ALGORITHMS FOR CLASSICAL STABILITY PROBLEMS

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Abstract

A differential equation is stable if the roots of the characteristic polynomial are in the interior of the left half-plane. Likewise a difference equation is stable if the roots of the characteristic polynomial are in the interior of the unit circle. This paper concerns algorithms which test polynomials for these properties. Also of concern is the relationship between the two problems. In particular special numerical integration formulae are developed which transform a differential equation into a difference equation. These formulae are such that the differential equation and the corresponding difference equation are both stable or else they are both unstable.

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Algorithms for Classical Stability Problems

1, Introduction, A classical stability problem requires testing a polynomial to see if all of its roots have negative real parts. Polynomials with this property are commonly termed Hurwitz polynomials and they correspond to stable systems. There are many ways in which they can arise but for the sake of being definite we shall suppose that the polynomial is the characteristic polynomial of a linear differential equation with constant coefficients. Thus all solutions of the associated homogeneous equation vanish at positive infinity if and only if the characteristic polynomial is a Hurwitz polynomial.

From the point of view of pure mathematics the stability of the differential equation is completely resolved by the above criteria. However in applied mathematics it often happens that the differential equation is to be approximated by a difference equation. It is quite possible that this difference equation may be unstable even though the differential equation is stable. In any case it is important to know the stability properties of both the differential equation and the difference equation.

The stability of a difference equation is also governed by an associated characteristic polynomial. Then the solutions of a difference equation vanish at positive infinity if all the roots of the characteristic polynomial are in absolute value smaller than unity. Polynomials of this type were studied by Schur and for this reason we term them "Schur polynomials".

In what follows a simple algorithm will be given to test polynomials for the Hurwitz property. Also a simple algorithm will be given to test polynomials for the Schur property. Other algorithms are to be found in the literature but many of these are complicated and ill suited for numerical evaluation.

A question of interest is the relationship between Hurwitz polynomials and Schur polynomials. It is found here that there is a matrix r which transforms the coefficients of Hurwitz polynomials into the coefficients of Schur polynomials. This matrix is idempotent ($r^2 = I$) so conversely Schur coefficients are transformed into Hurwitz coefficients. It is quite easy to evaluate r so this transformation is suited to numerical use. For example a polynomial could be tested for the Hurwitz property by first transforming it by the matrix r and then using the algorithm for the Schur property.

The approximation of a differential equation by a difference equation is often accomplished by employing a numerical integration formula such as the trapezoidal rule or Simpson's rule. In Section 8 a class of such rules is introduced termed <u>conservative integration formulae</u>. A conservative formula has the property that the differential equation and the difference equation have the same stability properties. The proof of this invariance is made to depend on a special conformal mapping.

Hurwitz polynomials and Schur polynomials have had application to a great range of problems both in pure and applied mathematics. Many authors have contributed to the development including Sturm, Routh, Hurwitz, Cauer, Foster, Nyquist, Schur, Wall, Frank, Harden, and Wilf. The main ideas in this paper may be regarded as a synthesis of the works of these authors. Nevertheless, many theorems given here are either new or else have new proofs. In order not to interfere with the train of ideas some of the proofs are confined to an appendix.

The writer wishes to thank Dr. Bruce Swanson of the Westinghouse Research Laboratories for discussion concerning the stability troubles encountered in practical numerical integration.

2. The stability of a differential equation.

The following elementary differential equation is probably the most commonly occurring differential equation in all of applied mathematics.

$$\sum_0^n a_j D^j u(t) = \sum_0^m b_j D^j w(t) .$$

Here D denotes d/dt and the coefficients a_j and b_j are constants. This equation can be interpreted as characterizing a "filter" in which w(t) is the input function and u(t) is the output function. Assuming sufficient differentiability we see that the right side uniquely determines a function v(t) and so the equation takes on the still simpler form

$$\sum_0^n a_j D^j u(t) = v(t) \quad . \tag{1}$$

This is the equation to be studied in this paper.

A stability question arising naturally is whether or not an error in initial conditions at time t = 0 will die out as $t \to +\infty$. If the effect of the error disappears we shall say that the equation is stable. A stable differential equation corresponds to a filter with "fading memory".

The stability of equation (1) is determined, of course, by the corresponding homogeneous equation

$$\sum_0^n a_j D^j u(t) = 0 . \tag{2}$$

The characteristic polynomial associated with this equation is

$$f(z) = \sum_0^n a_j z^j . \tag{3}$$

Then if f(z) = 0 there is a solution of the homogeneous equation (2) of the form $u(t) = e^{zt}$. Moreover if z is a root of multiplicity d there is a solution of the form $u(t) = z^d e^{zt}$. The general solution of (2) is a linear combination of such solutions. Thus the following theorem holds.

<u>Stability Criterion</u> H. <u>A linear differential equation with constant coefficients is stable if and only if all the roots of the characteristic polynomial have negative real parts</u>. Polynomials with this property are termed Hurwitz polynomials in the literature.

From the viewpoint of pure mathematics the above stability requirement resolves the problem. However, in applied mathematics there are further requirements. The first requirement is an algorithm for identifying Hurwitz polynomials. We use the word "algorithm" to mean a rule for resolving a question in a preassigned number of rational steps. The algorithms given in this paper have the following further properties: (1) Each step is the same form as the first. (2) No memory of preceding steps is retained. (3) The only decision is to stop.

3. Tests for Hurwitz polynomials.

The following lemma proves one part of an algorithmic test for Hurwitz polynomials.

Lemma 1. Let $f(z)$ be a Hurwitz polynomial of degree n precisely then

$$\mathrm{Re}\left[\frac{f'(0)}{f(0)}\right] > 0, \ldots, \mathrm{Re}\left[\frac{f^{(n)}(0)}{f^{(n-1)}(0)}\right] > 0 .$$

Proof. If $r_1, \ldots, r_n$ are the roots of the polynomial

$$\frac{f'(z)}{f(z)} = \sum_1^n \frac{1}{(z-r_j)} \quad .$$

Thus $f'(0)/f(0) = -\sum_1^n 1/r_j$ and since $\mathrm{Re}\, r_j < 0$ the first inequality is proved. By the well known Lucas lemma $f'(z)$ must also have all its roots in the convex region $\mathrm{Re}\, z < 0$. Thus the proof of the lemma is completed by induction.

It will be seen that this lemma proves condition (I) and condition (I') in the following tests.

Algorithm H_1. Let $f(z)$ be a polynomial

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

of degree $n \neq 0$ and with real coefficients. Let $f_1(z)$ be the "reduced" polynomial defined as

$$f_1(z) = a_1 a_1 + (a_1 a_2 - a_0 a_3) z + a_1 a_3 z^2 + (a_1 a_4 - a_0 a_5) z^3 + \cdots$$

and of degree $n - 1$. Then $f(z)$ is a Hurwitz polynomial if and only if:

(I) $a_0 a_1 > 0$.

(II) $f_1(z)$ is a Hurwitz polynomial.

If desired the condition (I) may be replaced by the more restrictive condition

$$(I') \quad a_0 a_j > 0 \qquad j = 1,\dots,n.$$

As an example of this algorithm let us test the polynomial

$$f(z) = 1 + 2z + 4z^2 + 5z^3 + 4z^4 + 9z^5 .$$

Since all coefficients are positive we must test

$$f_1(z) = 4 + 3z + 10z^2 - z^3 + 18z^4 .$$

Since the coefficient of z^3 is negative it follows that $f_1(z)$ is not Hurwitzian and hence $f(z)$ is not Hurwitzian. As another example consider

$$F(z) = 1 + 2z + 4z^2 + 4z^3 + 2z^4 + z^5$$

$$F_1(z) = 4 + 4z + 8z^2 + 3z^3 + 2z^4$$

$$F_2(z) = 16 + 20z + 12z^2 + 8z^3$$

$$F_3(z) = 400 + 48z + 160z^2 .$$

Hence $F(z)$ is a Hurwitz polynomial,

A polynomial whose roots have negative real parts is termed a "Hurwitz polynomial" in this paper even though the coefficients are not real. Hurwitz, himself, did not give a test for this case. The following algorithm applies for complex coefficients.

Algorithm H_2. *Let* $f(z)$ *be a polynomial*

$$f(z) = A_0 + A_1 z + A_2 z^2 + \dots + A_n z^n$$
$$+ i\,[B_1 z + B_2 z^2 + \dots + B_n z^n]$$

of degree $n \neq 0$ *where the coefficients* A_j *and* B_j *are real. Let* $f_1(z)$ *be the reduced polynomial defined as*

$$f_1(z) = A_1^2 + (A_1A_2 - A_0A_3)z + A_1A_3 z^2 + (A_1A_4 - A_0A_5)z^3 + \dots$$
$$+ i\,[(A_1B_1 - A_0B_2) + A_1B_2 z + (A_1B_3 - A_0B_4)z^2 + A_1B_4 z^3 + \dots$$

of degree $n - 1$. *Then* $f(z)$ *is a Hurwitz polynomial if and only if*

(I) $A_0A_1 > 0$

(II) $f_1(z)$ is a Hurwitz polynomial .

If desired the condition (I) can be replaced by the more stringent condition

(I') $A_jA_{j+1} + B_jB_{j+1} > 0 \quad j = 0, 1, \ldots, n-1$.

This condition is easily deduced from Lemma 1. The proof of condition II follows from Theorem 3 of Appendix A when the parameter K is taken to be zero.

In this algorithm the polynomial f(z) is required to have a real constant term. However, the constant term of the reduced polynomial $f_1(z)$ may not be real. In the following modified algorithm the reduced polynomial has a real constant term.

Algorithm H_3. Let f(z) be a polynomial

$$f(z) = A_0 + A_1 z + A_2 z^2 + \ldots + A_n z^n + i\,[B_1 z + B_2 z^2 + \ldots + B_n z^n]$$

of degree $n \neq 0$ where the coefficients A_j and B_j are real. Let $f_1(z)$ be the reduced polynomial defined as

$$\begin{aligned} f_1(z) = {} & pA_1 + (pA_2 - qA_3 + rB_2)z \\ & + pA_3 z^2 + (pA_4 - qA_5 + rB_4)z^3 + \ldots \\ & + i\,[pB_2 z + (pB_3 - qB_4 - rA_3)z^2 + \\ & pB_4 z^3 + (pB_5 - qB_6 - rA_5)z^4 + \ldots \end{aligned}$$

where $p = A_1^2$, $q = A_0A_1$ and $r = A_0B_2 - A_1B_1$. Then f(z) is a Hurwitz polynomial if and only if:

(I) $A_0A_1 > 0$

(II) $f_1(z)$ is a Hurwitz polynomial.

The proof of this algorithm follows from Theorem 3 by taking the parameter K to satisfy $KA_1^2 = A_0B_2 - A_1B_1$.

4. <u>The stability of a difference equation</u>.

The algorithms just given serve to determine the stability of the differential equation (1). However applied mathematics introduces an additional question which should be answered. This comes about because in numerical work the differential equation (1) is replaced by an approximating difference equation. But it may happen that the difference equation is unstable even though the differential equation is stable. Consequently it is seen necessary to analyze the stability of the difference equation as well as the stability of the differential equation.

There are various procedures for arriving at an approximating difference equation. Before considering refined methods it is instructive to look at the crude approximation in which the differential operator D is replaced by the difference operator $(T - 1)/h$. Here T is the forward translation operator defined as

$$TU(t) = U(t + h) \tag{4}$$

and h is the spacing of the grid points. Then the differential equation (2) is replaced by the difference equation

$$\sum_0^n a_j \left(\frac{T - 1}{h}\right)^j U = 0$$

Seek a solution of this equation of the form $U = W^t$. Then $(T - 1)W^t = W^t\,(W^h - 1)$ and the condition for a solution is

$$\sum_0^n a_j \left(\frac{w - 1}{h}\right)^j = 0\ , \quad w = W^h\ .$$

Suppose the differential equation is stable and let z be a root of the characteristic equation £(z) « 0, Write z » -A +iB where A > 0 and B is real. Then w • 1 + h(-A • iB). In order for the difference equation to be stable it is necessary that |w|<l and so 1 > $(1-hA_j)^2$ + $(hB_j.)^2$. Hence if the numerical solution is to be stable

$$h < \min \frac{2 M^A M}{A^z+B^z}.$$

where the minimization is over the roots of f(z).

Another crude approximation is to replace D by (1-T"^{1})/h. By analogous considerations the condition for stability is found to be

$$h > \max \frac{2 \%}{A^\wedge+B^\wedge} .$$

Thus the difference equation is stable whenever the differential equation is. But this inequality shows that it is possible for the difference equation to be stable even though the differential equation is unstable.

We now turn to more refined methods for obtaining an approximating difference equation. We shall employ the method of numerical integration formulae such as the trapezoidal rule, Simpson's rule, etc. They involve approximations of the type

(5) $$\int_t^{t+rh} u(x)dx \; 2f \sum_0^r V_m \; u(t*mh)$$

where h is the distance between grid points and r may be termed termed the degree of the approximation. The real constants k_m are chosen so that for certain functions the integration formula (5) is exact. For example if it is required that it be exact for u(t) a constant then

(6) $$\sum_0^r k_m = rh.$$

In many applications the symmetry condition

(7) $$k_{r-m} = k_m \qquad m = 0, 1, \ldots, r$$

seems natural. It is worth noting that (6) and (7) together insure that the integration formula is exact for all linear functions.

Let T again denote the operator giving a forward translation of amount h. Then the integration formula (5) may be expressed in operational form as

(8) $$(T^r - 1) \cong \sum_0^r k_m T^m D \ .$$

Repeated operation with this formula gives

(9) $$(T^r - 1)^j \cong (\sum_0^r k_m T^m)^j \, D^j \ .$$

Now apply the operator $(\sum_0^r k_m T^m)^n$ to differential equation (1) and obtain $\sum_0^n a_j (\sum_0^n k_m T^m)^n D^j u(t) = V(t)$ where

(10) $$V(t) = (\sum_0^r k_m T^m)^n \, v(t) \ .$$

Next make use of the approximation (9) to obtain

(11) $$\sum_0^n a_j (\sum_0^n k_m T^m)^{n-j} (T^r - 1)^j \, U(t) = V(t)$$

where U(t) is taken to be an approximation to u(t). Carrying out the indicated multiplication in (11) gives the difference equation

(12) $$\sum_0^N c_q T^q \, U(t) = V(t)$$

where $N = n + r$. In this equation we stipulate that t is to be restricted to the grid points $0, \pm h, \pm 2h$, etc. Intermediate values of t are not considered.

Equation (12) is the equation which gives the numerical solution of the problem. It is, of course, a linear difference equation and the stability properties are the same as those of the corresponding homogeneous equation

$$\sum_0^N c_q T^q U(t) = 0 . \tag{13}$$

In other words (12) is stable if and only if all solutions of (13) vanish as $t \to +\infty$ along the sequence of grid points.

Let w be a non-zero complex number and seek a solution of (13) such that U has the values $w^0, w^1, w^2, \cdots$ at the grid points $0, 1, 2, \cdots$. Then it follows from (13) that

$$g(w) = \sum_0^N c_q w^q = 0 . \tag{14}$$

We term g the <u>characteristic polynomial</u> (associated with forward translations). If g has a d-fold zero there is a solution of the form $U(mh) = m^d w^{-m}$. As is well known the general solution of equation (13) is a linear combination of such special solutions. This leads to the following statement.

<u>Stability Criterion</u> S. <u>A linear difference equation with constant coefficients is stable if and only if all the roots of the characteristic polynomial are inside the unit circle</u>.

Of course the characteristic polynomial corresponding to backward translations could have been used instead. However for the present analysis the forward translation results in notational convenience.

5. A test for Schur polynomials.

The stability of a difference equation is assured if the characteristic polynomial $g(w)$ is such that $g(w) = 0$ implies $|w| < 1$. We shall term $g(w)$ a Schur polynomial because Schur made an extensive study of such functions.

A testing algorithm for Schur polynomials will now be stated. In appearance it is somewhat simpler than the testing algorithm for Hurwitz polynomials. In using this test it must be kept in mind that zeros at the origin do not generate solutions of the difference equations and so are not significant.

Algorithm S. Let $g(w)$ be the polynomial

$$g(w) = c_0 + c_1 w + c_2 w^2 + \ldots + c_n w^n$$

where $c_0 \neq 0$, $c_n \neq 0$, and $n \neq 0$. Let $g_1(w)$ be the reduced polynomial

$$g_1(w) = (c_n^* c_1 - c_0 c_{n-1}^*) + (c_n^* c_2 - c_0 c_{n-2}^*) w + \ldots + (c_n^* c_n - c_0 c_0^*) w^{n-1}$$

of degree $n - 1$. Then $g(w)$ is a Schur polynomial if and only if:

(I) $|c_0| < |c_n|$.

(II) $g_1(w)$ is a Schur polynomial.

If desired the condition (I) can be replaced by the more restrictive condition

(I') $|c_j| > \binom{n}{j} |c_0| \qquad j = 1, 2, \ldots, n$.

As an example of the use of the algorithm let

$$g(w) = 1 + 2w + w^2 - w^3 + 3w^4$$

$$g_1(w) = 7 + 2w - 5w^2 + 8w^3$$

$$g_2(w) = 51 - 54w + 15w^2$$

Since $51 > 15$ it follows that g is not a Schur polynomial.

Proof of Algorithm S. First suppose $g(w)$ is a Schur polynomial. Then if the roots are $r_1, \dots, r_n$ we have $c_0 = \pm c_n \prod_{j=1}^{n} r_j$. Since $|r_j| < 1$ relation (I) is proved. (Relation (I') follows by a similar argument by comparing $c_n \prod (w - r_j)$ with $c_n (1+w)^n$.)

Now let $G(w) = w^n g^*(w^{-1})$ and so

$$w g_1(w) = c_n^* g(w) - c_0 G(w) .$$

On the circle $|w| = 1$ it is seen that

$$|c_n^* g(w)| > |c_0 g(w)| = |c_0 g^*(w^*)| = |c_0 G(w)| .$$

Thus by Rouche's theorem $w g_1(w)$ and $c_n^* g(w)$ have the same number of zeros for $|w| < 1$. Then $g_1(w)$ has n-1 zeros for $|w| < 1$ and since $g_1(w)$ is a polynomial of degree n-1 there are no other zeros. Thus $g_1(w)$ is a Schur polynomial.

Next suppose that (I) and (II) are true. Then Rouche's theorem can again be applied and it follows that $g(w)$ has n zeros inside the unit circle and the proof is complete.

6. Relating Schur polynomials and Hurwitz polynomials.

It is natural to ask if Schur polynomials can be related to Hurwitz polynomials in some direct way. To answer this question consider the following Möbius mapping

$$\frac{z+1}{z-1} = w \quad \text{and} \quad \frac{w+1}{w-1} = z .$$

This is a one to one mapping of the open region $\operatorname{Re} z < 0$ in the z-plane and the open region $|w| < 1$ in the w-plane. The imaginary axis, $\operatorname{Re} z = 0$, maps one to one on the circle $|w| = 1$ with point $w \equiv 1$ deleted. This mapping leads to the following relation between polynomials.

Lemma 2. Let $f(z)$ be a polynomial of degree n in z. Then

$$(15) \qquad g(w) = 2^{-n/2}(w-1)^n f\left(\frac{w+1}{w-1}\right)$$

is a polynomial of degree n in w and

$$(16) \qquad f(z) = 2^{-n/2}(z-1)^n g\left(\frac{z+1}{z-1}\right) .$$

For short we write these relations as $g(w) = \Gamma f(z)$ and $f(z) = \Gamma g(w)$ and term them gamma transforms.

Proof. If $f(z) = \Sigma a_i z^i$ then clearly

$$(17) \qquad g(w) = 2^{-n/2} \Sigma a_i (w+1)^i (w-1)^{n-i} = \Sigma_0^n c_j w^j .$$

Here the coefficients c_j are defined by carrying out the binomial products. Then

$$2^{-n/2}(z-1)^n g\left(\frac{z+1}{z-1}\right) = \left(\frac{z-1}{2}\right)^n \Sigma_0^n a_i \left(\frac{z+1}{z-1} + 1\right)^i \left(\frac{z+1}{z-1} - 1\right)^{n-i}$$

$$= 2^{-n} \Sigma a_i (2z)^i (2)^{n-i} = f(z)$$

and (16) is verified.

$$|\gamma_{ij}| = \begin{vmatrix} 1 & -4 & 6 & -4 & 1 \\ -1 & 2 & 0 & -2 & 1 \\ 1 & 0 & -2 & 0 & 1 \\ -1 & -2 & 0 & 2 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{vmatrix}$$

algorithm:
a • b + c + d
where

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

It is of some interest to note that

(21) $$\det(\gamma_{ij}) = \pm 2^{n(n+1)/2}.$$

The correct sign here is left to the curiosity of some reader.

Lemma 4. A polynomial is a Schur polynomial of degree precisely n if and only if the gamma transform is a Hurwitz polynomial of degree precisely n.

Proof. If g(w) is a Schur polynomial then formula (16) of Lemma 2 shows that f(z) cannot vanish if Re $z \geq 0$ but $z \neq 1$ because $|(z+1)/(z-1)| \geq 1$ and $(z-1)^n \neq 0$. Moreover formula (16) gives $f(1) = 2^{n/2} c_n \neq 0$ so f(z) is a Hurwitz polynomial.

Conversely suppose f(z) is a Hurwitz polynomial then formula (15) shows that if $|w| \geq 1$ but $w \neq 1$ then $g(w) \neq 0$. However $g(1) = 2^{n/2} a_n \neq 0$ so this completes the proof.

These lemmas prove the following theorem.

Theorem 1a. The sequence $c_0, c_1, \ldots, c_n$ with $c_n \neq 0$ are Schur coefficients if and only if the sequence $\{a_j = \sum_0^n c_i \gamma_{ij}\}$ are Hurwitz coefficients and $a_n \neq 0$.

By the idempotent property this can also be stated as

Theorem 1b. The sequence $a_0, a_1, \ldots, a_n$ with $a_n \neq 0$ are Hurwitz coefficients if and only if the sequence $\{c_j = \sum_0^n a_i \gamma_{ij}\}$ are Schur coefficients and $c_n \neq 0$.

7. The trapezoidal integration formula.

We have just seen that there is a correspondence set up between Hurwitz polynomials and Schur polynomials by a conformal mapping of Mbbius type. This raises the following question. Is there a numerical integration scheme such that the same correspondence exists between the differential equation and the approximating difference equation?

To investigate this question consider the trapezoidal integration formula

$$\int_0^h u(x)\,dx \cong \frac{u(h)+u(0)}{2}\,h .$$

This is a special case of the numerical integration formula (5) where $r = 1$, $k_0 = h/2$, and $k_1 = h/2$. The operational form is

$$(T-1)^j \cong (h/2)^j (T+1)^j D^j .$$

Then the differential equation $\sum_0^n a_j D^j u = 0$ is transformed into the difference equation

$$\sum_0^n a_j (h/2)^j (T+1)^{n-j} (T-1)^j U = 0 .$$

The associated characteristic polynomial is

$$g(w) = \sum_0^n 2^{-n/2} A_j (w-1)^{n-j} (w+1)^j$$

where $A_j = 2^{n/2}(h/2)^{n-j} a_{n-j}$. Then $g(w)$ is a gamma transform of $F(z) = \sum_0^n A_j z^j$. Obviously, however, $f(z) = \sum a_j z^j$ is a Hurwitz polynomial if and only if $F(z)$ is because $F(z) = 2^{n/2} z^n f(h/2z)$ and $h > 0$.

Corollary 1. Suppose that the trapezoidal integration formula is used. Then the resulting difference operator is stable and of order precisely n if and only if the differential operator is stable and of order precisely n.

Proof. According to Lemma 4 the characteristic polynomial $g(w)$ is a Schur polynomial of degree precisely n if and only if $F(z)$ is a Hurwitz polynomial of degree precisely n. But $F(z) = 2^{n/2}\, z^n\, f(h/2z)$ where f is the characteristic polynomial of the differential operator. Clearly $F(z)$ is a Hurwitz polynomial of degree precisely n if and only if $f(z)$ is a Hurwitz polynomial of degree precisely n. This completes the proof.

As an example of the above process let

$$f(z) = 2 + 3z + 8z^2 + 4z^3 + 4z^4 .$$

Taking $h = 2$ and neglecting constant multipliers gives

$$F(z) = 4 + 4z + 8z^2 + 3z^3 + 2z^4 .$$

Using the γ matrix tabulated above for $n = 4$ gives

$$g(w) = 7 - 6w + 20w^2 - 10w^3 + 21w^4 .$$

Both f and g are seen to be stable.

8. Conservative integration formulae.

It has just been shown that the trapezoidal integration formula preserves stability. This raises the question of whether or not there are other integration formulae with the same property. This question will now be treated by employing a generalization of the Möbius conformal mapping.

Definition. Consider the integration formula

$$\int_x^{x+rh} f(t)dt \cong \sum_0^r k_m\ f(x+mh)$$

and suppose that the associated polynomial $p(w) = \sum_0^r k_m w^m$ satisfies:

(I) If $w^r = 1$ then $p(w)$ is real and $p(w) > 0$.

(II) $2p'(1) - rp(1) \geq 0$.

Then the integration formula is termed "semi-conservative". However, if (II) is an equality it is termed "conservative". For example the trapezoidal formula corresponds to $p(w) = (w+1)h/2$. So $p(1) = h > 0$ and $2p'(1) - rp(1) = 0$ and it follows that the trapezoidal formula is conservative. It is also of interest to note that conservative formulae satisfy the symmetry condition (7). The proof of this will be omitted.

Theorem 2a. If a stable differential equation is converted to a difference equation by a semi-conservative integration formula then the difference equation is stable.

Proof. Let the differential equation have characteristic function $f(z) = \sum_0^n a_j z^j$. Then the characteristic function of the difference equation is

$$g(w) = \sum_0^n a_j\ [p(w)]^{n-j}\ (w^r-1)^j. \tag{21}$$

Lemma 5. If the hypotheses of Theorem 2 hold and $g(w_1) = 0$ then $w_1^r - 1 \neq 0$ and $p(w_1) \neq 0$.

Proof. If $w_1^r - 1 = 0$ then $a_0[p(w_1)]^n = 0$. Since $f(0) \neq 0$ it follows that $a_0 \neq 0$ and hence $p(w_1) = 0$ this contradicts (I) so $w_1^r - 1 \neq 0$. On the other hand if $p(w_1) = 0$ then $a_n(w_1^r - 1)^n = 0$. But $a_n \neq 0$ so $w_1^r - 1 = 0$ and we have just shown that this is impossible.

Now write $g(w)$ in the form

$$g(w) = [p(w)]^n \sum_0^n a_j \left[\frac{w^r-1}{p(w)}\right]^j . \tag{22}$$

By virtue of Lemma 5 it follows that if $g(w_1) = 0$ then $f(z_1) = 0$ where

$$z_1 = \frac{w_1^r - 1}{p(w_1)} , \tag{23}$$

moreover $z_1 \neq 0$. Now according to Theorem 4 of Appendix B if $|w_1| \geq 1$ then $\operatorname{Re} z_1^{-1} \geq 0$. Thus $\operatorname{Re} z_1 \geq 0$ which is a contradiction because the differential equation is assumed to be stable.

Theorem 2b. Suppose that a differential equation is converted to a difference equation by a conservative integration formula with at least three non-zero coefficients. Then the difference equation is stable if and only if the differential equation is stable.

Proof. If the differential equation is stable Theorem 2a shows that the difference equation is stable. Thus suppose the differential equation is unstable. Then $f(z_1) = 0$ for a z_1 such that $\operatorname{Re} z_1 \geq 0$. Then seek a w to satisfy the equation

$$(24) \qquad z_x\, p(w) - w^r + 1 = 0.$$

By hypothesis there are three values of m for which k_m *f* 0, Thus the above equation can be written as

$$(25) \qquad ({}^{z}l^{k}{}_{r} \text{ " } \wedge \wedge \text{ +\#\#**1" } {}^{z}i^{k}{}_{s}{}^{wS} \text{ + " \# + } ({}^{z}i^{k}o + 1)\ {}^{s}\ 0$$

where k_s ^ 0 and r > s > 0. Thus this equation is at least of degree s > 0 so there is at least one solution, say w_1, It follows by (I)+(24) that $p(w_x)$ *f* 0. Thus divide (24) by p to obtain (23). It now follows from Theorem 4 that $l^{w}ll$ i * because Re $zT_1^{1} \geq$ °* Thus the difference equation is unstable and the proof is complete.

The following is an example of a conservative integration formula of degree 2.

$$h^{-1} \int_0 f(t)dt \cong 1.21\ f(0) - .42\ f(h) + 1.21\ f(2h)\ .$$

The associated polynomial is

$$h^{-1}\ \gamma(w) \cdot 1.21 - .42w + 1.21\ w^2\ .$$

It is seen that conditions (I) and (II) hold because p(l) > 0 and p(-l) > 0. Since $3\pi/2 \cong 4.7 \cong (.21)^{-1}$ one sees that the formula is exact for a function of the form

$$f(t) = a + bt + c \cos(4.7t/h) + d \sin(4.7t/h).$$

Thus the class of exact functions is not independent of the mesh constant h. This appears undesirable from a practical point of viexv.

Simpson's formula is of degree 2 and lias the associated polynomial $p(w) = (1 + 4w + w^2)h/3$ however p(-l) < 0 so condition (I) is violated. Thus Simpson's rule is not conservative. Consequently a conservative formula of degree 2 cannot be exact for quadratic functions.

The conservative integration formulae of higher degree are certainly of interest from a theoretical point of view. However, the two examples just considered cast doubt on their practical advantage over the trapezoidal formula. There are, of course, various other methods for converting differential equations into difference equations such as the Runge-Kutta method and the Adams method. The choice of a method for a given problem depends on several factors but in any case stability is one of the factors to be considered.

Appendix A. A characterization of Hurwitz polynomials.

The following reduction theorem justifies the Algorithms H_{1f} H_2, and Hj.

Theorem 3, Let f(z) be a non-constant polynomial such that f(0) > 0, Let

h(z) . ££51. +(J+ iKz) *<*> . *< *'> .
z 2z*

Then f(z) is a Hurwitz polynomial if and only if

(I) Re $f^f(0) > 0$

(II) h(z) is a Hurwitz polynomial when J Re $f^f(0)$ + f(0) - 0 and K is a real constant.

Proof. The proof will be based on the following modification of Rouche*s theorem.

Lemma 5. Let F(z,X) be an analytic function of the complex variable z and a continuous function of the real variable X. Suppose that for 0 <^ X <^ 1 the function F(z,X) does not vanish on a finite contour C. Then F(z,0) and F(z,l) have the same number of zeros inside r.

Proof. Let N(X) be the number of zeros. Then

$$N(\lambda) = \frac{1}{2\pi i} \int_C \frac{F'}{F}\, dz .$$

Clearly the right side is a continuous function of A hence N(0) - N(l).

Lemma 6. Let f(z) be analytic on the imaginary axis and let

H(z) - f(z) + (J + iKz)[£(z) - f*(-z)]/2z

where J and K are real constants. Let y be real but y *1* 0 then H(iy) = 0 if and only if f(iy) - 0.

Proof, Let f(iy) = u + iv so f*(-iy) » u - iv and H(iy) • u + mv + iv where m is real. Thus H(iy) = 0 if and only if u • 0 and v ^{s} 0.

Lemma 7. Let f(z) be a polynomial such that f(0) > 0 and Re $f^{f}(0)$ > 0. Let

$$z \qquad \lambda(J+iKz)\frac{[f(z) - f^{*}(-z)]}{2z^{z}}$$

where J Re $f^{f}(0)$ + f(0) » 0 and K is real. Then there is a positive constant e such that for 0 <^ | z | <^ e and for Re z >_ 0 and for 0 <_ X <_ 1 the function h(z,A) *f* 0. Moreover h(0,l) is finite.

Proof. Let us write

$$f(z) = A_Q + (A_1 + iB_1)z + (A_2 + iB_2)z^2 +\bullet\bullet\bullet$$

where the A_j and B_j denote real constants. Thus

$$[f(z) - f^*(-z)]/2z= A_1 + iB_2 z + A_3 z^2 + iB_4 z^3 +\bullet\bullet\bullet$$

Hence we have

$$h(z,X) = (A_Q + XJA_1)z''^1 + (A_1 + iXJB_2 + iXKAj\text{^} + iB_i)$$
$$+ (A_2 + XJA_3 - XK3_2 + i^B 2\text{^}^{z} {}^{+\#\#*}$$

For short let h(z,X) = $h\text{^^}z''^{1}$ + h_Q + $h_1 z$ +••• . By hypothesis A_Q + XJA_1 ^ 0. Thus

$$\text{Re } h(z,X) \geq\text{^ } A_1. + \text{ReChjZ} +\bullet\bullet\bullet)$$

The coefficients h_j are uniformly bounded for 0 <^ X <_ 1. Thus

$$\text{Re } h(z,X) \text{ ^ } A_1 + 0|z|$$

The proof clearly follows from this inequality.

Lemma 8. If h(z,X) is defined as in Lemma 7 then there is an R such that for h(z, X) *f* 0 for | z | >_ R and 0 <^ X <^ 1.

<u>Proof</u>. Let a_n be the leading coefficient of f(z) so

$$\frac{h(z,\lambda)}{a_n z^{n-1}} = 1 + i\lambda K \left[1 + (-1)^n \frac{a^*_n}{a_n}\right] + 0(z^{-1}).$$

The constant term on the right may be written

$$1 + i\lambda K \left[1 + e^{i\theta}\right]$$

where $e^{i\theta} = (-1)^n\, a^*_n/a_n$ and θ is real. Thus if this term vanishes we have

$$1 + \cos\theta = 0 \quad \text{and} \quad 1 - \lambda K \sin\theta = 0.$$

Clearly these equations are contradictory for any value of λ. This shows that the constant term is bounded away from zero. The remaining term can be made uniformly small by making R large and the proof is complete.

Returning to the proof of Theorem 4 we first suppose that f(z) is a Hurwitz polynomial. Then by Lemma 1 it follows that Re $f'(0) > 0$ and so (I) holds. Thus J can be chosen to satisfy J Re $f'(0) + f(0) = 0$ and conditions for Lemmas 6, 7, and 8 hold. To apply the Lemma 5 take the contour C to be the two semi-circles $|z| = R$ and $|z| = \varepsilon$ for Re $z \geq 0$ together with the segments of the imaginary axis $(i\varepsilon, iR)$ and $(-i\varepsilon, -iR)$. It follows from Lemmas 6, 7, and 8 that $h(z,\lambda)$ does not vanish on the contour C.

Lemma 5 now states that h(z) and f(z)/z have the same number of zeros in C. Since ε can be arbitrarily small and and since R can be made arbitrarily large it follows that h(z) has no zeros for Re $z \geq 0$, $|z| \neq 0$. Moreover the proof of Lemma 7 shows that $h(0) \neq 0$. Thus h(z) is a Hurwitz polynomial.

To prove the second part of the theorem suppose that conditions (I) and (II) hold, then the conditions for Lemmas 6, 7, and 8 are again in force and Lemma 5 applies to the contour C just defined. Thus $h(z)$ and $f(z)/z$ have the same number of zeros in C. Thus $f(z)$ has no zeros in C and since $f(0) > 0$ the proof that $f(z)$ is a Hurwitz polynomial is complete.

It is worth noting that the algorithm defined by Theorem 3 makes sense for transforming a power series $f(z)$ into a "reduced" power series $h(z)$. Thus it may be possible to extend Theorem 3 to hold for entire functions of zero type as well as for polynomials.

To prove Algorithms H_1, H_2, and H_3 the coefficients of the series of $h(z)$ must be evaluated. It is seen that $h(z)$ is the sum of three series:

$$\begin{array}{ccccccc} (A_1+iB_1) & + & (A_2+iB_2)z & + & (A_3+iB_3)z^2 & +\ldots \\ iJ\;B_2 & + & J\;A_3\;z & + & iJ\;B_4z^2 & +\ldots \\ iK\;A_1 & - & K\;B_2\;z & + & iK\;A_3z^2 & -\ldots \end{array}$$

Separating real and imaginary parts gives

$$h(z) = A_1 + (A_2+JA_3-KB_2)z + A_1z^2 + \ldots$$
$$i[(B_1+JB_2+KA_1) + B_2z+(B_3+JB_4+KA_3)z^2 + \ldots$$

Here $J = -A_0/A_1$. Letting $K = 0$ and $f_1 = A_1h$ give algorithm H_1 and H_2. Letting

$$K = -(B_1+JB_2)/A_1 = (A_0B_2-A_1B_1)/A_1^2$$

and $f_1 = A_1^2\,h$ gives Algorithm H_3 .

Appendix B. A conformal mapping related to numerical integration.

A mapping of the circle into the half-plane is given here which is a generalization of the Möbius mapping.

Theorem 4. Let $z = Q(w)$ be a mapping of the w-plane into the z-plane where

$$Q(w) = \frac{p(w)}{w^r - 1}$$

where p is a polynomial of degree $r \geq 1$ such that:

(I) If $w^r = 1$ then $p(w)$ is real and $p(w) > 0$.

(II) $2p'(1) - rp(1) \geq 0$.

Then

(a) $\text{Re } z > 0$ if $|w| > 1$

(b) $\text{Re } z \geq 0$ if $|w| = 1$ and $w^r \neq 1$.

Moreover if (II) is an equality then

(c) $\text{Re } z = 0$ if $|w| = 1$ but $w^r \neq 1$

(d) $\text{Re } z < 0$ if $|w| < 1$.

Proof. Let $w = (s+1)/(s-1)$ and let $q(s) = Q(w)$ so

$$q(s) = \frac{(s-1)^r \, p[(s+1)/(s-1)]}{(s+1)^r - (s-1)^r} .$$

The denominator here is a polynomial of degree r-1. The denominator vanishes when

$$s = \frac{w+1}{w-1} \quad \text{and} \quad w^r = 1 \text{ but } w \neq 1 .$$

Let these vanishing points be $s_1, s_2, \ldots, s_{r-1}$. They are distinct and pure imaginary. The numerator of q(s) is a polynomial of degree r. Thus by the Lagrange interpolation formula we can write

$$q(s) = As + B + \sum_{j=0}^{r-1} \frac{C_j}{s - s_j} .$$

It is seen that A is defined by the limit

$$A = \lim q(s)/s \quad \text{as } s \to \infty .$$

But as $s \to \infty$ we see that $w \to 1$ so

$$A = \lim \frac{p(w)}{w^r - 1} \left(\frac{w-1}{w+1}\right) = \frac{p(1)}{2r} > 0 .$$

Likewise we see that B is defined by the limit

$$B = \lim [q(s) + q(-s)]/2 \quad \text{as } s \to \infty .$$

Clearly $q(-s) = p(w^{-1})/(w^{-r}-1)$ so

$$B = \lim \frac{p(w) - w^r p(w^{-1})}{2(w^r - 1)} \quad \text{as } w \to 1 .$$

Applying L'Hopital's rule gives

$$B = \frac{2p'(1) - rp(1)}{2r} \geq 0 .$$

The residue C_j is defined as

$$C_j = \lim (s-s_j)\, q(s) \quad \text{as } s \to s_j$$

$$C_j = \lim \frac{(s-s_j)(s-1)^r\, p}{(s+1)^r - (s-1)^r}$$

$$C_j = \frac{(1-s_j^2)\,p(w_j)}{2r} = \frac{(1+|s_j|^2)\,p(w_j)}{2r}$$

Since $(w_j)^r = 1$ it follows that $C_j > 0$.

The coefficients A, B, and C_j in the Lagrange formula are non-negative and the poles are on the imaginary axis. It follows that

$$\operatorname{Re} q(s) \geq 0 \quad \text{if} \quad \operatorname{Re} s \geq 0 .$$

(In the terminology of electrical network theory q(s) is a "positive real function".)

Since the coefficient A is positive

$$\text{Re } q(s) > 0 \quad \text{if} \quad \text{Re } s > 0.$$

But $|w| > 1$ implies Re $s > 0$ so statement (a) is seen to be proved. If $|w|$ « 1 but w ^ 1 then Re s • 0 and statement (b) follows.

If (II) is an equality then clearly

$$\text{Re } q(s) \div 0 \quad \text{if} \quad \text{Re } s = 0 \text{ and } s + s_j \ .$$

This is seen to prove (c). Again if (II) is an equality

$$\text{Re } q(s) < 0 \quad \text{if} \quad \text{Re } s < 0 \ .$$

But Re $s < 0$ if $|w| < 1$ so statement (d) follows and the proof is complete.

The book by Kaplan in the following list of references treats linear systems and their stability. The book by Harden gives a survey of the general problem of root location of polynomials. The book by Wall relates root location to continued fractions. The book by Hazony relates polynomials to network theory. The paper [1] treats the characteristic polynomial of systems of differential equations and also the characteristic polynomial of systems of difference equations. The paper [2] treats filters for time series and the underlying differential equation and difference equation. The paper [5] relates root location to determinants. In paper [10] tfilf applies a theorem of Schur to the stability to several integration methods. He indicates how the stability criteria have approximate validity even for differential equations which have variable coefficients or which are nonlinear.

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