ON THE CONVERSION OF HEAT INTO MECHANICAL WORK

W. A. Day

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Introduction.

Suppose that a material has been subjected to a known deformation and entropy history in the past so that energy has been stored in it. Suppose too that some amount H, say, of heat is available and that we subsequently take the material through a closed cycle in strain and entropy space in such a way that the heat absorbed is precisely H. This paper is concerned with determining how much mechanical work can be recovered from the two sources of energy, the energy stored in the material in the past and the heat absorbed, by taking the material around a closed cycle and with finding those cycles which maximize the mechanical work. In fact we will not admit to competition all the closed cycles which absorb heat H. Instead we suppose that two working temperatures θ^-, θ^+ with $0 < \theta^- < \theta^+$ are assigned and consider only those closed cycles for which the temperature lies always between the assigned working temperatures.

This problem has, of course, been discussed in a great many papers and texts on thermodynamics since the work of CARNOT [2]. The content of these works appears to me to be the claim that the maximum recoverable mechanical work has the value $\left(\frac{\theta^+ - \theta^-}{\theta^+}\right)H$ and that this amount of work can be obtained only in Carnot cycles performed 'quasi-statically'.

In this paper I show that, modulo a condition of regularity on the material, the maximum recoverable mechanical work is never less than the classical amount $\left(\frac{\theta^+ - \theta^-}{\theta^+}\right)H$ and that if the given strain and entropy histories are constant then the maximum recoverable mechanical work does indeed have this precise value. Also the maximum work has this value for hyper-elastic materials. However, I give an example of a material with memory in which the maximum work exceeds the classical value except for certain very special histories, for which it has the classical value. Furthermore I construct two one-parameter families of closed cycles depending on a parameter $A > 0$; the first family ultimately extracts precisely the amount $\left(\frac{8^{\prime\prime *\prime\prime} - 9}{?^{+}}\right)H$ of mechanical work as $A \to 0$ and has the interesting feature that these cycles are traversed more and more rapidly as $A \to 0$ rather than more and more slowly or 'quasi-statically'. The second family ultimately extracts all the recoverable mechanical work and has the feature that parts of the cycle are traversed more and more rapidly and other parts more and more slowly as $A \to 0$.

1. Preliminaries.

Throughout this paper the real numbers will be denoted by R, the non-negative reals by R^+ and the strictly positive reals by R^{++}. On occasion, and where the context permits, we speak of 'the time t' instead of 'the real number t'. The derivatives of functions whose domain is an open subset of R will be indicated by a superposed dot $\cdot$. The symbol X stands for a real finite dimensional inner product space, L(X) for the associated inner product space of all endomorphisms of X with inner product

$$\ell\cdot m = \mathrm{trace}\ \ell m^T \tag{1.1}$$

and $U \subset L(X)$ is an open subset. For the applications we have in mind two examples are important. In the first, X is the translation space of euclidean point space and U is the set of all endomorphisms of X with positive determinant. Any endomorphism in L(X) is a 'tensor' and endomorphisms in U are to be thought of as deformation gradient tensors. The second example arises in one-dimensional situations where we identify X and L(X) with R and U with R^{++}.

We suppose that we have at our disposal a collection $\mathcal{C}$ of pairs (φ,ψ) of continuous and piecewise smooth functions $\varphi\colon R^+ \to U$, $\psi\colon R^+ \to R$ with the properties

C1 <u>constant functions are in</u> $\mathcal{C}$: if $u\in U$ and $r\in R$ are arbitrary and (u^*,r^*) are the constant functions on R^+ with values u, r respectively then $(u^*,r^*)\in\mathcal{C}$, C2 C is closed under connection: if $(\varphi,\psi)\in\mathcal{C}$ and the continuous and piecewise smooth functions $\varphi'\colon R^+ \to U$, $\psi'\colon R^+ \to R$ have the property that there is a number $\tau > 0$ with $\varphi'(t) = \varphi(t-\tau)$ and $\psi'(t) = \psi(t-\tau)$ for each $t \geq \tau$ then $(\varphi',\psi')\in\mathcal{C}$.

A pair $(\varphi,\psi)\in\mathcal{C}$ is called a history. If the conditions in C2 hold we say that the history (φ',ψ') is a connection of the history (φ,ψ) and we call it a closed connection of (φ,ψ) if the additional requirements $\varphi'(0) = \varphi(0)$, $\psi'(0) = \psi(0)$ are met. A particularly important example of a closed connection of (φ,ψ) is its constant continuation by amount τ which is the history (φ_τ,ψ_τ) defined by setting $\varphi_\tau(t) = \varphi(t-\tau)$, $\psi_\tau(t) = \psi(t-\tau)$ for $t \geq \tau$ and $\varphi_\tau(t) = \varphi(0)$, $\varphi_\tau(t) = \varphi(0)$ for $0 \leq t \leq \tau$.

If α is any function whose domain is R and $t\in R$ is any number we define the function α^t on R^+ by $\alpha^t(s) = \alpha(t-s)$ and call α^t the history of α up to t. Any pair of functions (f,η) with $f\colon R \to U$, $\eta\colon R \to R$ and such that $(f^t,\eta^t)\in\mathcal{C}$ for every $t\in R$ will be called a process and we interpret f and η as describing the evolution with time of the deformation gradient and the entropy, respectively, at a material particle. For our purposes it suffices to define a material as an ordered triple (E,S,Θ) of functions $E\colon \mathcal{C} \to R$, $S\colon \mathcal{C} \to L(X)$ and $\Theta\colon \mathcal{C} \to R$ with the property that, for each process (f,η), the functions $e\colon R \to R$, $s\colon R \to L(X)$, $\theta\colon R \to R$ with values

$$e(t) = E(f^t,\eta^t), \quad s(t) = S(f^t,\eta^t), \quad \theta(t) = \Theta(f^t,\eta^t) \tag{1.2}$$

are continuous and piecewise smooth. The interpretation of the functions e,s,9 is that they describe the evolution with time of the internal energy, Piola-Kirchhoff stress and the temperature at the particle in the process (f^f?) and the functions E,S,© are the response functions for these quantities. A very special example of a class of materials is provided by the hyperelastic materials: a material (E,S,(9>) is hyperelastic† if there is a class C^2 function TT: UXR -• R with partial derivatives D_1TT: UXR -» L(X) and Dpir: UXR—*R such that, for each history (<p,0)€C,

E(<p,0) = 7f((p(O) ,0(0)) , S(<p,0) = 0^(^(0),0(0)) ,<3>(<p,0) = D_2TT(<O(0),0(0)). (1.3)

We say that the material has the energy relaxation property if, given any history (cp,0)€C, E(<p_T,0$_T$) -* E(cp(O)*, 0(0)*) as r -> oo, where (<p$_T$, 0$_T$) is the constant continuation of (<p, 0) by amount r, defined previously, and *(cp* (0)*,0(O)*) is the constant history with value (<p(O),0(O)). From here on, and without further mention, we confine attention to processes with 6 > 0.

The (real-valued) heat flux‡ h: R - R needed to support a process (f.,7?) is given by the energy balance equation

$$h = \dot{e} - s.\dot{f} \qquad (1.4)$$

and the corresponding internal dissipation a: R -• R by

† The word 'hyperelastic' is used here in a sense differing from that of the treatise of TRUESDELL and NOLL [9].

‡ Of course the heat flux is given by h = div q + r, where q is the heat flux vector and r the heat supply but we do not use this expression.

$$a = \dot{r}j - | \, . \qquad (1.5)$$

The definitions (1.4) and (1.5) are, of course, meaningful only where the derivatives $\dot{e}$, $\dot{f}$, *it* exist. We make the convention that if any of the derivatives e, f, 77 does not exist at t€R then we define e(t) = 0, f (t) = 0, r?(t) = 0, h(t) = 0, a(t) = 0. The material will be assumed compatible with thermodynamics in the sense that *a* >_ 0 for every process (f,??) . Various authors (see COLEMAN [3], COLEMAN and MIZEL [4], GURTIN [8];, WANG and BOWEN [10]) have shown that the inequality or >^ 0 follows from the Clausius-Duhem inequality for extensive classes of materials and in [7] I have shown how, for a broad class of materials, this inequality may be deduced from a work axiom. Hyperelastic materials, as defined here, are trivially compatible with thermodynamics since for them the internal dissipation is identically zero.

Equations (1.4) and (1.5) tell us that if the material is compatible with thermodynamics and if f (t) = 0 and rj(t) = 0 at some t€R then

$$\dot{e}(t) = -6(t)cr(t) \leq 0.$$

It follows that if (cpj$)€C is any history and (<p$_T^0_T$) is its constant continuation by amount r then

$$E(\varphi,\psi) \geq E(\varphi_T,\psi_T)$$

and thus, if the energy relaxation property holds,

$$E(\varphi,\psi) \geq E(\varphi(0)^*,\psi(0)^*) \qquad (1.6)$$

i.e. among all histories ending with a given value $(\varphi(0), \psi(0)$, the constant history $(\varphi(0)^*, \psi(0)^*)$ has the least internal energy. This result and its proof are due to COLEMAN [3].

For any process (f,η) and any closed interval $[a,b] \subset R$ we introduce the measurable sets

$$\Sigma^+(a,b) = \{t \in [a,b] \mid h(t) > 0\} \qquad (1.7)$$

$$\Sigma^-(a,b) = \{t \in [a,b] \mid h(t) \leq 0\} \qquad (1.8)$$

$$\Sigma(a,b) = \{t \in [a,b] \mid \dot{\eta}(t) = 0\}. \qquad (1.9)$$

If $t \in \Sigma(a,b)$ then equation (1.5) and the assumption of compatibility with thermodynamics imply that $h(t) = -\theta(t)\sigma(t) \leq 0$ and so $t \in \Sigma^-(a,b)$. In other words

$$\Sigma(a,b) \subset \Sigma^-(a,b). \qquad (1.10)$$

We say that heat is absorbed on $\Sigma^+(a,b)$, heat is emitted on $\Sigma^-(a,b)$ and heat is emitted at constant† entropy on $\Sigma(a,b)$. The number

$$H(a,b) = \int_{\Sigma^+(a,b)} h \geq 0. \qquad (1.11)$$

will be called the heat absorbed on $[a,b]$.

† Of course, the set $\Sigma(a,b)$ is not in general connected and the restriction of η to $\Sigma(a,b)$ need not, in fact, be constant.

2. Carnot Cycles. Regular Materials. The Main Result,

Let (<p,0)eC be any given history ending with the final value (<P(0), 0(0)), let 8+, 8" be any numbers with 0 < 8" < 8+, called the upper working temperature and the lower working temperature, respectively, and let H > 0 be any positive number. We consider processes (f,7?) for which the history of *(f,y)* up to time 0 coincides with (<p, *if))* , which assumes the value (<p (0) , 0(0)) again at some later time r > 0, for which the temperature on [0,T] lies between the assigned working temperatures and such that the heat absorbed on [0,T] is H. More precisely, the process (f^Tj) is a (6 +, 8",H)-admissible closed cycle for the history (<p, 0) ori the interval [0,T] if (1) (f°,T?°) = (<p,0), (2) (f(T),r?(r)) = (cp(0),</)(0)), (3) for each te[O,r], 9" < 8(t) < 9+, and (4) H(0,r) = H. Of course, for a given history (<p*0) and a given triple (8+ ,8",H) there may be no admissible closed cycles. This happens, trivially, for example if we choose an upper working temperature 6~ < 8(0) = @ (<pj$) *or* a lower working temperature 8" > 8 (0) .

If *(f,T))* is an admissible closed cycle on [0,T] the mechanical work done by the material in this cycle is - J s»f . Given the history (<p, 0) and the triple (8+ ,8~_,H) we wish to determine the maximum value taken by the mechanical work in admissible closed cycles and to find those closed cycles which maximize the mechanical work. With these aims in mind we define the maximum mechanical work recoverable from the history

(φ, ψ) <u>in</u> (θ^+, θ^-, H) - <u>admissible closed cycles</u>† to be

$$W(<p,0,0^+, 0\sim,H) = \sup\{-J^{T}_{\circ} s\text{-}f \mid (£,T)) \text{ is a } (6^+, O'\wedge H)\text{-admissible closed cycle for } (<p,ij)) \text{ on some interval } [O,T]\} , \tag{2.1}$$

whenever the set appearing on the right side of (2,1) is not empty.

It is convenient to introduce, for each $(8^+, 0\sim,H)$ - admissible closed cycle on [0,T] the, possibly empty, measurable sets

$$ir^+(O,r) = \{t€[O,T] \mid 6(t) = 9^+\}, \; TT"(O,T) = \{t€[O,r]|6(t)= 6-\} \tag{2.2}$$

on which the working temperatures are attained. In terms of these sets we define an important subclass of the admissible closed cycles - the Carnot cycles. A $(9^+,0\sim"\_,H)$- admissible closed cycle for (<p, 0) on the interval [0,r] is a <u>Carnot cycle</u> if

(1) heat is absorbed only at the upper working temperature i.e.

$$S^+(O,r) \subset ir^+(O,r), \tag{2.3}$$

(2) heat is emitted either at the lower working temperature or at constant entropy i.e.

$$2T(0,T) \subset 7r"(O,r) \cup S(O,T) . \tag{2.4}$$

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† The terminology *maximum recoverable work' was introduced in a purely mechanical context by BREUER and ONAT [1]. See also DAY [5], [6] and [7].

Before stating our main result we introduce a particular class of materials, the regular materials. To motivate the definition we note that for a hyperelastic material the final value $e(\tau)$ of the internal energy at the end of a closed cycle on $[0,\tau]$ coincides with the starting value $e(0)$. In addition the internal dissipation $\sigma \equiv 0$ and so $\int_0^\tau \sigma = 0$. Regular materials are materials close to being elastic in the following special sense: a material (E,S,Θ) is <u>regular</u> if it is compatible with thermodynamics and if, given any history $(\varphi,\psi) \in C$, given any pair (θ^+,θ^-) with $\theta^- \leq \theta(0) = \Theta(\varphi,\psi) \leq \theta^+$, given any $H > 0$ and given any $\epsilon > 0$ there is a (θ^+,θ^-,H)-admissible Carnot cycle for (φ,ψ) on some interval $[0,\tau]$ such that $e(\tau) < e(0) + \epsilon$ and $\int_0^\tau \sigma < \epsilon$. For hyperelastic materials regularity is equivalent to the existence of at least one (θ^+,θ^-,H)- admissible Carnot cycle for each history (φ,ψ). An example of a regular material which is not elastic is discussed in section 3. We turn to proving our main result.

<u>Theorem</u>. <u>Let</u> (E,S,Θ) <u>be a regular material</u>. <u>If</u> $(\varphi,\psi) \in C$ <u>is any history and</u> (θ^+,θ^-,H) <u>is any triple with</u> $0 < \theta^- \leq \Theta(\varphi,\psi) \leq \theta^+$ <u>and</u> $H > 0$ <u>then the maximum recoverable mechanical work has the lower bound</u>

$$W(\varphi,\psi,\theta^+,\theta^-,H) \geq \left(\frac{\theta^+ - \theta^-}{\theta^+}\right) H. \tag{2.5}$$

<u>The inequality sign in</u> (2.5) <u>can be replaced by equality if the material has the energy relaxation property and if</u> (φ,ψ) <u>is a constant history</u>.

Proof. If (f,η) is any (θ^+,θ^-,H)- admissible closed cycle for the history (φ,ψ) on some interval $[0,\tau]$ then the energy balance equation (1.4) and the observation that $\Sigma^+(0,\tau) \cup \Sigma^-(0,\tau) = [0,\tau]$ together imply

$$-\int_0^\tau s\cdot\dot{f} = e(0) - e(\tau) + \int_{\Sigma^+(0,\tau)} h + \int_{\Sigma^-(0,\tau)} h$$

$$= e(0) - e(\tau) + H + \int_{\Sigma^-(0,\tau)} h \ . \qquad (2.6)$$

Now

$$\int_{\Sigma^-(0,\tau)} h = \int_{\Sigma^-(0,\tau)} (1 - \theta^-/\theta)h + \theta^- \int_{\Sigma^-(0,\tau)} h/\theta$$

and substituting for h/θ from equation (1.5) gives

$$\int_{\Sigma^-(0,\tau)} h = \int_{\Sigma^-(0,\tau)} (1 - \theta^-/\theta)h - \theta^- \int_{\Sigma^-(0,\tau)} \sigma + \theta^- \int_{\Sigma^-(0,\tau)} \dot{\eta} \ . \qquad (2.7)$$

In a similar way we can also deduce that

$$H = \int_{\Sigma^+(0,\tau)} h = \int_{\Sigma^+(0,\tau)} (1 - \theta^+/\theta)h - \theta^+ \int_{\Sigma^+(0,\tau)} \sigma + \theta^+ \int_{\Sigma^+(0,\tau)} \dot{\eta} \ ,$$

which implies

$$\int_{\Sigma^-(0,\tau)} \dot{\eta} = \int_0^\tau \dot{\eta} - \int_{\Sigma^+(0,\tau)} \dot{\eta}$$

$$= - \int_{\Sigma^+(0,\tau)} \dot{\eta}$$

$$= - H/\theta^+ + 1/\theta^+ \int_{\Sigma^+(0,\tau)} (1 - \theta^+/\theta)h + \int_{\Sigma^+(0,\tau)} \sigma \ . \qquad (2.8)$$

Substituting for $\int_{\Sigma^-(0,\tau)} \dot\eta$ from equation (2.8) into (2.7) and then substituting for $\int_{\Sigma^-(0,\tau)} \dot h$ from (2.7) into (2.6) produces the key identity

$$-\int_0^\tau s\cdot\dot f = \left(\frac{\theta^+-\theta^-}{\theta^+}\right)H + e(0) - e(\tau) - \theta^-\int_0^\tau \sigma + \int_{\Sigma^+(0,\tau)} (1-\theta^+/\theta)\dot h + \int_{\Sigma^-(0,\tau)} (1-\theta^-/\theta)\dot h\,, \tag{2.9}$$

which holds for all (θ^+,θ^-,H)- admissible closed cycles on $[0,\tau]$. In the particular case of a Carnot cycle $\theta = \theta^+$ on $L^+(0,\tau)$ and $L^-(0,\tau) = (\Pi^-(0,\tau)\cap L^-(0,\tau))\cup L(0,\tau)$ where $\theta \le \theta^-$ on $\Pi^-(0,\tau)\cap\Sigma^-(0,\tau)$ and $\dot f \equiv 0$ on $\Sigma(0,\tau)$ and the identity (2.9) reduces to the identity

$$-\int_0^\tau s\cdot\dot f = \left(\frac{\theta^+-\theta^-}{\theta^+}\right)H + e(0) - e(\tau) - \theta^-\int_{[0,\tau]\cap\Sigma(0,\tau)}\sigma - \int_{\Sigma(0,\tau)}\theta\sigma \tag{2.10}$$

Given any $\epsilon > 0$ the regularity of the material enables us to choose a Carnot cycle with $e(\tau) < e(0) + \epsilon$ and $\int_0^\tau \sigma < \epsilon$. For this Carnot cycle

$$0 \le \theta^-\int_{[0,\tau]\cap L(0,\tau)}\sigma + \int_{\Sigma(0,\tau)}\theta\sigma \le (\theta^-+\theta^+)\int_0^\tau\sigma < (\theta^-+\theta^+)\,\epsilon\,,$$

and the identity (2.10) implies that

$$-\int_0^\tau s\cdot\dot f > \left(\frac{\theta^+-\theta^-}{\theta^+}\right)H - (1+\theta^-+\theta^+)\,\epsilon\,,$$

from which we deduce, since $\varepsilon > 0$ is arbitrary, that

$$W(\varphi,\psi,\theta^+,\theta^-,H) \ge \left(\theta^+ - \theta^-\right)H.$$

To prove the remainder of the theorem we return to the identity (2.9). On the set $\mathcal{E}^+(0,\tau)$ we have $h > 0$, by definition, and so $(1 - \frac{\theta}{\theta^+})h \le 0$ on $\mathcal{E}^+(0,\tau)$. In the same way $(1 - \frac{\theta}{\theta^-})h \le 0$ on $\mathcal{E}^-(0,\tau)$. On noting in addition that $\sigma \ge 0$, since compatibility with thermodynamics is assumed, we deduce from (2.9) that

$$-\int_0^\tau s\dot{\theta} \le \left(\frac{\theta^+ - \theta^-}{\theta^+}\right)H + e(0) - e(\tau), \qquad (2.11)$$

for all (θ^+,θ^-,H)- admissible closed cycles on $[0,\tau]$. In the case where the material has the energy relaxation property and (φ,ψ) is a constant history the history (φ^τ,ψ^τ) ends with the value $(\varphi(0), \theta(0))$ and thus the inequality (1.6) implies that $e(\tau) \ge e(0)$. It follows from (2.11) that

$$W(\varphi,\theta,\theta^+,\theta^-,H) \le \left(\frac{\theta^+ - \theta^-}{\theta^+}\right)H.$$

If the material is also regular the first part of the theorem applies and consequently

$$W((\varphi,\psi),\theta^+,\theta^-,H) \le \left(\frac{\theta^+ - \theta^-}{\theta^+}\right)H.$$ Q.E.D.

It should be recorded too that <u>equality holds in</u> (2.5) <u>for regular hyperelastic materials</u>. To see that this is so note that the inequality (2.11) and the assumption of hyper-elasticity, which implies *e(r)* = e(0), together yield

$$w(cp,<M^{+},e-,H) \leq \left(\frac{e^{+} \; n \; 9^{-}}{e^{+}}\right) H.$$

The assumption of regularity tells us that the first part of the theorem is applicable and the result follows•

3. An Example of a Regular Material.

This section is devoted to discussing a simple example of a one-dimensional regular material which is not elastic and for which the maximum mechanical work recoverable from a history can be computed explicitly.

Since we are dealing with the one-dimensional case we identify X and $L(X)$ with R and identify U with R^{++}. As the class of histories C we take the collection of all pairs (φ,ψ) with $\varphi: R^{+} \to R^{++}$ any continuous piecewise smooth function with $\int_0^\infty e^{-u}\varphi(u)du < \infty$ and with $\psi: R^{+} \to R$ any continuous and piecewise smooth function. Then the conditions C1 and C2 on C hold. The material with which we are concerned is the triple of response functions (E, S, Θ) defined by

$$\left.\begin{aligned}
E(\varphi,\psi) &= \tfrac{1}{2}\Big(\varphi(0) - \int_0^\infty e^{-u}\varphi(u)\,du\Big)^2 + k\varphi(0)\psi(0) + \tfrac{1}{2}c\psi(0)^2 \\
S(\varphi,\psi) &= \varphi(0) - \int_0^\infty e^{-u}\varphi(u)\,du + k\psi(0) \\
\Theta(\varphi,\psi) &= k\varphi(0) + c\psi(0),
\end{aligned}\right\} \quad (3.1)$$

where $k, c > 0$ are constants. In other words if (f,η) is a process and we define the function $\xi: R \to R$ by writing

$$\xi(t) = f(t) - e^{-t}\int_{-\infty}^{t} e^{u}f(u)\,du \qquad (3.2)$$

then the evolution with time t of the internal energy, stress and temperature is given by

$$e(t) = \frac{1}{2}\xi(t)^2 + kf(t)\eta(t) + \frac{1}{2}c\eta(t)^2 \tag{3.3}$$

$$s(t) = \xi(t) + k\eta(t) \tag{3.4}$$

$$\theta(t) = kf(t) + c\eta(t). \tag{3.5}$$

As before, we restrict attention to processes with $\theta > 0$. We prove the following proposition:

Proposition. The material defined by equations (3.1) is regular and it has the energy relaxation property. If the history $(\varphi,\psi)\in C$ and the working temperatures θ^-,θ^+ satisfy the inequalities

$$\theta^- \leq k\min(\varphi(0),\int_0^\infty e^{-u}\varphi(u)\,du) + c\psi(0)$$

$$\theta^+ \geq k\max(\varphi(0),\int_0^\infty e^{-u}\varphi(u)\,du) + c\psi(0)$$

then the maximum mechanical work recoverable from the history (φ,ψ) in (θ^+,θ^-,H)- admissible closed cycles is

$$W(\varphi,\psi,\theta^+,\theta^-,H) = \left(\frac{\theta^+-\theta^-}{\theta^+}\right)H + \frac{1}{2}\left(\varphi(0) - \int_0^\infty e^{-u}\varphi(u)\,du\right)^2$$

Proof. A straightforward computation using equations (1.4), (1.5), (3.3), (3.4) and (3.5) shows that the internal dissipation σ is given by $\theta\sigma = \xi^2$ and so compatibility with thermodynamics is assured.

We establish regularity in the following way. For each λ in $0 < \lambda < 4Hc^2/(k\theta^+)^2$ set $\alpha(\lambda) = \left(\frac{H}{\lambda} - \frac{1}{4}\left(\frac{k\theta^+}{c}\right)^2\right)^{1/2}$. Define the one-parameter family of processes (f_λ,η_λ) on the interval $(-\infty,5\lambda]$ by setting $(f_\lambda^0,\eta_\lambda^0) = (\varphi,\psi)$, which defines (f_λ,η_λ)

on $(-\infty, 0]$, by setting

$$f_\lambda(t) = \int_0^\infty e^{-u}\varphi(u)\,du + \xi_\lambda(t) + \int_0^t \xi_\lambda(u)\,du \qquad (3.6)$$

where

$$\xi_\lambda(t) = \begin{cases} \left(\varphi(0) - \int_0^\infty e^{-u}\varphi(u)\,du\right)e^{-t} + \frac{1}{k}(\theta^+ - \Theta(\varphi,\psi))\frac{1}{\lambda}(1-e^{-t}), & 0 \le t \le \lambda, \\ -\frac{1}{2}\frac{k\theta^+}{c} + \dfrac{\xi_\lambda(\lambda) + \frac{1}{2}\frac{k\theta^+}{c} - \alpha(\lambda)\tan\frac{c\alpha(\lambda)}{k\theta^+}(t-\lambda)}{1 - \frac{1}{\alpha(\lambda)}\left(\xi_\lambda(\lambda) + \frac{1}{2}\frac{k\theta^+}{c}\right)\tan\frac{c\alpha(\lambda)}{k\theta^+}(t-\lambda)}, & \lambda \le t \le 2\lambda \\ \xi_\lambda(2\lambda)e^{2\lambda - t} + \frac{1}{k}(\theta^- - \theta^+)\frac{1}{\lambda}(1-e^{2\lambda-t}), & 2\lambda \le t \le 3\lambda \\ \xi_\lambda(3\lambda)e^{3\lambda-t} + (f_\lambda(\lambda) - f_\lambda(2\lambda))\frac{1}{\lambda}(1-e^{3\lambda-t}), & 3\lambda \le t \le 4\lambda \\ \xi_\lambda(4\lambda)e^{4\lambda-t} + \frac{1}{k}(\Theta(\varphi,\psi) - \theta^-)\frac{1}{\lambda}(1-e^{4\lambda-t}), & 4\lambda \le t \le 5\lambda \end{cases} \qquad (3.7)$$

and by setting

$$\eta_\lambda(t) = \begin{cases} \psi(0), & 0 \le t \le \lambda \\ \frac{1}{c}(\theta^+ - kf_\lambda(t)), & \lambda \le t \le 2\lambda \\ \eta_\lambda(2\lambda), & 2\lambda \le t \le 3\lambda \\ \frac{1}{\lambda}\eta_\lambda(2\lambda)(4\lambda - t) + \frac{1}{\lambda}\psi(0)(t-3\lambda), & 3\lambda \le t \le 4\lambda \\ \psi(0), & 4\lambda \le t \le 5\lambda \end{cases} \qquad (3.8)$$

It should be noted that equation (3.6) results on solving the equation

$$\xi_\lambda(t) = f_\lambda(t) - e^{-t}\int_{-\infty}^{t} e^u f_\lambda(u)\,du.$$

subject to the condition $f_\lambda^0 = \varphi$. It should be noted too that there is a uniform bound K with $|\xi_\lambda(t)| < K$ for all t in $0 \le t \le 5\lambda$ and for all suitably small λ.

Straightforward but tedious computations elicit the following facts about the family $(f_\lambda, \eta_\lambda)$. First of all the corresponding temperature is piecewise linear on $[0,5\lambda]$ and has the values

$$\theta_\lambda(t) = \begin{cases} \frac{1}{\lambda}\Theta(\varphi,\psi)(\lambda - t) + \lambda\theta^+ t & , 0 \le t \le \lambda \\ \theta^+ & , \lambda \le t \le 2\lambda \\ \frac{1}{\lambda}\theta^+(3\lambda - t) + \frac{1}{\lambda}\theta^-(t - 2\lambda) & , 2\lambda \le t \le 3\lambda \\ \theta^- & , 3\lambda \le t \le 4\lambda \\ \frac{1}{\lambda}\Theta(\varphi,\psi)(t - 4\lambda) + \frac{1}{\lambda}\theta^-(5\lambda - t) & , 4\lambda \le t \le 5\lambda. \end{cases} \qquad (3.9)$$

Thus $\theta^- \le \theta_\lambda \le \theta^+$ on $[0,5\lambda]$ and $\theta_\lambda(5\lambda) = \Theta(\varphi,\psi) = \theta_\lambda(0)$. Since $\eta_\lambda(5\lambda) = \psi(0) = \eta_\lambda(0)$, equation (3.5) tells us that $f_\lambda(5\lambda) = f_\lambda(0) = \varphi(0)$ i.e. the process $(f_\lambda, \eta_\lambda)$ is closed on $[0,5\lambda]$. Secondly the heat flux $h_\lambda = \theta_\lambda\dot{\eta}_\lambda - \xi_\lambda{}^2$ is found to have the values

$$h_\lambda(t)=\begin{cases} -\xi_\lambda(t)^2 & , \ 0 < t < \lambda \\ \frac{1}{\lambda}H & , \ \lambda < t < 2\lambda \\ -\xi_\lambda(t)^2 & , \ 2\lambda < t < 3\lambda \\ \frac{1}{\lambda}(\psi(0) - \eta_\lambda(2\lambda)) - \xi_\lambda(t)^2 & , \ 3\lambda < t < 4\lambda \\ -\xi_\lambda(t)^2 & , \ 4\lambda < t < 5\lambda \end{cases} \tag{3.10}$$

Thirdly, as $\lambda \to 0+$,

$$\xi(\lambda) \to \varphi(0) - \int_0^\infty e^{-u}\varphi(u)\,du + \frac{1}{k}(\theta^{+} - \Theta(\varphi,0)) \tag{3.11}$$

$$\xi_\lambda(2\lambda) \to \varphi(0) - \int_0^\infty e^{-u}\varphi(u)\,du + \frac{1}{k}(\theta^{+} - \Theta(\varphi,\psi)) - \frac{2H}{k\theta^{+}} \tag{3.12}$$

$$\xi_\lambda(5\lambda) \to \varphi(0) - \int_0^\infty e^{-u}\varphi(u)\,du = \xi_\lambda(0). \tag{3.13}$$

It now follows from (3.6), (3.11) and (3.12) that

$$f_\lambda(2\lambda) - f_\lambda(\lambda) = \xi_\lambda(2\lambda) - \xi_\lambda(\lambda) + \int_\lambda^{2\lambda} h_\lambda(u)\,du \to -\frac{H}{k\theta^{+}}, \quad \text{as} \quad \lambda \to 0+,$$

and, since $\eta_\lambda(\lambda) = \psi(0)$ and $\theta_\lambda(\lambda) = \theta_\lambda(2\lambda) = \theta^{+}$, equation (3.5) implies

$$\eta_\lambda(2\lambda) = \eta_\lambda(\lambda) - \frac{1}{\theta^{+}}(f_\lambda(2\lambda) - f_\lambda(\lambda)) \to \psi(0) + \frac{H}{\theta^{+}} > \psi(0), \ \text{as} \ \lambda \to 0+.$$

Thus, for small enough λ, $\eta_\lambda(2\lambda) > \psi(0)$ and the expression (3.10) for the heat flux tells us that $h_\lambda \leq 0$ on $[3\lambda, 4\lambda]$. On examining (3.10) we see that in the process (f_λ, h_λ), with

A suitably small, heat is absorbed on the set £$^{+}$(0,5A)=(A,2A), emitted on the set ZT(0,5A)=[0,A]U[2A, 5A] and emitted at constant entropy on the set £(0,5A)=[0,A]U[2A,3A]U[4A,5A]. Furthermore the heat absorbed is H(0,5A) = $\int_A^{2A}$ r- = H and the working temperatures are attained on the sets TT^{+}(O,5A)=[A,2A] and TT"(O, 5A)=[3A,4A]. The conditions £$^{+}$(O,5?0<= TT^{+}(0,5A) and 2T(O, 5A) c TT~(O, 5A) U S(0,5.) are met and so, for small values of A, each of the processes (f..5TK) is a (6^{+},0",H)- admissible Carnot cycle for the history (ω,ψ) eC on the interval [O,5λ].

For this family of Carnot cycles equation (3.13) tells us that £..(5A) -* €\ (°) anc* so* from equation (3.3), the internal energy e_λ must satisfy

$$e_\lambda(5A)-\hat{}e(0)= \tfrac{1}{2}(<p(0) - \int_0 e^{-u}(p(u)du)^2 + k\omega(O)\psi(O)+ \tfrac{1}{2}c\psi(O)^2,$$

as A -» 0+. Thus to establish regularity it suffices to prove that the integral of the internal dissipation

$$\int_0^{5A} a_A \to 0, \quad \text{as } A \to 0+.$$

Since 9" ≤ 6$_A$ ≤ 9^{+} on [0,5A] and $\sigma_A = \frac{1}{\nu_A} n \text{£}_A^2$ it is sufficient just to show

$$\int_0^{5} \text{£}_A^2 \to 0, \quad \text{as } A \to 0+.$$

and this is clearly the case since (^(t)| < K for all te[0,5A] and all suitably small A,

Before proceeding further we pause to observe that the identity (2.10) and the regularity of the material imply that

$$-\int_0^{2\lambda} s_\lambda \dot{f}_\lambda \to \left(\frac{\theta^+ - \theta^-}{\theta^+}\right)H, \quad \text{as} \quad \lambda \to 0+, \tag{3.14}$$

that is to say the family of Carnot cycles $(f_\lambda, \eta_\lambda)$ ultimately extracts the classical amount of mechanical work and these cycles are traversed more and more rapidly as $\lambda \to 0+$.

The verification that the material has the energy relaxation property is trivial and so we turn, finally, to showing that the maximum recoverable mechanical work does indeed have the value stated provided the working temperatures meet the required conditions. To do this we construct Carnot cycles extracting not just the classical amount $\left(\frac{\theta^+ - \theta^-}{\theta^+}\right)H$ but also the mechanical work stored in the history (φ, θ). Consider the family of processes $(f'_\lambda, \eta'_\lambda)$ defined by setting

$$f'_\lambda(t) = \begin{cases} f_\lambda(t), & -\infty < t \le 5\lambda \\ \frac{1}{\lambda}\varphi(0)(6\lambda - t) + \frac{1}{\lambda}\left(\int_0^\infty e^{-u}\varphi(u)\,du\right)(t - 5\lambda), & 5\lambda < t < 6\lambda \\ \int_0^\infty e^{-u}\varphi(u)\,du, & 6\lambda \le t \le 6\lambda + \frac{1}{\lambda} \\ \lambda\varphi(0)\left(t - 6\lambda - \frac{1}{\lambda}\right) + \lambda\left(\int_0^\infty e^{-u}\varphi(u)\,du\right)\left(6\lambda + \frac{2}{\lambda} - t\right), & 6\lambda + \frac{1}{\lambda} \le t \le 6\lambda + \frac{2}{\lambda} \end{cases}$$

and

$$\eta'_\lambda(t) = \begin{cases} \eta_\lambda(t), & -\infty < t \le 5\lambda \\ \theta(0), & 5\lambda \le t \le 6\lambda + \frac{2}{\lambda} \end{cases}$$

where $(f_\lambda, \eta_\lambda)$ are as defined previously. Clearly $f'_\lambda(6\lambda + \frac{2}{\lambda}) = \varphi(0)$, $\eta'_\lambda(6\lambda + \frac{2}{\lambda}) = \theta(0)$ and the heat flux $h'_\lambda = -\vartheta_\lambda \xi_\lambda^2 \le 0$ on

$[5\lambda, 6\lambda + \frac{2}{\lambda}]$. Also the stated conditions on the working temperatures and equation (3.5) imply that $\theta^- < \theta'_\lambda < \theta^+$ on $[5\lambda, 6\lambda + \frac{2}{\lambda}]$ and so the processes $(\hat{f}_\lambda, \hat{\eta}_\lambda)$ form a family of (θ^+, θ^-, H) - admissible Carnot cycles for the history (φ, ψ) on the interval $[0, 6\lambda + \frac{2}{\lambda}]$, for small enough λ. It can be verified too that

$$-\int_{5\lambda}^{6\lambda+\frac{2}{\lambda}} s'_\lambda \dot{f}'_\lambda = -\int_{5\lambda}^{6\lambda+\frac{2}{\lambda}} (\xi'_\lambda + k\psi(0))\dot{f}'_\lambda$$

$$= -\int_{5\lambda}^{6\lambda+\frac{2}{\lambda}} \xi'_\lambda \dot{f}'_\lambda$$

$$= \left(\varphi(0) - \int_0^\infty e^{-u}\varphi(u)\,du\right)\left(\frac{1}{\lambda}\int_{5\lambda}^{6\lambda}\xi'_\lambda - \lambda\int_{6\lambda+\frac{1}{\lambda}}^{6\lambda+\frac{2}{\lambda}}\xi'_\lambda\right)$$

$$\to \frac{1}{2}\left(\varphi(0) - \int_0^\infty e^{-u}\varphi(u)\,du\right)^2, \quad \text{as } \lambda \to 0+. \qquad (3.15)$$

Combining equations (3.14) and (3.15) tells us that

$$-\int_0^{6\lambda+\frac{2}{\lambda}} s'\dot{f}' \longrightarrow \left(\frac{\theta^+ - \theta^-}{\theta^+}\right)H + \frac{1}{2}\left(\varphi(0) - \int_0^\infty e^{-u}\varphi(u)\,du\right)^2$$

and so

$$W(\varphi, \psi, \theta^+, \theta^-, H) \geq \left(\frac{\theta^+ - \theta^-}{\theta^+}\right)H + \frac{1}{2}\left(\varphi(0) - \int_0^\infty e^{-u}\varphi(u)\,du\right)^2. \qquad (3.16)$$

However, the converse inequality to (3.16) also holds and for the following reason. Let (f, η) be any (θ^+, θ^-, H)- admissible closed cycle for (φ, ψ) on some interval. For the material considered here

$$e(0) - e(\tau) = \frac{1}{2}\xi(0)^2 - \frac{1}{2}\xi(\tau)^2 \leq \frac{1}{2}\xi(0)^2 = \frac{1}{2}\left(\varphi(0) - \int_0^\infty e^{-u}\varphi(u)\,du\right)^2$$

and so the inequality (2.11) tells us that

$$W(\varphi,\psi,\theta^+,\theta^-,H) \leq \left(\frac{\theta^+ - \theta^-}{\theta^+}\right)H + \frac{1}{2}\left(\varphi(0) - \int_0^\infty e^{-u}\varphi(u)\,du\right)^2, \qquad (3.17)$$

which, together with (3.16), implies the required result. Q.E.D.

As mentioned in the introduction, the family of Carnot cycles (f_λ,η_λ) on $[0,5\lambda]$ is traversed faster and faster as $\lambda \to 0+$ whilst the family $(f'_\lambda,\eta'_\lambda)$ is traversed faster and faster on $[0,6\lambda]$ and more and more slowly on $[6\lambda,6\lambda+\frac{2}{\lambda}]$.

It would clearly be of considerable interest if, for a large class of materials with memory, regularity could be established, the maximum recoverable mechanical work evaluated explicitly and cycles maximizing the mechanical work constructed.

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