LINEAR DIFFERENCE EQUATIONS:

CLOSEDNESS OF COVARIANT SEQUENCES

By

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the transition operator $U(n_1, n_0)$ (defined by [1;(4.1)]) is compact for at least one pair n_0, n_1 of integers, $n_1 \geqq n_0 \geqq 0$. This assumption is verified, for instance, in applications to certain differential equations with delays in finite-dimensional spaces; this application will be discussed elsewhere.

The possibility of exploiting the compactness assumption to obtain further information on important covariant sequences was first suggested by C. V. Coffman, and the author is indebted to him for further useful discussions during the preparation of this paper.

We shall use the definitions and results of [1] freely.

2. Subcomplete manifolds in a Banach space.

If X is a Banach space, a linear manifold $Y \subset X$ is *subcomplete (in* X) if $Y = \Phi Z$ for some Banach space Z and some bounded linear mapping $\Phi : Z \to X$. If N is the kernel of Φ, then $\Phi = \Psi\Omega$, where $\Omega : Z \to Z/N$ is the canonical epimorphism and $\Psi : Z/N \to X$ is bounded and injective; since Z/N is a Banach space and $\Phi Z = \Psi(Z/N)$, replacement of Φ by Ψ shows that there is no loss in the definition of a subcomplete linear manifold if Φ is required to be injective. A subspace (closed linear manifold) of X is of course subcomplete in X. We do not propose to pursue the study of subcomplete manifolds here beyond the contents of the following lemma.

2.1. *Lemma. Let* X, Z *be Banach spaces,* $\Phi : Z \to X$ *a bounded linear mapping, and* Y *a subcomplete linear manifold in* X. *Then:*

(a): $\Phi^{-1}(Y)$ *is subcomplete in* Z;

(b): *if* $X = \Phi Z + Y$, *then* Y *is closed in* X *if and only if* $\Phi^{-1}(Y)$ *is closed in* Z;

(c): *if* $X = \Phi Z + Y$ *and* Φ *is compact, then* Y *is closed and has finite co-dimension in* X.

<u>Proof</u>. 1. In the proof, every linear manifold in a normed space will be assumed to carry the induced norm. By assumption, $Y = \Phi' Z'$, where Z' is a Banach space and $\Phi' : Z' \to X$ is bounded, linear, and injective. Let $W = Z \oplus Z'$ be the outer direct sum of Z, Z', with the maximum norm, i.e., $\Sigma(W) = \Sigma(Z) \oplus \Sigma(Z')$; this is a Banach space. Let $\Pi : W \to Z$, $\Pi' : W \to Z'$ be the canonical projections, and set $F = \Phi\Pi + \Phi'\Pi' : W \to X$.

Let Z'' be the kernel of F, a subspace of W, hence a Banach space, and let $J_0 : Z'' \to W$ be the inclusion map. We claim that $\Phi^{-1}(Y) = \Pi J_0 Z''$; since $\Pi J_0 : Z'' \to Z$ is bounded and linear, this will prove (a). Indeed, given $z \in Z$, we have $z \in \Phi^{-1}(Y)$, i.e., $\Phi z \in Y = \Phi' Z'$, if and only if there exists $z' \in Z'$ with $\Phi z = -\Phi' z'$, i.e., with $F(z \oplus z') = 0$, i.e., with $z \oplus z' \in Z''$; this happens if and only if $z \in \Pi Z'' = \Pi J_0 Z''$, and our claim is established.

We note that $FJ_0 = 0$, and therefore

$$\Phi\Pi J_0 = (F - \Phi'\Pi')J_0 = -\Phi'\Pi' J_0 . \tag{2.1}$$

If $z'' \in Z''$ and $\Pi J_0 z'' = 0$, we have $\Pi z'' = 0$ and, by (2.1), $\Phi'\Pi' z'' = \Phi'\Pi' J_0 z'' = 0$; since Φ' is injective, $\Pi' z'' = 0$; thus $z'' = 0$, and we conclude that ΠJ_0 is injective. We may write $\Phi' = J'\Psi'$ and $J_0 = J''\Psi''$, where $J' : Y \to X$ and

$J'': \Phi^{-1}(Y) \to Z$ are inclusion maps and $\Psi': Z' \to Y$, $\Psi'': Z'' \to \Phi^{-1}(Y)$ are bounded and bijective. From (2.1) we have, in particular,

$$(2.2) \qquad \Phi J'' = -\Phi'\Pi' J_0 \Psi''^{-1}: \Phi^{-1}(Y) \to X .$$

2. From now on, we assume that

$$(2.3) \qquad X = \Phi Z + Y = \Phi Z + \Phi' Z' = FW .$$

Since $F: W \to X$ is surjective, the Open-Mapping Theorem implies that $F\Sigma(W)$ is a 0-neighborhood in X; i.e., there exists $k > 0$ such that

$$(2.4) \qquad k\Sigma(X) \subset F\Sigma(W) \subset \Phi\Pi\Sigma(W) + \Phi'\Pi'\Sigma(W) = \Phi\Sigma(Z) + \Phi'\Sigma(Z') .$$

The "only if" part of the statement of (b) is trivial. To prove the "if" part, assume that $\Phi^{-1}(Y) = \Pi J_0 Z''$ is a subspace of Z, hence a Banach space. By the Open-Mapping Theorem, the linear mapping $\Psi''^{-1}: \Phi^{-1}(Y) \to Z''$ is bounded.

Let $u \in Y$, $u \neq 0$, be given. By (2.4) there exist $z \in \Sigma(Z)$, $z' \in \Sigma(Z')$ such that $ku = \|u\|(\Phi z + \Phi' z')$. Since $u, \Phi' z' \in Y$ and $\|u\| \neq 0$, we have $z \in \Phi^{-1}(Y)$, and (2.2) implies $ku = \|u\|\Phi'(-\Pi' J_0 \Psi''^{-1} z + z')$; applying Ψ'^{-1} and taking norms, we find $\|\Psi'^{-1}u\| \leqq k^{-1}(\|\Psi''^{-1}\| + 1)\|u\|$. Thus Ψ'^{-1} is bounded, $\Psi': Z' \to Y$ is an isomorphism, Y is a Banach space; hence Y is a subspace of X, and (b) is proved.

3. Assume now, in addition to (2.3), that Φ is compact, so that $\Phi\Sigma(Z)$ is precompact, hence totally bounded in X: there exists a finite set $M \subset X$ such that

$$\Phi\Sigma(Z) \subset M + \frac{1}{2} k\,\Sigma(X) \subset M + \frac{1}{2}\,\Phi\Sigma(Z) + \frac{1}{2}\,\Phi'\,\Sigma(Z')\,, \tag{2.5}$$

where k is as in (2.4). Denote by W the finite-dimensional subspace spanned by M.

Let $z_0 \in \Sigma(Z)$ be given. By (2.5) there exist sequences (z_n) in $\Sigma(Z)$, (z_n') in $\Sigma(Z')$, and (y_n) in M, such that

$$\Phi z_n = y_n + \frac{1}{2}\Phi\, z_{n+1} + \frac{1}{2}\Phi'\, z_n' \qquad n = 0,1,\ldots\,. \tag{2.6}$$

From (2.6) we get

$$\Phi z_0 = \sum_{j=0}^{n} 2^{-j} y_j + 2^{-n-1}\Phi z_{n+1} + \Phi' \sum_{j=0}^{n} 2^{-j-1} z_j' \qquad n = 0,1,\ldots\,. \tag{2.7}$$

Now M is bounded in the subspace W, Z' is a Banach space, and Φ, Φ' are continuous. Therefore the limit of each term of (2.7) exists in X as $n \to \infty$, and $\Phi z_0 \in W + \Phi' Z' = W + Y$. But $z_0 \in \Sigma(Z)$ was arbitrary, so that, by linearity, $\Phi Z \subset W + Y$; and by (2.3),

$$X = W + Y\;; \tag{2.8}$$

thus Y has finite co-dimension in X.

Finally, if $J_W: W \to X$ is the inclusion map, the assumptions of (b) are satisfied with Z, Φ replaced by W, J_W : indeed, (2.3) is replaced by (2.8), and $J_W^{-1}(Y) = W \cap Y$ is finite dimensional and trivially closed in W. It follows by (b) that Y is a subspace of X. This completes the proof of (c).

<u>Remark</u>. Parts (b) and (c) of Lemma 2.1 are related to the properties of "dihedra" as defined and studied in [2;pp. 4, 10-13]. (2.4) implies that a dihedron formed by subcomplete manifolds is "gaping"; and (b) is a generalization for such dihedra of the fact that the manifolds forming a gaping dihedron are closed if and only if their intersection is closed [2;11.J].

3. Subcomplete covariant sequences.

We consider equations (I), (II) with a given $A \in \underset{\sim}{S}_{[1]}(\tilde{X})$. We recall that a sequence Y of linear manifolds in X is a *covariant sequence* (*for* A) if it satisfies [1;(5.1)], i.e.,

$$Y(n-1) = (A(n))^{-1}(Y(n)) \qquad n = 1,2,\ldots ,$$

whence, equivalently,

$$(3.1) \qquad Y(n_0) = (U(n,n_0))^{-1}(Y(n)) \qquad n \geq n_0 \geq 0 .$$

A covariant sequence is *closed* if its terms are subspaces. We shall similarly say that a covariant sequence is *subcomplete* if its terms are subcomplete in X. It is sometimes useful to restrict verification to a subset without further assumptions:

3.1. *Lemma*. *A covariant sequence* Y *is closed* [subcomplete] *if* Y(n) *is closed* [*subcomplete*] *for each* n *in an infinite set*.

Proof. If Y(n) is closed [subcomplete] and $n_0 \leq n$, then $Y(n_0)$ is closed [subcomplete] by (3.1) [and Lemma 2.1, (a) with $Z = X$, $\Phi = U(n,n_0)$].

In [1;Lemma 5.2] a number of equivalent conditions for closed covariant sequences were given, and such a sequence that satisfied them was called a *regular* covariant sequence. A Remark to that lemma points out that in their purely algebraic forms these conditions remain equivalent for covariant sequences that are not necessarily closed: we have, in particular, the following equivalence.

3.2. Lemma. *If* Y *is a covariant sequence and* $m \in \omega_{[1]}$, *the following statements are equivalent*:

(a): *for every* $u \in X$, (II) *with* $f = \chi^m_{[1]} u$ *has a solution that lies eventually in* Y;

(b): $U(m,0)X + Y(m) = X$;

(c): $U(n,n_0)X + Y(n) = X$ *if* $0 \leqq n_0 \leqq n \leqq m$.

A covariant sequence satisfying these equivalent conditions for all $m \in \omega_{[1]}$ shall be termed *algebraically regular*; thus a covariant sequence is regular when it is both closed and algebraically regular.

3.3. Lemma. *A covariant sequence* Y *is algebraically regular if and only if condition* (a) or (b) *of Lemma 3.2 holds for each* m *in an infinite subset of* ω .

Proof. Lemma 3.2 (cf. [1;Lemma 5.3]).

The fundamental application of Lemma 2.1 is the following pair of closedness criteria for subcomplete algebraically regular covariant sequences.

3.4. Lemma. *Assume that the subcomplete covariant sequence* Y *is algebraically regular*. *Then*:

(a): Y *is closed, hence regular, if and only if* Y(m) *is closed in* X *for some* $m \in \omega$;

(b): *if* $U(n_1,n_0)$ *is compact for some* $n_0,n_1 \in \omega$, $n_1 \geqq n_0 \geqq 0$, *then* Y *is closed, hence regular, and its terms have constant finite co-dimension in* X.

Proof. *Proof of* (a): It will be enough to show that, for every fixed $m \in \omega_{[1]}$, $Y(m)$ is closed if and only if $Y(0)$ is closed. But this follows from Lemma 2.1 with Z, Φ, Y replaced by X, $U(m,0)$, $Y(m)$, respectively: by assumption, $Y(m)$ is subcomplete in X; by the assumption and Lemma 3.2, (b), $U(m,0)X + Y(m) = X$; and by (3.1), $(U(m,0))^{-1}(Y(m)) = Y(0)$.

Proof of (b): We apply Lemma 2.1, (c) with Z, Φ, Y replaced by X, $U(n_1,n_0)$, $Y(n_1)$, and use the assumptions, Lemma 3.2, (c), and (3.1) as in the first part of the proof; we conclude that $Y(n_1)$ is closed and has finite co-dimension. By Part (a), Y is closed, hence regular. The terms of Y then have constant finite co-dimension in X by [1;Lemma 5.4].

4. *Subcomplete* $(\underset{\sim}{b},\underset{\sim}{d})$-*sequences*.

We refer to the concepts discussed in [1;Sections 6,8]. If a Banach sequence space $\underset{\sim}{d} \in b\ell_K$ is given [1;Section 3], we have the corresponding sequence $X_{O\underset{\sim}{d}}$ defined by $X_{O\underset{\sim}{d}}(m) =$ $= \{x(m) : x$ is a $\underset{\sim}{d}$-solution of $(I_{[m]})\}$. $X_{O\underset{\sim}{d}}$ is a covariant sequence [1;Lemma 6.2] and it is subcomplete [1;Lemma 6.3]. This allows an immediate application of Lemma 3.4 (see Theorem 4.3).

Let $(\underset{\sim}{b},\underset{\sim}{d})$ be an $\mathfrak{f}$- pair of Banach sequence spaces. We recall that a covariant sequence Y is a $(\underset{\sim}{b},\underset{\sim}{d})$-*sequence for* A if $Y \subset X_{O\underset{\sim}{d}}$ and if there exists a number $K_Y \geqq 0$ such that for every $f \in k_O\underset{\sim}{b}_{[1]}(X)$ and every $\rho > 1$ there is a solution x of (II) that lies eventually in Y (hence is a $\underset{\sim}{d}$-solution) and satisfies $\|x\|_{\underset{\sim}{d}} \leqq \rho K_Y \|f\|_{\underset{\sim}{b}}$. We then have the following application of Lemma 3.4.

4.1. *Theorem*. *Let* Y *be a subcomplete* $(\underset{\sim}{b},\underset{\sim}{d})$-*sequence for an* $\mathfrak{f}$-*pair* $(\underset{\sim}{b},\underset{\sim}{d})$ *such that* supp $(\underset{\sim}{b})$ *is an infinite set* (*in particular*, *for a* $\mathfrak{t}$-*pair or a* $\mathfrak{t}^{\rightarrow}$ *pair* $(\underset{\sim}{b},\underset{\sim}{d})$). *Then*:

(a): Y *is closed*, *hence regular*, *if and only if* $Y(m)$ *is closed in* X *for some* $m \in \omega$;

(b): *if* $U(n_1,n_O)$ *is compact for some* $n_O, n_1 \in \omega$, $n_1 \geqq n_O \geqq 0$, *then* Y *is closed*, *hence regular*, *and its terms have constant finite co-dimension in* X.

Proof. For every $u \in X$ and every $m \neq 0$ in supp $(\underset{\sim}{b})$ we have $f = \chi^{m}_{[1]} u \in k_0 \underset{\sim}{b}_{[1]}(X)$, and by assumption there exists a solution of (II) with this f that lies eventually in Y. By Lemmas 3.2, 3.3, Y is algebraically regular (this is the algebraic form of [1;Lemma 8.6]). The conclusion follows from Lemma 3.4.

This theorem applies in particular to $X_{O\underset{\sim}{d}}$, which is a $(\underset{\sim}{b},\underset{\sim}{d})$-sequence provided some such sequence exists at all [1;Lemma 8.4]; but we can give a somewhat more instructive form to the result about $X_{O\underset{\sim}{d}}$, by making it independent of the choice of $\underset{\sim}{b}$.

4.2. _Lemma_. _For given_ $\underset{\sim}{d} \in b\ell_K$ _the following statements are equivalent_:

(a): _the covariant sequence_ $X_{O\underset{\sim}{d}}$ _is algebraically regular_;

(b): _for every_ $m \in \omega_{[1]}$ _there exists a number_ $K_m \geqq 0$ _such that for every_ $\rho > 1$ _and every_ $u \in X$ _there exists a_ $\underset{\sim}{d}$-_solution_ x _of_ (II) _with_ $f = \chi^{m}_{[1]} u$ _such that_

$\|x\|_{\underset{\sim}{d}} \leqq \rho K_m \|u\|$;

(c): _there exists a space_ $\underset{\sim}{b} \in b\ell_K$ _such that_ $(\underset{\sim}{b},\underset{\sim}{d})$ _is admissible_;

(d): *there exists a space* $\underset{\sim}{b}\in b\mathcal{f}$ *such that* supp $(\underset{\sim}{b})$ *is an infinite set and such that* $(\underset{\sim}{b},\underset{\sim}{d})$ *is admissible*.

Proof. (a) *implies* (b): Let $m\in\omega_{[1]}$ be fixed and let $\underset{\sim}{b}_m\in b\mathcal{f}$ be the space $\{\varphi\in\underset{\sim}{s}: \varphi(n) = 0,\ n \neq m\}$, with $\|\varphi\|_{\underset{\sim}{b}_m} = |\varphi(m)|$: $\underset{\sim}{b}_m$ is congruent to R and its elements are of the form $\varphi = \chi^m\varphi(m)$. We claim that $(\underset{\sim}{b}_m,\underset{\sim}{d})$ is admissible for A. Indeed, $f\in\underset{\sim}{b}_{m[1]}(X)$ if and only if $f = \chi^m_{[1]}u$, $u\in X$, and $\|f\|_{\underset{\sim}{b}_m} = \|u\|$. Since $X_{O\underset{\sim}{d}}$ is algebraically regular, (II) has, for this f, a solution x that lies eventually in $X_{O\underset{\sim}{d}}$, hence is a $\underset{\sim}{d}$-solution. Statement (b) then follows from [1;Theorem 8.1]. (Alternatively, Lemma 3.2, (b) and the implication (2.3) ⇒ (2.4) in the proof of Lemma 2.1 could have been used.)

(b) *implies* (c): Set $K_0 = 1$ and let K_m, $m\in\omega_{[1]}$, be as in (b). Define the space $\underset{\sim}{b}\in b\mathcal{f}_K$ as the set $\{\varphi\in\underset{\sim}{s}: \sum_{n=0}^{\infty} K_n|\varphi(n)|<\infty\}$ with $\|\varphi\|_{\underset{\sim}{b}} = \sum_{n=0}^{\infty} K_n|\varphi(n)|$; it is obvious that $\underset{\sim}{b}$ is in $\mathcal{f}_K$, and that it is congruent to $\underset{\sim}{\ell}^1$, hence complete; so it is indeed in $b\mathcal{f}_K$. It is further clear that $k_O\underset{\sim}{b}$ is dense in $\underset{\sim}{b}$, i.e., $k\underset{\sim}{b} = \underset{\sim}{b}$ ($\underset{\sim}{b}$ is lean). To show that $(\underset{\sim}{b},\underset{\sim}{d}) = (k\underset{\sim}{b},\underset{\sim}{d})$ is admissible it is sufficient to show, by [1;Theorem 8.8], that $X_{O\underset{\sim}{d}}$ is a $(\underset{\sim}{b},\underset{\sim}{d})$-sequence for A.

Let $\rho > 1$ and $f \in k_{0\underset{\sim}{b}[1]}(X)$ be given; thus $f = \sum_{n=1}^{s} \chi^{n}_{[1]} f(n)$, $s = s(f)$; by (b) there exists, for $m = 1,\ldots,s$, a $\underset{\sim}{d}$-solution x_m of (II) with $\chi^{m}_{[1]} f(m)$ instead of f, such that $\|x_m\|_{\underset{\sim}{d}} \leqq \rho K_m \|f(m)\|$. Thus $x = \sum_{m=1}^{s} x_m$ is a $\underset{\sim}{d}$-solution of (II) with the given f, so that x is eventually in $X_{O\underset{\sim}{d}}$, and $\|x\|_{\underset{\sim}{d}} \leqq \sum_{m=1}^{s} \|x_m\|_{\underset{\sim}{d}} \leqq \rho \sum_{m=1}^{s} K_m \|f(m)\| = \rho \|f\|_{\underset{\sim}{b}}$. Thus $X_{O\underset{\sim}{d}}$ is indeed a $(\underset{\sim}{b},\underset{\sim}{d})$-sequence.

(c) <u>implies</u> (d): Trivial, since $\underset{\sim}{b} \in b\ell_K$ signifies $\underset{\sim}{b} \in b\ell$ and supp $(\underset{\sim}{b}) = \omega$.

(d) <u>implies</u> (a): (d) implies, by [1;Theorem 8.8], that $X_{O\underset{\sim}{d}}$ is a $(\underset{\sim}{b},\underset{\sim}{d})$-sequence. (a) then follows as in the proof of Theorem 4.1.

4.3. <u>Theorem</u>. <u>For given</u> $\underset{\sim}{d} \in b\ell_K$, <u>assume that any one of the equivalent statements of Lemma 4.2 holds</u>. <u>Then</u>:

(a): $X_{O\underset{\sim}{d}}$ <u>is closed, hence regular, if and only if</u> $X_{O\underset{\sim}{d}}(m)$ <u>is closed in</u> X <u>for some</u> $m \in \omega$;

(b): <u>if</u> $U(n_1,n_0)$ <u>is compact for some</u> $n_0, n_1 \in \omega$, $n_1 \geqq n_0 \geqq 0$, <u>then</u> $X_{O\underset{\sim}{d}}$ <u>is closed, hence regular and its terms have constant finite co-dimension in</u> X.

Proof. As noted above, $X_{0\underset{\sim}{d}}$ is a subcomplete covariant sequence [1;Lemmas 6.2,6.3]. If it is algebraically regular (Lemma 4.2,(a)), the conclusion follows by Lemma 3.4.

References

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