CONTINUOUS UNIFORMITIES

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by

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The concept of extending to the whole space a continuous pseudometric defined on a subspace has proven to be very useful. Among other topological notions, it has been used to characterize collectionwise normal spaces, normal spaces, and a form of paracompact subspaces (see [1], [2], and [9]). In this paper we introduce an analogous concept - a continuous uniformity. A uniformity U on a topological space $(X,\mathcal{T})$ is said to be <u>continuous</u> in case the topology $\mathcal{T}(U)$ generated by U is a subcollection of the original topology $\mathcal{T}$.

Since any collection of pseudometrics on a topological space give rise to a uniformity, it is natural to compare results on extending pseudometrics with results on extending uniformities. Some interesting facts in this line have already been shown (see [4]). In particular we have the following concepts and results. A subspace S of a topological space X is <u>P-embedded in X</u> in case every continuous pseudometric on S can be extended to a continuous pseudometric on X. Rephrasing the definition, it can be said that S is P-embedded in X if and only if every continuous pseudometric d on S can be extended to a pseudometric d^* on X in such a way that the topology $\mathcal{T}_{d^*}$ generated by d^* is such that $\mathcal{T}_{d^*} \subset \mathcal{T}$. It can now be shown that S is P-embedded in X if and only if every admissible uniformity U on S can be extended to a uniformity $\mathcal{U}^*$ on X such that $\mathcal{T}(\mathcal{U}^*) \subset \mathcal{T}$.

In a similar fashion, new characterizations of C-embedding and C^*-embedding may be given. In particular it will be possible to state that a completely regular space X is collectionwise normal if and only if for every closed subset F of X, every admissible uniformity on F has a continuous extension to X. Also a completely regular space is normal if and only if for every closed subset F of X, every admissible precompact uniformity has a continuous precompact extension. It will also be possible to say that the Stone-Cech compactification βX of a space X is that unique compact Hausdorff space containing X densely such that every admissible precompact uniformity on X has a continuous extension. In a like manner the Hewitt realcompactification υX of X is that unique realcompact Hausdorff space containing X densely such that every admissible uniformity on X generated by a collection of continuous real valued functions has a continuous extension.

The notions and terminology used in this paper, with one exception, is that of [5]. The exception is that of the definition of a uniform space where entourages are used as in [8]. <u>All topological spaces are assumed to be Tichonov spaces</u> (i.e. <u>completely regular and</u> T_1) .

As mentioned above any collection of pseudometrics $\mathcal{D}$ on a topological space $(X, \mathcal{T})$ gives rise to a uniformity. The sets

$$U(d,\epsilon) = \{(x,y) \in X \times X : d(x,y) < \epsilon\}$$

for all d in & and all *e* > 0 form a subbase for some uniformity. A subcollection P of X X X <u>generates</u> a uniformity U on X if P is a subbase for U . To every real valued function f on X is associated a pseudometric 0_f on X defined by

$$tf)_f(x,y) = |f(x) - f(y)| \quad (x,y e X) .$$

A family of functions Q generates a uniformity U if {U(0_f,€> : feQ and e > 0} generates U # The pseudometric topology associated with a pseudometric d is denoted by JTd . A pseudometric d is said to be *JjT* $_o$-<u>separable</u> in case the pseudometric space *(X,3d)* has a countable dense subset. The pseudometric is <u>totally</u> <u>bounded</u> if for every *e* > 0 there is a finite subset F of X such that X is the union of the. d-spheres of radius e centered at the points of F .

<u>Definitions</u>* A <u>precompact</u> <u>uniformity</u> is a uniformity U on X that is generated by a collection of bounded continuous real valued functions on X . If the uniformity is generated by a collection of continuous real valued functions then the uniformity is said to be <u>prerealcompact</u>.

Every uniformity U on a non-empty set X yields a unique topology 3(10 . This topology is obtained by taking as a base for the open sets the collection of sets U[x] for all U in U and all x in X . If 3T(U) is a sub-collection of the original topology JI on X then we say that U is a *continuous uniformity*. If 3(U) agrees with £ then U is called an *admissible uniformity*. The admissible uniformity on X generated by the collection of all continuous pseudometrics (respectively, all continuous real valued functions, fill bounded continuous real valued functions) on X is denoted by $U_Q(X)$ (respectively, C(X) , C*(X)).

If S is a subset of X and if Q is a collection of subsets of X , then by Q|s is meant the collection (G fl S : G∈Q} . A uniformity U* on X is an extension of a uniformity U on S in case U*|S X S = U . The subset S is *uniformly embedded in X* in case every admissible uniformity U on S has an extension that is a continuous uniformity on X . It is said to be *prerealcompact uniformly embedded in X* in case every prerealcompact admissible uniformity U on X has an extension that is a continuous uniformity on X . The subset S is *precompact uniformly embedded in X* if every precompact admissible uniformity U on X can be extended to a continuous uniformity on • X .

In the definiton of a prerealcompact uniformly embedded subset and a precompact uniformly embedded subset we required only that the extension be a continuous uniformity. Our first theorem shows that these extensions can be taken to be prereal-compact in the case of a prerealcompact uniformly embedded subset and likewise the extension can be taken to be precompact in the case of the precompact uniformly embedded subsets.

Theorem 1. *If* S *jis £i subset of a, topological space* (X, 2T) *then for the following conditions,* (1) *is equivalent to* (2) *and*

(3) ij3 equivalent (4) .

(1) S ij3 prerealcompact uniformly embedded in X •

(2) Every prerealcompact admissible uniformity on S has £i continuous extension that is prerealcompact.

(3) ^ *%*L* precompact uniformly embedded in X .

(4) Every precompact admissible uniformity on S has £ continuous extension that is precompact.

Proof. Trivially (1) is implied by (2) and (3) is implied by (4). To see that (l) implies (2), let U be an admissible prerealcompact uniformity on S . By (1), there exists a continuous uniformity U* on X that is an extension of U . Let G be the collection of continuous real valued functions on S that generate U , let B be the collection of £T(U*)-continuous real valued functions f on X such that f | SeG , and let V* be the uniformity on X generated by 6 . Now (S, U) is a uniform subspace of the uniform space (X,U*) and in [10, Theorem l] it was shown that every uniformly continuous real valued function on (S, U) has a continuous real valued extension to (X,U*) . Thus, since every function in G is uniformly continuous with respect to U *9* it follows that its continuous real valued extension to (X,U*) is a member of IB and hence is uniformly continuous with respect to If* . Thus the uniformity U is a subcollection of U*B X S . On the other hand it is clear that V*|s X S ^{c} U and thus V* is an extension of U . Since U* is a continuous uniformity on X and each function in B is ff(lt*)-continuous, it follows that they are also JT-continuous. Hence U* is a continuous uniformity and it is also prerealcompact. Hence (1) is equivalent to (2).

For the case that (3) implies (4) the following adjustments in the above proof may be made . For U an admissible precompact uniformity on S , let G be the collection of bounded continuous real valued functions on S that generate U , let IB be the collection of bounded 3'(U^)-continuous real valued functions on X whose traces belong to G and let \s* be as above. In [6, Theorem 3] it was shown that every bounded uniformly continuous real valued function on (S,U) has a bounded uniformly continuous real valued extension to (X^li*) . Using this stronger result the proof proceeds exactly as above.

Remark. In the last proof the uniformity If* may be chosen to be a subcollection of U* . This can be done by choosing for IB only those bounded U*-uniformly continuous functions on X whose traces belong to G .

The following result will be needed for the proof of the main theorem. The universal uniformity on a topological space is the largest continuous uniformity on the space.

Lemma. If U is a continuous uniformity on (X, JT) then Ʋ c $U_o(x)$.

Proof. Let G be the collection of all uniformly continuous pseudometrics on X with respect to If . For any Ve\s there is a deG and e > 0 such that

$$W = \{(x,y) \in X \times X : d(x,y) < \epsilon\} \subset V .$$

Since d is also continuous relative to *3* and *3* = 3(U_o (X)) ,
it follows that d is uniformly continuous relative to $\wedge^o$(X) [5, 15G.4]. Therefore W is an entourage in U^o(X) and hence V is an entourage in U^o(X) .

We can now state and prove our main results.

Theorem 2, Let S be a subset of a topological space (X,2) . Then S is^ P-embedded in X if and only if S is^ uniformly embedded in X .

Proof. To prove sufficiency, it is necessary to recall that S is P-embedded in X if and only if \(*S*) = U_Q(X)|S X S (see [4,Theorem 7.5]) and that for any subspace S of X , U_O(X)IS x S is always contained in U_O(S) . Since U_O(S) is an admissible uniformity, S uniformly embedded in X implies that there is a continuous uniformity *s* on X such that U|s X S = $^U{}_O$(S) . For any U *e* U_Q(S) there is a Velr so that V- 0 (S X S) = U . The lemma above then yields that U is a member of U_o(X)|S X S and it follows that S is P-embedded in X .

For the necessity of the condition, let U be an admissible uniformity on S , let P be the set of all continuous pseudo-metrics on (X,7) satisfying:

if deP then d|S X S is uniformly continuous
on S relative to U ,

and let U* be the uniformity on X generated by P . The uniformity U* is a continuous extension of U ; in fact the subbasic elements of 2(11*) are the d-spheres about each xeX for deP and *e* > 0 • The continuity of each deP relative to 3 implies that IT_d c 3T and hence 3(U*) c 3 . Moreover, U* | S X S = U . If U is any member of U then there is a continuous pseudometric d on S and an e > 0 so that

W = Ux, y) e S X S;d(x,y) < €.} c U .

Since S is P- embedded in X , d has a continuous pseudo-metric extension d* in P . Let

W* = {(x,y) € X X X : d*(x,y) < *e]* .

Then W*eU* and

w^ n (s x s) = w c u € u*|s x s .

Hence U c U^|S X S . Conversely, if UeU|s x S , then there is U*eU* such that U* fl (S X S) = U . Hence there is a deP and *e* > 0 for which

V = {(x,y) e X X X : d(x,y) < *e*} c U^ .

Thus V fl (S X S) c U^ n (S X S) and since d|S X S is uniformly continuous on S relative to U , v|s X Sell and hence UeU . This completes the proof.

Corollary. A completely regular space (X,?) is collection-wise normal if and only if every closed subset is uniformly embedded in X .

Proof. This follows from the Theorem and [9, Theorem 5.2].

Corollary. *Let* f *be a closed continuous function from a topological space* X *onto a topological space* Y . *Then* X *is paracompact if and only if* Y *is paracompact and* $f^{-1}(y)$ *is paracompact and uniformly embedded in* X *for each* y *in* Y .

Proof. This follows from the Theorem and [11, Theorem 1.]

Theorem 3. *If* S *is a subset of a topological space* $(X, \mathcal{T})$, *then the following statements are equivalent*:

(1) S *is* C-*embedded in* X .

(2) *Every admissible uniformity on* S *generated by a collection of* $\aleph_0$-*separable continuous pseudometrics has a continuous extension to* X .

(3) S *is prerealcompact uniformly embedded in* X .

Proof. (1) *implies* (2). Let U be an admissible uniformity on S generated by a collection P of $\aleph_0$-separable continuous pseudometrics on S and let U* be the uniformity on X generated by the collection P* of all continuous pseudometric extensions of members of P . In [4, Theorem 10.3], it was shown that if S is C-embedded in X then every $\aleph_0$-separable continuous pseudometric on S has a continuous extension to X . From this it follows that U* is an extension of U . Moreover, the topology $\mathcal{T}(U^*)$ is the union of the pseudometric topologies $\mathcal{T}_{d^*}$ and since each d* is a continuous pseudometric, it follows that U* is a continuous uniformity on X .

(2) *implies* (3). Let U be an admissible uniformity on S generated by a collection G of continuous real valued

functions on S . Since the usual metric, e , on the real numbers is j) $_o$-separable, for each feG , the continuous pseudometric θ_f = e o (f X f) is ^$_o$-separable- Hence u is generated by a collection of ft$_o$-separable continuous pseudometrics and (2) applies to give the desired result.

(3) implies (1). Suppose that every admissible uniformity on S generated by a collection of continuous real-valued functions has a continuous extension. In particular, since C(S) is an admissible uniformity, there is a continuous uniformity U|s X S = C(S) . Hence (S,C(S)) is a uniform subspace of (X,U) . As referred to in the proof of Theorem 1, every uniformly continuous real valued function on (S,C(S)) has a continuous real valued extension to (X,U) . Any continuous real valued function f on S is uniformly continuous with respect to C(S) and hence has a 2(U)-continuous real valued extension f* to X . Since U is a continuous uniformity, f* is also JT-continuous and it follows that S is C-embedded in X .

Corollary* The Hewitt realcompactification vTL of X is that unique realcompact Hausdorff space containing X densely such that every admissible uniformity on X generated by a, collection of continuous real valued functions has a. continuous extension.

Theorem 4. If S is a subset of a topological space (X,!J) , then the following statements are equivalent:

(1) S i£ C*-embedded in X .

(2) Every admissible uniformity on S generated by a^ collection of totally bounded continuous pseudometrics has a. continuous extension to X .

(3) S ijs precompact uniformly embedded in X .

Proof. Let U be an admissible uniformity on S generated by a collection of totally bounded continuous pseudometrics. In [l, Theorem 3.7], the authors have shown that if S is C*-embedded in X then every totally bounded continuous pseudometric on S has a continuous extension to X . Proceeding as in the proof of the first implication in Theorem 3, we have that statement (1) implies statement (2) of the present Theorem.

If U is an admissible uniformity on S generated by a collection of bounded continuous real valued functions J5 on S then the associated pseudometrics i/)f for feJf are totally bounded. Thus U is a uniformity generated by a collection of totally bounded continuous pseudometrics. Hence statement (2) implies statement (3).

It remains to show that statement (3) implies statement (1). Here the proof proceeds the same as in the proof of the final implication of Theorem 3 with the following modifications. Of course C(S) is replaced by C*(S) and 'continuous real valued function' by 'bounded continuous real valued function' . The stronger result of Katetov (also referred to in the proof of Theorem 1) is needed. In particular every bounded uniformly continuous real valued function on (S,C*(S)) has a bounded uniformly continuous real valued extension to (X,U) . This completes the proof.

Corollary. *The Stone-Cech compactification* j8x *of* X *is that unique compact Hausdorff space containing* X *densely such that every admissible precompact uniformity on* X *has a continuous extension*.

Corollary. *For completely regular spaces* (X,ID *the following statements are equivalent*:

(1) (X,?) *is normal,*

(2) *Every closed subset is precompact uniformly embedded in* X .

(3) *Every closed subset is prerealcompact uniformly*' *embedded in* X .

Proof.• This is just a restatement of the well-known Tietze Extension Theorem in conjunction with Theorems 3 and 4.

In closing we remark that for infinite cardinal numbers *y* one can define the concepts of P^{γ}-embedding (see [9]) and y-uniformly-embedding using the uniformities U_{γ} (see [4] and [3]). Here a y-uniformly embedded subset would be one for which every admissible uniformity on S generated by a collection of y-separable continuous pseudometrics on S has a continuous extension. With an appropriate modification of the lemma, it is possible to show that S *is* y-*uniformly embedded in* X *if and only if* S *is* P^-*embedded in* X .

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