$, $k_1=n$) and disfocality ($\ell=n$, $k_1=\ldots=k_\ell=1$). (The differential equation (1.1) is said to be disfocal in a given region R, if for every choice of n points $a_1,\ldots,a_n$ of R, the only solution of (1.1) which satisfies

$$(4.2)\qquad y(a_1) = y'(a_2) = \ldots = y^{(n-1)}(a_n) = 0$$

is the trivial solution $y(z) \equiv 0$. Disfocality has been considered in various papers [10], [12], [4], [7].)

To obtain sufficient conditions for strong disconjugacy of equation (1.1) in a convex region R of diameter d, we consider the implications of the hypothesis that there exists a non-trivial solution $y(z)$ of equation (1.1) which satisfies (4.1). Applying Lemma 3, we shall find bounds for $|y^{(j)}(z)|$ in terms of the diameter d and $M_n = \text{Max}|y^{(n)}(\zeta)|$, $\zeta \in R$.

The analogous problem in polynomial interpolation can be stated in the following way: Let $F(z)$ be analytic in a region R, and let $p(z)$ be a polynomial of degree at most $(n-1)$, (i.e. $p(z) \in P_{n-1}$) which satisfies

$$F(a_1) = p(a_1), \ \ldots \ , F(a_{k_1}) = p(a_{k_1}),$$

$$F^{(k_1)}(a_{k_1+1}) = p^{(k_1)}(a_{k_1+1}), \ldots, F^{(k_1)}(a_{k_1+k_2}) = p^{(k_1)}(a_{k_1+k_2}), \ldots, \tag{4.3}$$

$$F^{(k_1+\ldots+k_{\ell-1})}(a_{k_1+\ldots+k_{\ell-1}+1}) = p^{(k_1+\ldots+k_{\ell-1})}(a_{k_1+\ldots+k_{\ell-1}+1}), \ldots,$$

$$F^{(k_1+\ldots+k_{\ell-1})}(a_n) = p^{(k_1+\ldots+k_{\ell-1})}(a_n),$$

where $a_i = R$, $i = 1,\ldots,n$, and $k_1,\ldots,k_\ell$ is a sequence of positive integers such that $k_1+\ldots+k_\ell = n$. If

$$r(z) = F(z) - p(z)$$

is the remainder, find bounds for $|r^{(j)}(z)|$, $z \in R$, $j = 0,\ldots,n-1$, in terms of $\text{Max}|F^{(n)}(\zeta)|$, $\zeta \in R$. (We note that the existence and uniqueness of $p(z) \in P_{n-1}$ satisfying (4.3) follow from the fact that equation $y^{(n)}(z) = 0$ is strongly disconjugate in the whole plane.)

Lemma 3'.

Let f(z) *be analytic in a finite closed convex region* R *of diameter* d. *Let* $k_1,\dots,k_\ell$ *be a sequence of positive integers such that* $k_1+\dots+k_\ell = n$. *Assume that* f(z) *satisfies* (4.1) (*with* y(z) *replaced by* f(z)) *where* $a_1,\dots,a_n$ *are points of* R. *If* $k_1+\dots+k_{t-1} \le j < k_1+\dots+k_t$, $1 \le t \le \ell$, *then*

$$(4.4) \qquad |f^{(j)}(z)| \le M_j \le \frac{M_n d^{n-j}}{(k_1+\dots+k_t-j)!\,k_{t+1}!\dots k_\ell!}, \quad z \in R,$$

where

$$M_j = \max_{\zeta\in R}|f^{(j)}(\zeta)|, \quad j = 0,\dots,n .$$

Proof.

Consider the functions $f(z), f^{(k_1)}(z),\dots,f^{(k_1+\dots+k_{\ell-1})}(z)$. In view of (4.1), $f^{(k_1+\dots+k_{t-1})}(z)$, $1 \le t \le \ell$, vanishes at k_t points of R. Applying Lemma 3 to the function $f^{(k_1+\dots+k_{t-1})}(z)$, we find by (3.1) that for $k_1+\dots+k_{t-1} \le j < k_1+\dots+k_t$,

$$(4.5) \qquad M_j \le \frac{M_{k_1+\dots+k_t}\, d^{k_1+\dots+k_t-j}}{(k_1+\dots+k_t-j)!} .$$

For $j = k_1+\dots+k_{t-1}$, $t = 1,\dots,\ell$, (4.5) yields

$$(4.5)' \qquad M_o \le \frac{M_{k_1} d^{k_1}}{k_1!} \le \frac{M_{k_1+k_2} d^{k_1+k_2}}{k_1!k_2!} \le \dots \le \frac{M_n d^n}{k_1!\dots k_\ell!} .$$

Combining (4.5) and (4.5)' we obtain (4.4).

Remark. For $\ell = n$, $k_1 = k_2 = \dots = k_n = 1$, (4.5)' yields

$$M_o \le M_1 d \le \dots \le M_j d^j \le \dots \le M_n d^n . \tag{4.6}$$

We now state

Lemma 4.

Let $f(z)$, R *and the sequence* $k_1, \dots, k_\ell$ *be as in Lemma* 3'. *If* $f(z)$ *satisfies* (4.1), *where* $a_i \in R$, $i = 1, \dots, n$, *then* (4.6) *holds*.

Proof.

We prove (4.6) for the case $\ell = 1$, $k_1 = n$. The proof for the other cases will follow similarly. Since $f(z)$ has n zeros in R, we may replace n in (3.1) by any integer m such that $1 \le m \le n$. This leads us to

$$M_j \le \frac{M_m d^{m-j}}{(m-j)!}, \quad j = 0, \dots, m, \quad 1 \le m \le n. \tag{4.7}$$

Setting now $j = m-1$ and $m = 1, 2, \dots, n$ in (4.7), we obtain (4.6).

Theorem 2'.

Let $q_j(z)$, $j = 0, \dots, n-1$, *be analytic in a finite closed convex region* R *of diameter* d. *If*

$$\sum_{j=0}^{n-1} |q_j(z)| d^{n-j} < 1, \quad z \in R \tag{4.8}$$

then the differential equation (1.1) *is strongly disconjugate in* R.

Proof.

Suppose that equation (1.1) has a non-trivial solution $y(z)$, such that for some sequence of positive integers $k_1,\ldots,k_\ell$, $k_1+\ldots+k_\ell = n$, $y(z)$ satisfies (4.1) where $a_i \epsilon R$, $i = 1,\ldots,n$. Then, by Lemma 4

$$|y^{(j)}(z)| \leq M_n d^{n-j}, \quad j = 0,\ldots,n-1, \quad z \in R,$$

where $M_n = \text{Max}|y^{(n)}(\zeta)|$ for $\zeta \in R$. Since $M_n = |y^{(n)}(z^*)|$ for some $z^* \in R$ and since $M_n > 0$, the result follows as in Theorem 2.

Theorem 3'.

Let $q_j(z)$, $j = 0,\ldots,n-1$, be analytic in a finite closed convex region R, whose boundary is a piecewise smooth curve C, and whose diameter is d. If

$$(4.9) \qquad \sum_{j=0}^{n-1} d^{n-j-1} \int_C |q_j(w)\,dw| < 2 ,$$

then the differential equation (1.1) is strongly disconjugate in R.

Proof.

Suppose that equation (1.1) has a non-trivial solution $y(z)$ which satisfies (4.1), where $a_i \in R$, $i = 1,\ldots,n$, and $k_1,\ldots,k_\ell$ are positive integers such that $k_1 + \ldots + k_\ell = n$. Then, by Lemma 4

$$(4.6)' \quad M_j \leq M_{n-1} d^{n-j-1}, \quad j = 0,\ldots,n-1, \quad M_j = \max_{\zeta \in R} |y^{(j)}(\zeta)|.$$

To complete the proof we now show that

$$M_{n-1} \le \frac{1}{2} \int_C |y^{(n)}(w)dw| \tag{3.6}$$

remains true under our present assumptions. Indeed, by (4.1), the function $\eta(z) = y^{(n-k_\ell)}(z)$ $(n-k_\ell = k_1+\dots+k_{\ell-1})$ vanishes at k_ℓ $(1 \le k_\ell \le n)$ points of R. Applying now (3.6) to $\eta(z)$ (with n replaced by k_ℓ), we obtain that

$$\max_{\zeta\in R} |\eta^{(k_\ell-1)}(\zeta)| \le \frac{1}{2} \int_C |\eta^{(k_\ell)}(w)dw| .$$

Hence, if $y(z)$ is regular in R and satisfies (4.1) for $a_i \in R$, $i = 1,\dots,n$, then (3.6) holds. It now follows from (4.6)' and (3.6) that inequality (4.9) is incompatible with the existence of a non-trivial solution of (1.1) which satisfies (4.1).

Remark. Sufficient conditions for disfocality of a similar character have been established by Kim [4, Theorem 3.1] and Schwarz [12].

We state now the analogous results for polynomial interpolations.

Theorem 4'.

Let $F(z)$ be analytic in a finite closed convex region R of diameter d. Let $p(z)$ be the unique polynomial in the set P_{n-1} which satisfies (4.3), where $a_i \in R$, $i = 1,\dots,n$, and $k_1,\dots,k_\ell$ is a sequence of positive integers such that $k_1+\dots+k_\ell = n$.

If $r(z) = F(z) - p(z)$ *is the remainder, then for* j *such that* $k_1+\ldots+k_{t-1} \le j < k_1+\ldots+k_t$, $1 \le t \le \ell$,

$$(4.10) \qquad |r^{(j)}(z)| \le \frac{M_n d^{n-j}}{(k_1+\ldots+k_t-j)!k_{t+1}!\ldots k_\ell!}, \quad z \in R,$$

where $M_n = \underset{\zeta \in R}{\mathrm{Max}}|F^{(n)}(\zeta)|$.

If the boundary of R *is a piecewise smooth curve* C, *then*

$$(4.11) \quad |r^{(j)}(z)| \le \frac{d^{n-j-1}}{2(k_1+\ldots+k_t-j)!k_{t+1}!\ldots(k_\ell-1)!}\int_C |F^{(n)}(w)dw|, \quad z \in R.$$

Remark. Among all possible choices of the sequences $k_1,\ldots,k_\ell$, $\ell = 1$ and $\ell = n$ are the extreme cases. If $\ell = 1$, then $k_1 = n$, and we have the Lagrange interpolation. (4.10) and (4.11) reduce in this case to (3.10) and (3.11) respectively, and the right-hand sides of (4.10) and (4.11) attain their minimum values. If $\ell = n$, then $k_1 = k_2 = \ldots = k_n = 1$, and we have the Abel-Gontscharoff [2, p.28] interpolation. (4.10) and (4.11) yield

$$(4.10)' \qquad |r^{(j)}(z)| \le M_n d^{n-j}, \quad z \in R, \quad j = 0,\ldots,n-1$$

and

$$(4.11)' \quad |r^{(j)}(z)| \le \frac{d^{n-j-1}}{2}\int_C |F^{(n)}(w)dw|, \quad z \in R, \quad j = 0,\ldots,n-1,$$

and the right-hand sides of (4.10) and (4.11) attain their maximum values. Furthermore, (by Lemma 4), (4.10)' and (4.11)' hold for all types of polynomial interpolations discussed in Theorem 4', regardless of the choice of the k's and the a's.

5. Disconjugacy in the unit disk.

In this section we consider the special case where the convex region is the unit disk.

Lemma 5.

Let $f(z)$ *be analytic in* $|z| < 1$ *and let* $f(a_i) = 0$, $|a_i| < 1$, $i = 1,\ldots,n$, $n \geq 2$. *Let*

(5.1) $$\tilde{M}_n = \text{Max}|f^{(n)}(\zeta)|, \quad \zeta \in H(a_1,\ldots,a_n)$$

where $H(a_1,\ldots,a_n)$ *is the convex hull of* $a_1,\ldots,a_n$, *and assume that* $\tilde{M}_n > 0$.

(i) *If* z *is a point of the closed linear segment* $[a_\ell a_k]$, $\ell \neq k$, $\ell,k = 1,\ldots,n$, *then*

(5.2) $$|f(z)| < \frac{\tilde{M}_n(1+|z|)^{n-j-1}[n-(n-2j)|z|]}{(n-j)!n}, \quad j = 0,\ldots,n-1.$$

(ii) *If* $z \in H(a_1,\ldots,a_n)$, *then*

(5.3) $$|f^{(j)}(z)| < \tilde{M}_n C_{jn} 2^{n-j}, \quad j = 0,\ldots,n-1.$$

where

(5.4) $$C_{on} = \frac{1}{nn!}\left(\frac{n-1}{n}\right)^{n-1}, \quad C_{jn} = \frac{j}{n(n-j)!}, \quad j = 1,\ldots,n-1.$$

(Cf. [1, p. 737].).

Proof.

Assume that $a_1 \neq a_2$, and let $z \in [a_1 a_2]$. We now apply Theorem 1 to the closed convex region $H(a_1,\ldots,a_n)$ and find upper bounds for $\sum_{s\in Q_{n-j,n}} \prod_{t=1}^{n-j} |z-a_{i_t}|$, $j = 0,\ldots,n-1$, where

$z \in [a_1,a_2]$ and $|a_i| < 1$, $i = 1,\ldots,n$. Since

$$|z-a| < 1 + |z|, \quad |z| < 1, \quad |a| < 1,$$

and by elementary geometry

$$|z-a_1||z-a_2| < (1+|z|)(1-|z|), \quad z \in [a_1,a_2], \ |a_1|,|a_2| < 1,$$

it follows that

$$(5.5) \qquad \prod_{i=1}^{n} |z-a_i| < (1+|z|)^{n-1}(1-|z|), \quad n \geq 2, \quad z\in[a_1,a_2],$$

[9, Theorem 2], [3, Theorem 2]. (2.2) and (5.5) now imply (5.2) for $j = 0$. In order to obtain upper bounds for $\prod_{t=1}^{n-j} |z-a_{i_t}|$, where $(i_1,\ldots,i_{n-j}) = s \in Q_{n-j,n}$, $2 \leq j \leq n-2$, we distinquish four cases. We recall that $(i_1,\ldots,i_{n-j})$ is an increasing sequence of integers such that $1 \leq i_1 < i_2 < \ldots < i_{n-j} \leq n$.

(a) If $i_1 = 1$, $i_2 = 2$ then, similarly to (5.5),

$$\prod_{t=1}^{n-j} |z-a_{i_t}| < (1+|z|)^{n-j-1}(1-|z|).$$

There are $\binom{n-2}{j}$ sequences of this type in $Q_{n-j,n}$.

(b) If $i_1 = 1$, $i_2 \neq 2$, then

$$\prod_{t=1}^{n-j} |z-a_{i_t}| < |z-a_1|(1+|z|)^{n-j-1} .$$

There are $\binom{n-2}{j-1}$ elements of this type in $Q_{n-j,n}$.

(c) If $i_1 = 2$, there are $\binom{n-2}{j-1}$ elements of this type in $Q_{n-j,n}$ and

$$\prod_{t=1}^{n-j} |z-a_{i_t}| < |z-a_2|(1+|z|)^{n-j-1} .$$

(d) If $i_1 \geq 3$, there are $\binom{n-2}{j-2}$ elements of this type in $Q_{n-j,n}$ and

$$\prod_{t=1}^{n-j} |z-a_{i_t}| < (1+|z|)^{n-j} .$$

Using the trivial inequality

$$|z-a_1| + |z-a_2| < 2, \quad z \in [a_1,a_2], \; |a_1|,|a_2| < 1,$$

we find that for $2 \leq j \leq n-2$

$$\sum_{s\in Q_{n-j,n}} \prod_{t=1}^{n-j} |z-a_{i_t}| < \frac{(n-2)!(1+|z|)^{n-j-1}(1-|z|)}{j!(n-j-2)!} +$$

$$+ \frac{2(n-2)!(1+|z|)^{n-j-1}}{(j-1)!(n-j-1)!} + \frac{(n-2)!(1+|z|)^{n-j}}{(j-2)!(n-j)!}$$

$$= \frac{(n-1)!(1+|z|)^{n-j-1}[n-(n-2j)|z|]}{j!(n-j)!} .$$

By (2.2), this leads to (5.2). For $j = 1, n-1$ the result follows in a similar way.

(ii). We note that if $a_1 = a_2 = \ldots = a_n = a^*$, then $H(a_1,\ldots,a_n) = a^*$, and since $f(a^*) = f'(a^*) = \ldots = f^{(n-1)}(a^*) = 0$, (5.3) is trivial. If $H(a_1,\ldots,a_n)$ is a linear segment $[a_\ell,a_k]$, $\ell \neq k$, $1 \leq \ell, k \leq m$ then (5.3) and (5.4) follow from (5.2) by observing that

$$(1+\rho)^{n-1}(1-\rho) \leq \left(\frac{n-1}{n}\right)^{n-1} \frac{2^n}{n}, \quad 0 \leq \rho \leq 1$$

and

$$(1+\rho)^{n-j-1}[n-(n-2j)\rho] \leq j\, 2^{n-j}, \; 0 \leq \rho \leq 1, \quad j = 1,\ldots,n-1.$$

If $H(a_1,\ldots,a_n)$ is a polygon then, by the maximum principle, for every $1 \leq j \leq n-1$, there exists a point z_j on the boundary of $H(a_1,\ldots,a_n)$ such that $|f^{(j)}(z_j)| = \mathrm{Max}|f^{(j)}(z)|$ for $z \in H(a_1,\ldots,a_n)$. Since the boundary of $H(a_1,\ldots,a_n)$ consists of segments of the type $[a_\ell a_k]$, $\ell \neq k$, it follows that $z_j \in [a_\ell, a_k]$ for some ℓ and k, $\ell \neq k$. (5.3) and (5.4) now follow from the above observation.

Remark.

If $f(z)$ is analytic in the closed unit disk $|z| \leq 1$ and vanishes at a_i, $|a_i| \leq 1$, $i = 1,\ldots,n$, then Lemma 5 holds with non-strict inequalities in (5.2) and (5.3). In this case the results are sharp and equalities hold if $f(z) = c(e^{i\gamma}+z)^{n-1}(e^{i\gamma}-z)$, where c is a constant and $0 \leq \gamma < 2\pi$.

In the next theorem we apply Lemma 5 to obtain disconjugacy criteria in $|z| < 1$.

Theorem 5.

Let $q_j(z)$, $j = 0,\ldots,n-1$, be analytic in $|z| < 1$.

(i) If

$$\sum_{j=0}^{n-1} \frac{|q_j(z)|(1+|z|)^{n-j-1}[n-(n-2j)|z|]}{n(n-j)!} \leq 1, \quad |z| < 1 \tag{5.6}$$

then equation (1.1) is disconjugate in $|z| < 1$.

(ii) If

$$\sum_{j=0}^{n-1} |q_j(z)| C_{jn} 2^{n-j} \leq 1, \quad |z| < 1, \tag{5.7}$$

where C_{jn}, $j = 0,\ldots,n-1$, *are given by* (5.4), *then equation* (1.1) *is disconjugate in* $|z| < 1$.

Note that assumption (5.6) implies that the functions $q_1(z),\ldots,q_{n-1}(z)$ are bounded in $|z| < 1$, but this does not necessarily hold for $q_0(z)$.

Proof.

(i). Suppose that equation (1.1) has a non-trivial solution $y(z)$ which vanishes at a_i, $|a_i| < 1$, $i = 1,\ldots,n$. If $z^* \in H(a_1,\ldots,a_n)$ is such that

$$|y^{(n)}(z^*)| = \underset{\zeta\in H(a_1,\ldots,a_n)}{\mathrm{Max}} |y^{(n)}(\zeta)| = \tilde{M}_n ,$$

then z^* is a point of some segment $[a_\ell, a_k]$, $\ell \neq k$. Hence, we may apply (5.2) in order to obtain an upper bound for $|y^{(j)}(z^*)|$, $j = 0,\ldots,n-1$. Equation (1.1) and inequality (5.2) lead us to

$$\tilde{M}_n = |y^{(n)}(z^*)| \leq \sum_{j=0}^{n-1} |q_j(z^*)y^{(j)}(z^*)| <$$

$$< \tilde{M}_n \sum_{j=0}^{n-1} \frac{|q_j(z^*)|(1+|z^*|)^{n-j-1}[n-(n-2j)|z^*|]}{n(n-j)!} .$$

Since $\tilde{M}_n > 0$, the last inequality contradicts (5.6).

(ii). (5.7) follows in a similar way from (5.3).

Remarks.

(i). If we replace the non-strict inequalities (5.6) and (5.7) by strict inequalities, we obtain disconjugacy criteria in the closed unit disk $|z| \leq 1$.

(ii). Since $C_{jn} < 1/(n-j)!$, $j = 0,\ldots,n-1$, it follows that (5.7) is sharper than the restriction of (3.2) to the unit disk.

(iii). Lemma 5 part (ii) can also be applied to estimate the remainder $r(z)$ in the Lagrange interpolation in the convex hull $H(a_1,\ldots,a_n)$. Let $F(z)$ be regular in a disk D of radius ρ and let $p(z) \in P_{n-1}$ satisfy $p(a_i) = F(a_i)$, $a_i \in D$, $i = 1,\ldots,n$. If $r(z) = F(z)-p(z)$, then

$$|r^{(j)}(z)| < \mathfrak{M}_n C_{jn}(2\rho)^{n-j}, \quad j = 0,\ldots,n-1, \quad z \in H(a_1,\ldots,a_n),$$

where $\mathfrak{M}_n = \mathrm{Max}|F^{(n)}(\zeta)|$ for $\zeta \in H(a_1,\ldots,a_n)$. (See also (3.10).)

We conclude with the following corollary of Theorem 5.

<u>Let</u> $q_j(z)$, $j = 0,\ldots,n-1$, $n \geq 3$, <u>be analytic in</u> $|z| < 1$, <u>and assume that there exists a positive constant</u> $A < \infty$, <u>such that</u>

$$|q_0(z)| < \frac{A}{1-|z|}, \quad |q_j(z)| < A, \quad j = 1,\ldots,n-1, \quad |z| < 1.$$

<u>Then the differential equation</u> (1.1) <u>is non-oscillatory in</u> $|z| < 1$, <u>i.e., every non-trivial solution of</u> (1.1) <u>has a finite number of zeros in</u> $|z| < 1$. (Cf.[3, Theorem 4].)

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Carnegie-Mellon University
Pittsburgh, Pennsylvania