ADDITIVE FUNCTIONALS ON SPACES WITH NON-ABSOLUTELY-CONTINUOUS NORM

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Let (T, Σ, μ) be a complete finite positive measure space, and let $(X, ||\cdot||)$ be a seminormed vector space of real valued measurable functions defined on T. We suppose that (a) The space $(\mathbf{X}, ||\cdot||)$ obtained by identifying functions of X which are equal a.e. is a Banach space. (b) If $x \in X$ and E is a measurable set then $x I_E \in X$ where I_E is the characteristic function of E, (c) If $x \in X$ and y is a real valued measurable function satisfying $|y|(t) \leq |x|(t)$ a.e. then $y \in X$, (d) $\mathcal{L}^{\infty}(\mu) \subset X$.

The problem of characterizing nonlinear functionals F on X which admit integral representations of the form $F(x) = \int_T \varphi(x(t), t)\, d\mu(t)$ where φ is a Caratheodory function has been discussed in Drewnowski and Orlicz [1] and Mizel [3]. In [3] the case $X = \mathcal{L}^p(\mu)$, $1 \leq p \leq \infty$, is discussed while in [1] X is assumed to be of absolutely continuous norm. The purpose of the present note is to extend these results to the case when X is not necessarily of absolutely continuous norm.

We recall a few definitions and notations before presenting the main results of the paper.

In the sequel the set of all measurable real-valued functions on the complete finite positive measure space (T, Σ, μ) is denoted by $\mathfrak{m}$, $\mathcal{L}^{\infty}(\mu) \subset \mathfrak{m}$ is the subspace of essentially bounded functions, and $X \subset \mathfrak{m}$ denotes a fixed subspace satisfying (a), (b), (c) and (d) above. We denote the real line by R. A functional $F: X \to R$ is said to be additive (orthogonally additive) if $F(x + y) = F(x) + F(y)$ whenever x,y are of disjoint support i.e. $\mu\{t | x(t)y(t) \neq 0\} = 0$. A function $\varphi: R \times T \to R$ is a Caratheodory function if for each $\gamma \in R$, $\varphi(\gamma,\cdot)$ is a measurable function, and for t a.e., $\varphi(\cdot,t)$ is a continuous function on R. If x is a real valued function then $\varphi \circ x(t) = \varphi(x(t),t)$. If x is a constant function taking the value h, $\varphi \circ x$ will be denoted by φ_h.

We proceed to the representation theorem of the paper. We present separately the necessary and sufficient conditions guaranteeing the representation. We recall a lemma given in [3] (see also Krasnoselskii [2]).

Lemma 1. If φ is a Caratheodory function and η is a positive real number then there exists a measurable set S_η such that (1) $\mu(T-S_\eta) < \eta$ and (2) $\varphi(\cdot,t)$ is uniformly continuous on each bounded interval $J \subset R$, uniformly for $t \in S_\eta$.

Theorem 1. Let F be a real valued function on X such that (i) F is additive, (ii) F is uniformly continuous on every ball in $(\mathcal{L}^\infty(\mu), \|\cdot\|_\infty)$, (iii) F is continuous with respect to dominated a.e. convergence, i.e. whenever $\{x_n\}$, $x,y \in X$ satisfy $x_n \to x$ a.e. and $|x_n(t)| \leq |y(t)|$ a.e. for all n, then $F(x_n) \to F(x)$.

Then there exists a Caratheodory function φ on $R \times T$ such that for all $x \in X$

$$(*) \qquad F(x) = \int_T \varphi \circ x \, d\mu.$$

Proof. Since the restriction of F to $\mathcal{L}^{\infty}(\mu)$ satisfies the hypotheses in theorem 1 of [3], it follows that there exists a Caratheodory function φ such that for all $x \in \mathcal{L}^{\infty}(\mu)$ the representation (*) is valid. Let now $x \in X$. It follows from conditions (ii) and (iii) that the set function $F(xI_A)$ on Σ is absolutely continuous with respect to μ. Thus it has a Radon-Nikodym derivative $g \in \mathcal{L}^1(\mu)$, unique up to a null set, such that (a) $F(xI_A) = \int_A g\, d\mu$, for $A \in \Sigma$. We proceed to verify that $g = \varphi \circ x$ a.e. Let for each real number $C \geq 0$,

$A_C = \{t \mid |x(t)| \leq C\}$. Given a fixed sequence of real numbers $\{\eta_m\}$ converging to 0 let us denote the corresponding measurable sets S_{η_m} whose existence is assured by lemma 1, by S_m. It is easily verified that the sequences of measurable sets $\{A_n\}$ and $\{S_m\}$ both converge to T as $n, m \to \infty$. Let $B_{m,n} = S_m \cap A_n$. We proceed to show that $g = \varphi \circ x$ a.e. on $B_{m,n}$. Since $|x(t)| \leq n$ on $B_{m,n}$ there exists a sequence $\{y_k\}$ of simple functions such that $y_k \to x$ uniformly on $B_{m,n}$ and $|y_k| \leq |x|$. By lemma 1, it follows that $\varphi \circ y_k \to \varphi \circ x$ uniformly on $B_{m,n}$ Hence if E is any measurable subset of $B_{m,n}$ it follows that

$$\lim \int_E \varphi \circ y_k \, d\mu = \int_E \varphi \circ x \, d\mu .$$

Further since $y_k I_E \to x I_E$ a.e. and $|y_k| \leq |x|$ it follows from property (b) of X and condition (iii) that $F(y_k I_E) \to F(x I_E)$ for each $E \in \Sigma$. Thus $\int_E g\, d\mu = \int_E \varphi \circ x\, d\mu$ for all $E \subset B_{m,n}$. Hence $g = \varphi \circ x$ a.e. on $B_{m,n}$. Since $T - \bigcup_{m,n} B_{m,n}$ is a null set, this completes the proof of the theorem.

We proceed next to the converse of Theorem 1.

Theorem 2. If φ is a Caratheodory function on $R \times T$ such that $\varphi(o, t) = o$ for t a.e. and $\varphi \circ x \in \mathcal{L}^1(\mu)$ for each $x \in X$, then the functional $F(x) = \int_T \varphi \circ x \, d\mu$ satisfies conditions (i), (ii) and (iii) of Theorem 1.

Before proceeding to the proof we note that by the converse assertion in Theorem 1 of [3] F certainly satisfies conditions (i) and (ii) above. We verify that F also satisfies the condition (iii) after establishing the following lemmas.

Lemma 2. If φ is a Caratheodory function and $x \in \mathcal{L}^\infty(\mu)$ then the (almost everywhere finite) function α_x defined below is measurable.

$$\alpha_x(\xi) = \sup \{ |\varphi_h(\xi)| \ \big| \ |h| \leq |x(\xi)| \}$$

Further for each $\epsilon > o$ there is a function $y \in \mathcal{L}^\infty(\mu)$ such that (1) $|y| \leq |x|$ and (2) $\big| |(\varphi \circ y)(\xi)| - \alpha_x(\xi) \big| \leq \epsilon$ a.e.

Proof. First let x be a simple function of the form $\sum_{m=1}^{n} c_m I_{A_m}$. Let $\alpha'_x(\xi) = \sup_{|h| \leq |x(\xi)|} |\varphi_h(\xi)|$, h rational. Clearly α'_x is a measurable function since on each of the sets A_m it is the supremum of a countable family of measurable functions. Moreover since $\varphi(\cdot, \xi)$ is continuous for ξ a.e. it is verified that $\alpha_x = \alpha'_x$ a.e. Thus α_x is also measurable. Now each $x \in \mathcal{L}^\infty(\mu)$ is the pointwise limit of a sequence of simple functions $\{x_n\}$ satisfying $|x_n| \uparrow |x|$ a.e. Therefore $\alpha_{x_n} \uparrow \alpha_x$ a.e. and the measurability result holds for such x as well. Notice that by a similar argument the functions $\alpha_x^{\pm}$ defined by

$$\alpha_x^+(\xi) = \sup_{o < h \leq |x(\xi)|} |\varphi_h(\xi)|, \quad \alpha_x^-(\xi) = \sup_{-|x(\xi)| \leq h < o} |\varphi_h(\xi)|$$

are both measurable.

Now let $\{\eta_m\}$ be a sequence of positive reals such that $\eta_m \to 0$ and let the corresponding sets $\{S_{\eta_m}\}$ whose existence is assured by lemma 1 be denoted by $\{S_m\}$. For each η_m there exists by lemma 1 a δ^m such that $|\varphi(h, \xi) - \varphi(h', \xi)| < \epsilon$ for all $\xi \in S_m$ whenever $|h-h'| \leq \delta^m$ and $|h|, |h'| \leq \| x \|_\infty$. Let $A^+ = \{\xi | \alpha_x(\xi) = \sup_{o \leq h \leq |x(\xi)|} |\varphi(h, \xi)|\} = \{\xi | \alpha_x(\xi) = \alpha_x^+(\xi)\}$. By the results above A^+ is a measurable set. Now define sets $E_{mj}^{\pm}$ as follows.

$$E_{mo}^{\pm} = \{\xi \in S_m | |\alpha_x(\xi)| < \epsilon\}, \quad E_{m1}^+ = \{\xi \in S_m \cap A^+ | ||\varphi(\delta^m, \xi)| - \alpha_x(\xi)| < \epsilon\} \sim E_{mo}$$

$E_{m1}^- = \{\xi \in S_m \sim A^+ | ||\varphi(-\delta^m, \xi)| - \alpha_x(\xi)| < \epsilon\} \sim E_{mo}$, and more generally

$$E_{mj}^+ = \{\xi \in S_m \cap A^+ | ||\varphi(j\delta^m, \xi)| - \alpha_x(\xi)| < \epsilon\} \sim \bigcup_{i \leq j-1} (E_{mi}^+ \cup E_{mi}^-).$$

$$E_{mj}^- = \{\xi \in S_m \sim A^+ | ||\varphi(-j\delta^m, \xi)| - \alpha_x(\xi)| < \epsilon\} \sim \bigcup_{i \leq j-1} (E_{mi}^+ \cup E_{mi}^-).$$

Clearly each set $E_{mj}^{\pm}$ is measurable and if $y_m^\epsilon = \sum_{1 \leq j} (j\,\delta^m I_{E_{mj}} + -j\delta^m I_{E_{mj}})$ then $||\varphi \circ y_m^\epsilon(\xi)| - \alpha_x(\xi)| < \epsilon$ for a.e. $\xi \in S_{\eta_m}$. Now let $y^\epsilon = \sum_{m \geq 1} y_m^\epsilon I_{S_m - S_{m-1}}$. From the construction of the $\{y_m^\epsilon\}$ it is verified that $|y^\epsilon| \leq |x|$ and satisfies $||\varphi o y^\epsilon| - \alpha_x| < \epsilon$ except on the null set $T \sim \bigcup_m S_{\eta_m}$. By construction each y_m^ϵ is a $k\delta^m$-valued function such that $\varphi \circ y_m^\epsilon$ approximates α_x within ϵ on S_{η_m}.

<u>Lemma 3</u>. If $x \in \mathcal{L}^{\infty}(\mu)$ then there exists a function $y_x \in \mathcal{L}^{\infty}(\mu)$ such that (1) $|y_x| \leq |x|$ a.e. and (2) $|\varphi \circ y_x| = \alpha_x$ a.e..

Proof. Let $\{\epsilon_n\}$ be a sequence of positive real numbers such that (1) $\epsilon_n \to 0$. Then the functions y^{ϵ_n} (following the notation in lemma 2) are all dominated by x and by construction converge pointwise on each set S_{η_m}. In fact, denoting $\cup S_{n_m} = S$, one has

$$(**) \quad \lim_{n \to \infty} y^{\epsilon_n}(\xi) = \begin{cases} \text{minimum } c \geq 0 \text{ s.t. } |\varphi(c,\xi)| = \alpha_x(\xi) & \xi \in S \cap A^+ \\ \text{maximum } c \leq 0 \text{ s.t. } |\varphi(c,\xi)| = \alpha_x(\xi) & \xi \in S \sim A^+ \end{cases}$$

Let $y_x(\xi) = \lim_{n \to \infty} y^{\epsilon_n}(\xi)$ for $\xi \in S$ and $= 0$ otherwise. Clearly y_x is measurable and $|y_x| \leq |x|$ since $|y^{\epsilon_n}| \leq |x|$ for all $n \geq 1$. From (**) it follows that $|\varphi \circ y_x| = \alpha_x$ a.e.

<u>Lemma 4</u>. For each $x \in X$ there exists a function $y_x \in X$ such that (1) $|y_x| \leq |x|$ a.e. and (2) $|\varphi \circ y_x| = \alpha_x$ a.e.

Proof. Define recursively a sequence of pairwise disjoint measurable sets $\{E_n\}$ as follows.

$E_1 = \{\xi \mid |x(\xi)| \leq 1\}$, $E_2 = \{\xi \mid |x(\xi)| \leq 2\} \sim E_1$

and in general

$E_n = \{\xi \mid |x(\xi)| \leq n\} \sim \bigcup_{j=1}^{n-1} E_j$. It follows from the preceding lemma that for each integer $n \geq 1$ there is a function y_n in $\mathcal{L}^{\infty}(\mu)$ such that $|\varphi \circ y_n(\xi)| = \alpha_x(\xi)$ for all $\xi \in E_n$ and $y_n(\xi) = 0$ otherwise. Let $y_x = \Sigma\, y_n I_{E_n}$. Then on each set E_n $|y_x| \leq |x|$ and $|\varphi \circ y_x| = \alpha_x$. Since $\mu(T \sim \cup E_n) = 0$ it follows that $|y_x| \leq |x|$ a.e., so that $y_x \in X$, and $|\varphi \circ y_x| = \alpha_x$ a.e.

We complete the proof of Theorem 2 by proving the following corollary to lemma 4.

Corollary. If φ is a Caratheodary function and if for each $x \in X$, $\varphi \circ x \in \mathcal{L}^1(\mu)$ then the functional $F(x) = \int \varphi \circ x \, d\mu$ has the property (iii) of Theorem 1.

Proof. Let $z \in X$ and $\{x_n\}$ be a sequence in X such that $|x_n| \leq |z|$. Then by the construction of y_z it follows that $|\varphi o x_n| \leq |\varphi o y_z|$ and $\varphi o y_z \in \mathcal{L}^1(\mu)$. If further the sequence $\{x_n\}$ converges to some function x a.e. where $x \in X$ then by the continuity of $\varphi(\cdot, \xi)$ it follows that $\varphi \circ x_n \to \varphi \circ x$ a.e. Since $|\varphi \circ x_n| \leq |\varphi \circ y_z|$ we have by the dominated convergence theorem

$$F(x_n) \to F(x).$$

In conclusion we mention that Theorems 1 and 2 can be extended to the case when (T, Σ, μ) is σ-finite measure space. As this generalization is straightforward and the proof is very similar to that of Theorem 2 in [3] once the results for the finite case are obtained, we content ourselves by stating the theorem without proof. In the next theorem (T, Σ, μ) is a complete σ-finite measure space.

Theorem 3. Let $(X, ||\cdot||)$ be as in the introduction except that X satisfies instead of condition (c) the following condition

(c') $f \in \mathcal{L}^\infty(\mu)$, $E \in \Sigma$ s.t. $\mu(E) < \infty$ implies $f I_E \in X$.

Suppose the function $F: X \to R$ satisfies conditions (i) and (iii) of theorem 1 as well as the condition,

(ii') F is uniformly continuous on each set of the form $(X \cap Y, ||\cdot||_\infty)$ where Y is a bounded subset of $\mathcal{L}^\infty(u)$ supported by a set of finite measure.

Then there exists a Caratheodory function φ, satisfying $\varphi \circ x \in \mathcal{L}^1(\mu)$ for each $x \in X$, for which the following representation holds

(*) $$F(x) = \int_T \varphi \circ x \, d\mu.$$

Conversely, each Caratheodory function φ which satisfies (1) $\varphi \circ x \in \mathcal{L}^1(u)$ for each $x \in X$, and (2) $\varphi(0,\xi) = 0$ a.e., determines by means of (*) a function F which satisfies (i), (iii) and (ii').

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