INEQUALITIES FOR THE COEFFICIENTS OF
UNIVALENT FUNCTIONS

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the set of coefficients of a function of S, but this is not necessarily helpful if it is desired to obtain specific information about a given coefficient, or a given finite set of coefficients. The best-known example for this state of affairs is the Bieberbach conjecture $|a_n| \leq n$ which, in addition to the elementary case $n = 2$, has so far been shown to be accessible via the Grunsky inequalities only for $n = 4,6$. For $n = 4$, this result is due to Charzynski and Schiffer [1] (the original proof of $|a_4| \leq 4$, by Garabedian and Schiffer [2], had used much more difficult considerations), and the case $n = 6$ has been settled in a recent paper by R. N. Pederson [13].

Although the inequalities (1.3) are sharp for every n, they can be strengthened to

$$\sum_{\nu=1}^{\infty} \left| \sum_{\mu=1}^{n} a_{\nu\mu}\alpha_\mu \right|^2 \leq \sum_{\mu=1}^{n} \frac{|\alpha_\mu|^2}{\mu}, \tag{1.4}$$

as was shown by Milin [11], Jenkins [8], and Pommerenke [14]; an inequality equivalent to (1.4) had previously been obtained by Golusin [5,9]. (The fact that (1.4) contains (1.3) becomes apparent if it is noted that

$$\left| \sum_{\nu=1}^{n} \alpha_\nu \left(\sum_{\mu=1}^{n} a_{\nu\mu}\alpha_\mu \right) \right|^2 \leq \sum_{\nu=1}^{n} \frac{|\alpha_\nu|^2}{\nu} \sum_{\nu=1}^{n} \left| \sum_{\mu=1}^{n} a_{\nu\mu}\alpha_\mu \right|^2 .)$$

Another generalization of the Grunsky inequalities was given in a recent paper by Garabedian and Schiffer [4] who discovered a set of inequalities of a type similar to (1.3) which characterizes the coefficients of functions belonging to the subclass S_d of

S, consisting of those functions of S which do not take the value d in $|z| < 1$. While in [4] these inequalities are obtained with the help of Schiffer's variational method, a later paper by Hammel and Schiffer [7] gives an elementary derivation based on Jenkins' version [8] of the classical area principle.

In the present paper we shall show that this set of inequalities by Schiffer and Garabedian is but one example of a whole class of 'Grunsky-type inequalities' which hold for functions of S, of S_d, and of other subclasses of S. Our proofs will be based on procedures which are closely related to the area principle and, indeed, are equivalent to the latter if the analytic functions involved are single-valued. While it is possible to formulate these procedures in terms of properly generalized notions of area even if the functions in question are not single-valued, this is neither natural nor helpful, and we shall therefore refrain from doing so.

The following notations will be used throughout this paper. D will denote the image of the conformal mapping $z \rightarrow f(z)$, $f \in S$. D_ρ will stand for the image of $|z| < \rho$ $(0 < \rho < 1)$ under the same mapping, and C_ρ will denote the boundary of D_ρ (i.e., the image of $|z| = \rho$). $\overline{D}$ and $\overline{D}_\rho$ are the complements of the domains in question with respect to the extended plane. The statement that a real function $\phi(w)$ is continuous, or subharmonic near the point $w = \infty$ will mean that the function $\Psi(w) = \phi(w^{-1})$ possesses these properties near the point $w = 0$.

2. In this section, we derive some lemmas which will be used repeatedly later on.

Lemma 2.1. *Let* R *be a bounded simply-connected domain in the w-plane whose boundary* Γ *is a closed analytic curve, and let* $\bar{R}$ *denote the complement of* R *with respect to the extended plane. Let* $\sigma(w)$ *be analytic in* R *with the possible exception of a finite number of points, and let* $|\sigma(w)|$ *be single-valued and continuous in* R. *Then,*

$$\mathrm{Re}\{\frac{1}{i}\int_{\Gamma}\overline{\sigma(w)}\,d\sigma(w)\}\leq 0, \tag{2.1}$$

if Γ *is positively oriented with respect to* R.

Lemma 2.2. *Let* Γ_1 *be a closed analytic Jordan curve whose interior* R_1 *contains the curve* Γ *of Lemma* 2.1. *Let* $\sigma(w)$ *satisfy the same assumptions as in Lemma* 2.1, *except that* $\bar{R}$ *is now taken to be the complement of* R *with respect to* $R_1+\Gamma_1$. *Then*

$$\mathrm{Re}\{\frac{1}{i}\int_{\Gamma}\overline{\sigma(w)}\,d\sigma(w)\}\leq \mathrm{Re}\{\frac{1}{i}\int_{\Gamma_1}\sigma(w)\,d\sigma(w)\}, \tag{2.2}$$

where both Γ *and* Γ_1 *are positively oriented with respect to their interiors.*

If $\sigma(w)$ is single-valued in $\bar{R}$, these results are immediate consequences of Green's formula. Indeed, if R is defined as in Lemma 2.2, we have

$$\frac{1}{2i}\int_{\Gamma}\bar{\sigma}\,d\sigma-\frac{1}{2i}\int_{\Gamma_1}\bar{\sigma}\,d\sigma=-\iint_{R}|\sigma'|^2\,dA, \tag{2.3}$$

where dA is the area element, and this establishes (2.2). If $\Gamma_1 = \{w: |w| = r\}$, where r is sufficiently large, (2.1) follows from (2.3) on letting $r \to \infty$ and using the fact that $w = \infty$ is necessarily a regular point of σ.

If σ is not single-valued in $\bar{R}$, we cut $\bar{R}$ along a non-selfintersecting chain γ of smooth arcs which passes through all singularities of $\log \sigma$ in $\bar{R}$ and has a point in common with the boundary of $\bar{R}$. Since $\bar{R} - \gamma$ is simply-connected, the harmonic conjugate $\arg \sigma$ of $\log |\sigma|$ is single-valued in this region, and we may therefore use Green's formula. From the single-valuedness of $\log |\sigma|$ in R we infer that, on a section γ_0 of γ between two singularities of $\log \sigma$, the values of $\arg \sigma$ on opposite edges of γ_0 differ by a constant. Hence, the contribution of both edges of γ_0 to

$$\int_{\partial(\bar{R}-\gamma)}' |\sigma|^2 d(\arg \sigma)$$

is zero. But $d(\arg \sigma) = \mathrm{Re}\{-i\, d\sigma/\sigma\}$, and it follows therefore that

$$\mathrm{Re}\{\frac{1}{i}\int_{\partial(\bar{R}-\gamma)}' \bar{\sigma}\, d\sigma\} = \mathrm{Re}\{\frac{1}{i}\int_{\partial\bar{R}}' \bar{\sigma}\, d\sigma\}$$

$$= \mathrm{Re}\{\frac{1}{i}\int_{\Gamma_1}' \bar{\sigma}\, d\sigma\} - \mathrm{Re}\{\frac{1}{i}\int_{\Gamma} \bar{\sigma}\, d\sigma\}.$$

By Green's formula, this is equal to $2\iint |\sigma'|^2 dA$ over $\bar{R} - \gamma$, and this establishes (2.2). If σ is regular at $w = \infty$, (2.1) follows as before. If $w = \infty$ is a singular point of σ, we necessarily have $\sigma(w) = w^{\lambda}\tau(w)$, where τ is regular at infinity and λ is real. If we assume (as we may) that $\tau(\infty) \neq 0$, the continuity of $|\sigma|$ requires that $\lambda < 0$. Using this in the computation of the right-hand integral in (2.2) over a large circle $|w| = r$, we obtain (2.1) for $r \to \infty$.

Lemmas 2.1 and 2.2 may also be expressed in terms of real functions, since, by the Cauchy-Riemann equations,

$$
\begin{aligned}
\operatorname{Re}\left\{\frac{1}{i}\int_{\Gamma} \bar{\sigma}\, d\sigma\right\} &= \operatorname{Re}\left\{\frac{1}{2i}\int_{\Gamma} |\sigma|^2 \frac{d(\sigma^2)}{\sigma^2}\right\} = \frac{1}{2}\int_{\Gamma} |\sigma|^2 d(\arg \sigma^2) \\
&= \frac{1}{2}\int_{\Gamma} |\sigma|^2 \frac{\partial \log|\sigma|^2}{\partial n}\, ds = \frac{1}{2}\int_{\Gamma} \frac{\partial |\sigma|^2}{\partial n}\, ds,
\end{aligned}
\tag{2.4}
$$

where $ds = |dw|$ and $\partial/\partial n$ denotes differentiation in the direction of the outwards pointing normal. We shall now show that, in this formulation, the conclusions of Lemmas 2.1 and 2.2 do not depend at all on the assumption that $|\sigma|^2$ is the square of the modulus of an analytic function; all that is required is the assumption that this function be subharmonic.

Lemma 2.3. *Let Γ, R and $\bar{R}$ have the same meaning as in Lemma 2.1, and let $Q(w)$ be a non-constant subharmonic function in $\bar{R}$ which is continuously differentiable on Γ. Then*

$$
\int_{\Gamma} \frac{\partial Q}{\partial n}\, ds \leq 0,
\tag{2.5}
$$

with equality only if Q is harmonic in $\bar{R}$.

Lemma 2.4. *If* Γ, R, Γ_1 *and* $\bar{R}$ *are defined as in Lemma* 2.2, *and if* Q(w) *is subharmonic in* $\bar{R}$ *and continuously differentiable on* Γ *and* Γ_1, *then*

$$\int_{\Gamma} \frac{\partial Q}{\partial n}\, ds \leq \int_{\Gamma_1} \frac{\partial Q}{\partial n}\, ds, \tag{2.6}$$

with equality only if Q *is harmonic in* $\bar{R}$.

If it were known that the harmonic function u in R whose boundary values coincide with those of Q has a continuous normal derivative on these curves, (2.6) would follow very easily. Indeed, we have $Q \leq u$ in R, and the derivatives are taken in the direction which points away from the interiors of these curves. Hence, if w is a point on Γ and w_1 a nearby point of $\bar{R}$ on the normal at w, we have

$$\frac{Q(w_1) - Q(w)}{|w_1 - w|} \leq \frac{u(w_1) - u(w)}{|w_1 - w|},$$

and thus $\partial Q/\partial n \leq \partial u/\partial n$ on Γ. Similary. $\partial Q/\partial n \geq \partial u/\partial n$ on Γ_1. Since

$$\int_{\Gamma} \frac{\partial u}{\partial n}\, ds = \int_{\Gamma_1} \frac{\partial u}{\partial n}\, ds,$$

it follows that

$$\int_{\Gamma} \frac{\partial Q}{\partial n}\, ds \leq \int_{\Gamma} \frac{\partial u}{\partial n}\, ds = \int_{\Gamma_1} \frac{\partial u}{\partial n}\, ds \leq \int_{\Gamma_1} \frac{\partial Q}{\partial n}\, ds,$$

and (2.6) is proved.

In order to avoid the use of $\partial u/\partial n$, we map $\bar{R}$ conformally onto an annulus $\rho_1 < |z| < \rho_2$, and we note that, because of the analyticity of the arcs Γ and Γ_1, the mapping function

$w = p(z)$ is analytic in $\rho_1 \leq |z| \leq \rho_2$. Since the mapping is conformal on the boundary of R, the normal direction on Γ and Γ_1 is transformed into the radial direction on $|z| = \rho_1$ and $|z| = \rho_2$. If we write $Q(w) = T(z)$, $|z| = r$, we thus have

$$|p'(z)|\frac{\partial Q}{\partial n} = \frac{\partial T}{\partial r}$$

and therefore, because of $ds = |p'(z)| r\, d\theta$,

$$\int_{\Gamma} \frac{\partial Q}{\partial n}\, ds = r \frac{\partial}{\partial r} \int_0^{2\pi} T(re^{i\theta})\, d\theta, \qquad r = \rho_1, \tag{2.7}$$

and a similar expression for $w \in \Gamma_1$. The function $T(z)$ is subharmonic in $\rho_1 \leq |z| \leq \rho_2$. Hence, by a classical result [10], the function

$$M(r) = \frac{1}{2\pi} \int_0^{2\pi} T(re^{i\theta})\, d\theta \tag{2.8}$$

is a convex function of $\log r$ for $\rho_1 \leq r \leq \rho_2$. This implies

$$\frac{\partial M(r)}{\partial \log r}\Big|_{r=\rho_1^+} \leq \frac{\partial M(r)}{\partial \log r}\Big|_{r=\rho_2^-}. \tag{2.9}$$

Because of (2.7) and (2.8), this establishes (2.6).

To prove (2.5), we apply (2.6) to the case in which Γ_1 is a circumference $|w| = r$ with r sufficiently large, and we obtain

$$\int_{\Gamma} \frac{\partial Q}{\partial n}\, ds \leq \frac{\partial M(r)}{\partial \log r}, \tag{2.10}$$

where $M(r)$ is the mean value (2.8) (with Q instead of T). If the right-hand side of (2.10) were positive for any value of r, it would follow from the convexity of $M(r)$ (as a function of $\log r$) that $M(r) \to \infty$ for $r \to \infty$. Since this would violate the hypotheses of Lemma 2.3, the right-hand side of (2.10) cannot be positive. This proves (2.5).

If there is equality in (2.6), the same is true in (2.9). Because of the convexity of $M(r)$ as a function of $\log r$ this implies that $M(r) \equiv c \log r$ $(\rho_1 \leq r \leq \rho_2)$, where c is a constant. If $u(z)$ is the harmonic function in $\rho_1 < |z| < \rho_2$ whose values on $|z| = \rho_1$ and $|z| = \rho_2$ coincide with those of $T(z)$, the function $T - u$ is also subharmonic and the mean value

$$(2.11) \qquad M_u(r) = \frac{1}{2\pi} \int_0^{2\pi} [T(re^{i\theta}) - u(re^{i\theta})]\,d\theta$$

vanishes for $r = \rho_1$ and $r = \rho_2$. By the argument just used, we have $M_u(r) = c_1 \log r$. Since now, evidently, $c_1 = 0$, we find that $M_u(r) \equiv 0$. On the other hand, $T(z) - u(z) \leq 0$ throughout $\rho_1 \leq |z| \leq \rho_2$, and (2.11) shows that a contradiction with $M_u(r) \equiv 0$ can be avoided only if $T(z) \equiv u(z)$. This establishes the statement regarding equality in Lemma 2.4 (and, similarly, in Lemma 2.3).

It may be noted that equality in (2.5) and (2.6) is excluded if the subharmonic function Q is the modulus of an analytic function, since the latter, for elementary reasons, cannot be harmonic unless it reduces to a constant.

On the other hand, there may be equality in (2.5) and (2.6) (and also in (2.1) and (2.2)) if the conditions imposed on Γ are relaxed. For instance, if Γ is allowed to consist of a finite number of analytic arcs, equality will hold if R is a slit-domain and Γ_1 (in Lemmas 2.2 and 2.4) is the 'outer boundary' of R. That this is indeed the case becomes evident if it is observed that the contributions of the two edges of each slit to the integrals on the right-hand sides of (2.2) and (2.6) cancel each other. Similarly, there will be equality in (2.1) and (2.5) if R is bounded by analytic slits (and, thus, $\bar{R}$ has zero area).

In our applications of these lemmas, Γ will be identified with a level curve C_ρ, and the desired result will then be obtained by letting $\rho \to 1$. This detour is necessary since C is subject to no smoothness assumptions, and the lemmas may not be applicable to the case $\Gamma = C$. In all these cases, it is evident that, in the inequalities obtained in this way, the sign of equality is possible only if the complement of C (with respect to the extended plane or with respect to $R_1 + \Gamma_1$, as the case may be) has zero area. If C is bounded by analytic arcs (or, at least, piecewise differentiable arcs) there will indeed be equality in these cases. If C is subject to no smoothness assumptions, the question of equality becomes more subtle. To avoid difficulties we shall therefore, in the formulation of our results, confine ourselves to the statement that, in the inequalities under consideration, equality is possible only if the complement of D has zero area.

We add here a restatement of Lemma 2.3 in a form which is particularly well suited to applications involving univalent functions.

Lemma 2.5. *If* $f \in S$ *and* $Q(w)$ *is subharmonic in* $\bar{D}_{\varsigma}$, *then the function*

$$H(r) \equiv \frac{1}{2\pi} \int_0^{2\pi} Q[f(re^{i\theta})]\,d\theta \tag{2.12}$$

is non-increasing on $(\varsigma, 1)$.

To see that this is equivalent to Lemma 2.3 we only have to note that $H(r)$ is essentially identical with the function $M(r)$ defined in (2.8) (with $f(z)$ playing the part of the function $p(z)$ used in the definition of $T(z)$) and that, by (2.7), (2.5) expresses the fact that $H(r)$ is non-increasing.

To illustrate the use of Lemma 2.5, we set $Q(w) = |P(w^{-1})|^2$, where $P(t)$ is a polynomial of degree n. If

$$P[\frac{1}{f(z)}] = \sum_{\nu=-n}^{\infty} c_\nu z^\nu,$$

the function $H(r)$ of (2.12) is found to be of the form

$$H(r) = \sum_{\nu=-n}^{\infty} |c_\nu|^2 r^{2\nu}.$$

Hence,

$$\frac{1}{2} \lim_{r \to 1} H'(r) = \sum_{\nu=-n}^{\infty} \nu |c_\nu|^2 \leq 0,$$

and this is equivalent to the 'strong' Grunsky inequalities (1.4) [11].

3. In this section, we show that the 'strong' Grunsky inequalities (1.4) are a special case of a general set of inequalities which depend on a parameter $\delta\in[0,1]$. For $\delta = 0$, this set reduces to (1.4). To formulate our result, we associate with the function $f(z)$ a set of polynomials $P_n^{(\delta)}(t)$, $n = 1,2,\ldots,P_n^{(\delta)}(0) = 0$, which reduce to the classical Faber polynomials for $\delta = 0$. $P_n^{(\delta)}(t)$ is defined by the requirement that the Laurent expansion of the function $[z^{-1}f(z)]^{\delta}P_n^{(\delta)}[f^{-1}(z)]$ should contain no negative powers of z except for the term z^{-n}, i.e.,

$$(3.1) \qquad [\frac{f(z)}{z}]^{\delta}P_n^{(\delta)}[\frac{1}{f(z)}] = \frac{1}{z^n} + \sum_{\nu=0}^{\infty} b_{\nu n}z^{\nu}.$$

The polynomials $P_n^{(\delta)}$ are easily computed. We have

$$(3.2) \qquad P_1^{(\delta)}(t) = t,\ P_2^{(\delta)}(t) = t^2 + (2-\delta)a_2t,$$

etc., where the a_k are the coefficients of the expansion (1.1).

There is a simple relation between $P_n^{(\delta)}$ and the ordinary Faber polynomial F_n of the function f. Since, by the definition of F_n, the function $F_n[1/f(z)] - z^{-n}$ remains bounded near $z = 0$, it follows from (3.1) that

$$[\frac{f(z)}{z}]^{\delta}P_n^{(\delta)}[\frac{1}{f(z)}] = F_n[\frac{1}{f(z)}] + O(1)$$

and thus

$$P_n^{(\delta)}[\frac{1}{f(z)}] = [\frac{z}{f(z)}]^{\delta}F_n[\frac{1}{f(z)}] + O(1).$$

If $z = g(w)$ is the function inverse to $w = f(z)$, this may be written

$$P_n^{(\delta)}[\tfrac{1}{w}] = [\tfrac{g(w)}{w}]^{\delta} F_n[\tfrac{1}{w}] + O(1),$$

where $O(1)$ now denotes terms which are bounded near $w = 0$. $P_n^{(\delta)}(w^{-1})$ is thus found to be the meromorphic part of

$$[\tfrac{g(w)}{w}]^{\delta} F_n[\tfrac{1}{w}]$$

near $w = 0$, where F_n is the Faber polynomial and $z = g(w)$ is the inverse of $w = f(z)$.

We now state our result.

Theorem 3.1. _Let_ $f \epsilon S$, _let_ $0 \leq \delta \leq 1$ _and let_ $P_n^{(\delta)}(t)$ _be the polynomial of degree_ n _defined by_ $P_n^{(\delta)}(0) = 0$ _and the expansion_ (3.1). _If_ $\alpha_1, \alpha_2, \ldots, \alpha_n$ _are arbitrary complex numbers and the_ $b_{\nu n}$ _are the coefficients appearing in_ (3.1), _then_

$$(3.3) \qquad \sum_{\nu=0}^{\infty} (\nu + \delta) \left| \sum_{\mu=1}^{n} b_{\nu\mu} \alpha_\mu \right|^2 \leq \sum_{\mu=1}^{n} (\mu - \delta) |\alpha_\mu|^2 .$$

Equality in (3.3) _is possible only if_ $\bar{D}$ _has zero area_.

For $\delta = 0$, (3.3) reduces to (1.4). Indeed, $P_n^{(0)}(t)$ is the Faber polynomial $F_n(t)$ and, in this case, $b_{\nu\mu} = -\mu a_{\nu\mu}$, where the $a_{\nu\mu}$ are given by the generating function (1.2) [16]. If the arbitrary constants α_μ in (3.3) are replaced by $\mu^{-1}\alpha_\mu$, (3.3) becomes identical with (1.4).

We also note that, for $n = 1$, (3.3) yields the inequality

$$(3.4) \qquad \sum_{\nu=0}^{\infty} (\nu + \delta) |b_{\nu 1}|^2 \leq 1 - \delta,$$

where, by (3.1) and (3.2), the $b_{\nu 1}$ are the coefficients of the expansion

$$\left[\frac{f(z)}{z}\right]^{\delta-1} = 1 + \sum_{\nu=0}^{\infty} b_{\nu 1} z^{\nu+1}.$$

(3.4) is the well-known inequality of Prawitz [15], which is thus found to be but the first of a countable set of related inequalities.

It may also be of interest to point out that for $\delta = \frac{1}{2}$ the inequalities (3.3) are equivalent to the ordinary Grunsky inequalities for the odd univalent function $\sqrt{f(z^2)}$, which have been used to such good effect in [1], [3], [13]. Since the Bieberbach inequalities for $n = 4,6$ are accessible via these 'odd' inequalities, but apparently not via the ordinary Grunsky inequalities, it may perhaps be rewarding to study the possible contributions of the general inequalities (3.3) to the coefficient problem.

To prove Theorem 3.1, we consider the function

$$\sigma(w) = w^{\delta} \sum_{\mu=1}^{n} \alpha_\mu P_\mu^{(\delta)}\left(\frac{1}{w}\right), \tag{3.5}$$

where $\alpha_1, \ldots, \alpha_n$ are arbitrary complex constants. In $\bar{D}_\rho$ $(0 < \rho < 1)$, $|\sigma(w)|$ is single-valued and continuous, and we may therefore apply Lemma 2.1. With $w = f(z)$ we have, by (3.5) and (3.1),

$$\sigma(w) = \sum_{\mu=1}^{n} \frac{\alpha_\mu}{z^{\mu-\delta}} + \sum_{\nu=0}^{\infty} \left(\sum_{\mu=1}^{n} b_{\nu\mu} \alpha_\mu \right) z^{\nu+\delta}.$$

Hence, with the abbreviation $\gamma_\nu = \sum_{\mu=1}^{n} b_{\nu\mu} \alpha_\mu$,

$$\frac{1}{2\pi i}\int_{C_\rho}\bar{\sigma}\,d\sigma = \frac{1}{2\pi i}\int_{|z|=\rho}\overline{\left[\sum_{\mu=1}^{n}\frac{\alpha_\mu}{z^{\mu-\delta}} + \sum_{\nu=0}^{\infty}\mathfrak{b}_\nu z^{\nu+\delta}\right]}\left[-\sum_{\mu=1}^{n}\frac{(\mu-\delta)}{z^{\mu-\delta}} + \right.$$

$$\left. + \sum_{\nu=0}^{\infty}(\nu+\delta)\mathfrak{b}_\nu z^{\nu+\delta}\right]\frac{dz}{z}$$

$$= \frac{\rho^{2\delta}}{2\pi i}\int_{|z|=\rho}\overline{\left[\sum_{\mu=1}^{n}\frac{\alpha_\mu}{z^{\mu}} + \sum_{\nu=0}^{\infty}\mathfrak{b}_\nu z^{\nu}\right]}\left[-\sum_{\mu=1}^{n}\frac{(\mu-\delta)\alpha_\mu}{z^{\mu}} + \sum_{\nu=0}^{\infty}(\nu+\delta)\mathfrak{b}_\nu z^{\nu}\right]\frac{dz}{z}$$

$$= -\sum_{\mu=1}^{n}(\mu-\delta)|\alpha_\mu|^2\rho^{2\delta-2\mu} + \sum_{\nu=0}^{\infty}(\nu+\delta)|\mathfrak{b}_\nu|^2\rho^{2\delta+2\nu}.$$

Using (2.1) and letting $\rho \to 1$, we obtain (3.3).

It is also possible to obtain a set of inequalities essentially equivalent to (3.3) via a generating function. We define the coefficients $c_{\nu\mu}$ by means of the expansion

$$[f(z) - f(\zeta)]^\delta - [f(z)]^\delta = z^\delta\left[\frac{f(z)}{z}\right]^\delta\left\{\left[1 - \frac{f(\zeta)}{f(z)}\right]^\delta - 1\right\}$$

(3.6)
$$= z^\delta\left[\frac{f(z)}{z}\right]^\delta\sum_{\nu=1}^{\infty}(-1)^\nu\binom{\delta}{\nu}\frac{f^\nu(\zeta)}{f^\nu(z)} = z^\delta\sum_{\mu=1}^{\infty}\left(\sum_{\nu=-\mu}^{\infty}c_{\nu\mu}z^\nu\right)\zeta^\mu,$$

which will converge if $|\zeta|$ is small enough so that $\max_{|\zeta|=\rho}|f(\zeta)| < \min_{|z|=r}|f(z)|$ $(0 < \rho < r < 1)$. We then have the following result.

Theorem 3.2. *If* $f(z)\in S$, $0 \le \delta \le 1$, *and the coefficients* $c_{\nu\mu}$ *are defined by the expansion* (3.6), *then*

(3.7)
$$\sum_{\nu=-n}^{\infty}(\nu+\delta)\left|\sum_{\mu=1}^{n}c_{\nu\mu}\alpha_\mu\right|^2 \le 0$$

where $\alpha_0,\ldots,\alpha_n$ *are arbitrary complex constants and* $c_{\nu\mu} = 0$ *for* $\nu < -\mu$.

The result follows again from Lemma 2.1, but this time we set

$$(3.8) \qquad \sigma(w) = \int_{|\zeta|=\rho} \{[w - f(\zeta)]^{\delta} - w^{\delta}\} dm(\zeta),$$

where

$$(3.9) \qquad dm(\zeta) = \frac{1}{2\pi i}\Big(\sum_{\mu=1}^{n} \frac{\alpha_\mu}{\zeta^\mu}\Big)\frac{d\zeta}{\zeta},$$

and ρ is small enough so that $|f(\zeta)| < |f(z)|$ for all $|\zeta| = \rho$ and $|z| = 1$. To show that Lemma 2.1 is applicable, we choose a number r in $(\rho, 1)$ and join a point on C_r with $w = \infty$ by an analytic curve γ in $\bar{D}_r$. We then extend γ by adding an analytic arc γ_ζ which joins $w = f(\zeta)$ and the endpoint of γ on C_r and, except for the latter point, is contained in $D_r \cap \bar{D}_\rho$. The function $p_\zeta(w) \equiv [w - f(\zeta)]^{\delta} - w^{\delta}$ is single-valued in the domain obtained by cutting the w-plane along $\gamma + \gamma_\zeta$ and, if $p_\zeta^{(1)}(w_o)$, $p_\zeta^{(2)}(w_o)$ $(w_o \in \gamma)$ denote the values taken by $p_\zeta(w)$ if w approaches w_o from opposite sides of γ, we have $p_\zeta^{(2)}(w_o) = e^{2\pi i \delta} p_\zeta^{(1)}(w_o)$ (or $p_\zeta^{(2)}(w_o) = e^{-2\pi i \delta} p_\zeta^{(1)}(w_o)$, depending on the identification of the two edges of γ). Since this is true for all values of ζ on $|\zeta| = \rho$, the function $\sigma(w)$ defined by (3.8) and (3.9) has the same behavior. Hence, $|\sigma(w)|$ is single-valued in D. Since $|\sigma(w)|$ is also continuous in D, we may apply Lemma 2.1. If we set $w = f(z)$ we have, by (3.6), (3.8) and (3.9),

$$\sigma(w) = \sum_{\nu=-n}^{\infty} \Big(\sum_{\mu=1}^{n} c_{\nu\mu}\alpha_\mu\Big) z^{\nu+\delta},$$

and thus, for $r < s < 1$,

$$\frac{1}{2\pi i}\int_{C_s}\overline{\sigma(w)}\,d\sigma(w) = \sum_{\nu=-n}^{\infty}(\nu+\delta)\left|\sum_{\mu=1}^{n} c_{\nu\mu}\alpha_\mu\right|^2 s^{2\nu+2\delta}.$$

The inequality (3.7) now follows by using (2.1) and letting $s \to 1$.

4. We now turn to the consideration of certain subclasses of S, beginning with the class S_d of functions $f(z)\in S$ for which $f(z) \neq d$ in $|z| < 1$. We shall show that there exists a one-parameter family of sets of Grunsky-type inequalities for functions of this class. Like in the case discussed in the preceding section, the sets of inequalities depend on a parameter $\delta\in[0,1]$. The inequalities obtained by Garabedian and Schiffer [4] (and, by a simpler method, by Hummel and Schiffer [7]) will be contained in our result for the case $\delta = \frac{1}{2}$.

The case $\delta = \frac{1}{2}$ presents some particular features which make it possible to treat it in a much simpler manner than the case of general δ. We shall therefore denote this section to a separate treatment of this case, leaving the general case for the following section.

Theorem 4.1. *Let $f\in S_d$, and $a_{\nu\mu}$ be the coefficients defined by (1.2) and let $b_{\nu\mu}$ be given by the expansion*

$$\log[\sqrt{f(z) - d} + \sqrt{f(\zeta) - d}] = \sum_{\nu,\mu=0}^{\infty} b_{\nu\mu} z^\nu \zeta^\mu, \tag{4.1}$$

where in both cases the same branch of the radical is taken. Let $\alpha_1,\alpha_2,\ldots,\alpha_n$ be arbitrary complex constants, let α_0 be real,

<u>and</u> <u>set</u> $c_{\nu\mu} = a_{\nu\mu} - 2b_{\nu\mu}$. <u>Then</u>

$$(4.2) \qquad 2\alpha_0 \operatorname{Re}\{\sum_{\mu=0}^{n} c_{0\mu}\alpha_\mu\} + \sum_{\nu=1}^{\infty}\nu|\sum_{\mu=0}^{n} c_{\nu\mu}\alpha_\mu|^2 \leq \sum_{\mu=1}^{n}\frac{|\alpha_\mu|^2}{\mu}.$$

<u>Equality</u> <u>in</u> (4.2) <u>is</u> <u>possible</u> <u>only</u> <u>if</u> $\bar{D}$ <u>has</u> <u>zero</u> <u>area</u>.

In our notation, the result of Garabedian and Schiffer [4] referred to above is equivalent to

$$(4.3) \qquad \operatorname{Re}\{\sum_{\nu,\mu=0}^{n} c_{\nu\mu}\alpha_\nu\alpha_\mu\} \leq \sum_{\nu=1}^{n}\frac{|\alpha_\nu|^2}{\nu}.$$

While these inequalities are sharp, they are weaker than (4.2). Indeed, if we denote the left-hand and right-hand sides of (4.3) by A and B, respectively, and we write 2C for the first term on the left-hand side of (4.2), we have

$$\begin{aligned} A &= C + \sum_{\nu=1}^{n}(\sum_{\mu=0}^{n} c_{\nu\mu}\alpha_\mu)\alpha_\nu \\ &\leq C + [\sum_{\nu=1}^{n}\nu|\sum_{\mu=0}^{n} c_{\nu\mu}\alpha_\mu|^2]^{1/2}B^{1/2}. \end{aligned}$$

Hence, by (4.2),

$$\begin{aligned} A &\leq C + (B - 2C)^{1/2}B^{1/2} = C + [(B - C)^2 - C^2]^{1/2} \\ &\leq C + B - C = B, \end{aligned}$$

and this is inequality (4.3). Incidentally, this also shows that equality in (4.3) is possible only if $\sum_{\mu=0}^{n} c_{\nu\mu}\alpha_\mu = 0$ for $\nu = n + 1$, $n + 2, \ldots$, and $C = 0$, i.e., $\alpha_0 \operatorname{Re}\{\sum_{\mu=0}^{n} c_{0\mu}\alpha_\mu\} = 0$.

To prove Theorem 4.1, we utilize the function

$$(4.4) \qquad \sigma_\zeta(w) = \log \frac{\sqrt{d - f(\zeta)} - \sqrt{d - w}}{\sqrt{d - f(\zeta)} + \sqrt{d - w}}, \quad |\zeta| < 1.$$

If $|\zeta| < \rho < 1$, the function $u(w) = \mathrm{Re}\{\sigma_\zeta(w)\}$ is harmonic and single-valued on the two-sheeted surface Δ_ρ obtained by removing from the Riemann surface of $\sqrt{d - w}$ the parts covering the domain D_ρ. If $u_1(w)$ and $u_2(w)$ are the two branches of $u(w)$ in $\bar{D}_\rho$, we evidently have $u_1(w) = -u_2(w)$, and this property extends to the harmonic function $U(w)$ defined by

$$(4.5) \qquad U(w) = \mathrm{Re}\{\sigma(w)\} = \int_{|\zeta|=r} u(w)\, dm(\zeta) = \mathrm{Re}\{\int_{|\zeta|=r} \sigma_\zeta(w)\, dm(\zeta)\}$$

where $dm(\zeta)$ is the real differential

$$(4.6) \qquad dm(\zeta) = \frac{1}{2\pi i}\left[\sum_{\mu=1}^{n} \frac{r^{\mu}\alpha_\mu}{\zeta^\mu} + \alpha_0 + \sum_{\mu=1}^{n} \frac{\bar{\alpha}_\mu \zeta^\mu}{r^\mu}\right]\frac{d\zeta}{\zeta}, \quad |\zeta| = r,\ \alpha_0 \text{ real},$$

and $0 < r < \rho < 1$.

Since $U_1(w) = -U_2(w)$, the function $U^2(w)$ is single-valued in $\bar{D}_\rho$ and, as the square of a harmonic function, it is subharmonic. It therefore follows from Lemma 2.3 that

$$(4.7) \qquad \int_{C_\zeta} U(w) \frac{\partial U(w)}{\partial n}\, ds \leq 0.$$

To express the left-hand side of (4.7) in terms of the coefficients of f we note that, with $w = f(z)$, (4.4) may be written in the form

$$\sigma_\zeta(w) = \log \frac{f(z) - f(\zeta)}{z - \zeta} - 2 \log[\sqrt{d - f(\zeta)} + \sqrt{d - f(z)}]$$
$$+ \log(1 - \frac{\zeta}{z}) + \log z$$

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$$\sigma_\zeta(w) = \log \frac{f(z) - f(\zeta)}{z - \zeta} - 2\log[\sqrt{d - f(\zeta)} + \sqrt{d - f(z)}]$$
$$+ \log(1 - \frac{\zeta}{z}) + \log z$$

and thus, by (1.2) and (4.1),

$$\sigma_\zeta(w) = \sum_{\nu,\mu=0}^{\infty}(a_{\nu\mu} - 2b_{\nu\mu})z^\nu \zeta^\mu - \sum_{\mu=1}^{\infty}\frac{\zeta^\mu}{\mu z^\mu} + \log z,$$

where the series converge uniformly for $|z| = \rho$ and $|\zeta| = r$. Hence, by (4.5) and (4.6),

$$(4.8)\quad U[f(z)] = \mathrm{Re}\{\sum_{\nu=0}^{\infty}[\sum_{\mu=0}^{n}(a_{\nu\mu} - 2b_{\nu\mu})r^\mu\alpha_\mu]z^\nu - \sum_{\mu=1}^{n}\frac{r^\mu\alpha_\mu}{\mu z^\mu}\} + \alpha_0\log|z|$$

$$= \mathrm{Re}\{H(z)\} + \alpha_0\log|z|.$$

If U^* is the harmonic conjugate of U, the left-hand side of (4.7) may be written $\int_{C_\rho} U dU^*$. If $H(z) = H_1 + iH_2$, where H_1, H_2 are real and $H(z)$ is defined in (4.8), (4.7) is therefore equivalent to

$$\int_{|z|=\rho}(H_1 + \alpha_0\log\rho)(dH_2 + \alpha_0 d\theta) \le 0.$$

Since H_2 is single-valued, it follows that

$$\int_{|z|=\rho} H_1 dH_2 + \alpha_0\int_{|z|=\rho} H_1 d\theta + 2\pi\alpha_0^2\log\rho \le 0,$$

or, equivalently,

$$(4.9)\quad \mathrm{Re}\{\frac{1}{2i}\int_{|z|=\rho}\bar{H}dH + \frac{\alpha_0}{i}\int_{|z|=\rho} H\frac{dz}{z}\} + 2\pi\alpha_0^2\log\rho \le 0.$$

Computing this by means of the Laurent expansion (4.8) of H and, in the result, letting first $\rho \to 1$ and then $r \to 1$, we obtain (4.2).

We note here an interesting special case of (4.2). If $\alpha_0 = 1$ and all the other α_k are zero, (4.2) reduces to

$$[\ \mathrm{Re}\{c_{oo}\} + \sum_{\nu=1}^{\infty} |c_{\nu o}|^2 \leq 0,$$

where $c_{oo} = -\log(4d)$ and

$$\text{(4.10)} \quad \sum_{\nu=1}^{\infty} c_{\nu o} z^{\nu} = \log \frac{f(z)}{z} - 2 \log \frac{1}{2}[1 + \sqrt{1 - \frac{f(z)}{d}}]$$

$$= (a_2 + \frac{1}{2d}) z + (d_3 - \frac{a_2^2}{2} + \frac{a_2}{2d} + \frac{3}{16d^2}) z^2 + \ldots .$$

This leads to the following strengthened version of the Koebe $\frac{1}{4}$-theorem.

<u>Theorem</u> 4.2. <u>If</u> $f \in S$ <u>and</u> $f \neq d$ <u>in</u> $|z| < 1$, <u>then</u>

$$\text{(4.11)} \qquad |d| \geq \frac{1}{4} \exp\{\frac{1}{2} \sum_{\nu=1}^{\infty} \nu |c_{\nu o}|^2\},$$

<u>where the</u> $c_{\nu o}$ <u>are the coefficients of the function</u> (4.10).

Utilizing only the first of these coefficients, we obtain

$$|d| \geq \frac{1}{4} e^{\frac{1}{2}|a_2 + \frac{1}{2d}|^2}.$$

In [7], inequality (4.3) is formulated as a result for the class B of Bieberbach-Eilenberg functions, i.e., univalent functions in $|z| < 1$ for which $f(z_1) f(z_2) \neq 1$ for $|z_1| < 1$, $|z_2| < 1$, and $f(0) = 0$. It is easy to see that, for any $f \in S_d$ the function

$$g = \frac{1 - \sqrt{1 - \frac{f}{d}}}{1 + \sqrt{1 - \frac{f}{d}}}$$

is in B and, vice versa, for any $g \in B$, the function

(4.12) $$f = \frac{4dg}{(1+g)^2} \qquad ((4d)^{-1} = g'(0))$$

is in S_d [7]. Since $c_{\nu\mu} = a_{\nu\mu} - 2b_{\nu\mu}$, where $a_{\nu\mu}$ and $b_{\nu\mu}$ were defined by the expansion (1.2) and (4.1), respectively, the $c_{\nu\mu}$ have the generating function

$$\sum_{\nu,\mu=0}^{\infty} c_{\nu\mu} z^{\nu}\zeta^{\mu} = \log \frac{f(z) - f(\zeta)}{(z-\zeta)[\sqrt{d-f(\zeta)}+\sqrt{d-f(z)}]^2}$$

$$= \log \frac{\sqrt{d - f(\zeta)} - \sqrt{d - f(z)}}{(z-\zeta)[\sqrt{d-f(\zeta)}+\sqrt{d-f(z)}]}.$$

In view of (4.12), this is equivalent to

(4.13) $$\sum_{\nu,\mu=0}^{\infty} c_{\nu\mu} z^{\nu}\zeta^{\mu} = \log \frac{g(z) - g(\zeta)}{(z-\zeta)[1-g(z)g(\zeta)]},$$

which is the generating function used by Hummel and Schiffer [7]. Theorem 4.1 may therefore also be formulated as a result for Bieberbach-Eilenberg functions: If $g \in B$ and the $c_{\nu\mu}$ are defined by (4.13), then (4.2) holds.

In [7], it is also shown that the coefficients of a function g of B satisfy the additional set of inequalities

(4.14) $$\left|\sum_{\nu,\mu=1}^{n} d_{\nu\mu}\alpha_{\nu}\alpha_{\mu}\right| \le \sum_{\mu=1}^{n} \frac{|\alpha_{\mu}|^2}{\mu},$$

where the $d_{\nu\mu}$ are given by the generating function

(4.15) $$\phi(z,\zeta) \equiv \log \frac{[g(z)-g(\zeta)][1-g(\zeta)g(z)]}{z-\zeta} = \sum_{\nu,\mu=0}^{\infty} d_{\nu\mu} z^{\nu}\zeta^{\mu}.$$

However, (4.14) is only a restatement of the classical Grunsky inequalities (1.3) for a function $f \epsilon S_d$. Indeed, if g and f are related by (4.12), we have

$$\log \frac{f(z)-f(\zeta)}{z-\zeta} = \log 4d - 2\log[1+g(z)] - 2\log[1+g(\zeta)] + \phi(z,\zeta),$$

and a comparison of (1.2) and (4.15) shows that $d_{\nu\mu} = a_{\nu\mu}$ for $\nu,\mu \leq 1$. Hence, (1.3) and (4.14) are equivalent.

5. We now turn to the discussion of the generalization of Theorem 4.1 mentioned at the beginning of Section 4. As in a similar case in Section 3, the desired Grunsky-type inequalities can be obtained via two different routes, either by way of suitable generalized Faber polynomials or through the use of a generating function. We shall see that the proof of these inequalities lies on two different levels of difficulty, according as--in the language of Theorem 4.1--$\alpha_0 = 0$ or $\alpha_0 \neq 0$ in (4.2). If $\alpha_0 = 0$, the result can be obtained very easily by the use of generalized Faber polynomials. If $\alpha_0 \neq 0$, more subtle considerations are required.

We first treat the elementary case.

Theorem 5.1. *Let* $f \epsilon S_d$, $\delta \epsilon [0,1]$, *and let* $P_\mu(t)$ *be the polynomial of degree* μ *defined by the Laurent expansion*

$$[1 - \frac{f(z)}{d}]^\delta P_\mu[\frac{1}{f(z)}] = \frac{1}{z^\mu} - \mu \sum_{\nu=0}^{\infty} c_{\nu\mu} z^\nu \tag{5.1}$$

and the condition $P_\mu(0) = 0$. *Then*

$$\sum_{\nu=1}^{\infty} \left| \sum_{\mu=1}^{n} c_{\nu\mu}\alpha_\mu \right|^2 \leq \sum_{\mu=1}^{n} \frac{|\alpha_\mu|^2}{\mu}, \tag{5.2}$$

where the $c_{\nu\mu}$ *are the coefficients appearing in* (5.1), *and* $\alpha_1,\ldots,\alpha_n$ *are arbitrary complex numbers. Equality in* (5.2) *is possible only if* $\overline{D}$ *has zero area.*

It may be noted that, because of

$$\left| \sum_{\nu=1}^{n} \left(\sum_{\mu=1}^{n} c_{\nu\mu}\alpha_\mu \right)\alpha_\nu \right|^2 \leq \sum_{\mu=1}^{n} \nu \left| \sum_{\mu=1}^{n} c_{\nu\mu}\alpha_\mu \right|^2 \sum_{\nu=1}^{n} \frac{|\alpha_\nu|^2}{\nu},$$

(5.2) contains the weaker inequality

$$\left| \sum_{\nu,\mu=1}^{n} c_{\nu\mu}\alpha_\nu\alpha_\mu \right| \leq \sum_{\nu=1}^{n} \frac{|\alpha_\nu|^2}{\nu}.$$

The polynomials $P_\mu(t)$ are easily computed from the condition that the Laurent expansion (5.1) should contain no negative powers of z except for $z^{-\mu}$. Thus, $P_1(t) = t$, $P_2(t) = t^2 + (2a_2 + \delta d^{-1})t$, etc. The computation of $P_\mu(t)$ is particularly easy if the ordinary Faber polynomial $F_\mu(t)$ of f is known. Since, by (5.1),

$$\left[1 - \frac{f(z)}{d}\right]^{\delta} P_\mu\left[\frac{1}{f(z)}\right] = F_\mu\left[\frac{1}{f(z)}\right] + O(1),$$

where $O(1)$ denotes terms which are bounded near $f = 0$, we have

$$P_\mu\left(\frac{1}{t}\right) = \left(1 - \frac{t}{d}\right)^{-\delta} F_\mu\left(\frac{1}{t}\right) + O(1),$$

i.e., $P_\mu(\frac{1}{t})$ is the meromorphic part of

$$\left(1 - \frac{t}{d}\right)^{-\delta} F_\mu\left(\frac{1}{t}\right)$$

near $t = 0$.

For $\mu = 1$, (5.2) yields the inequality

$$(5.4) \qquad \sum_{\nu=1}^{\infty} \nu \, |c_{\nu 1}|^2 \leq 1,$$

where the $c_{\nu 1}$ are given by the expansion

$$\frac{1}{f(z)} - \frac{\delta}{d} + \sum_{\nu=2}^{\infty} (-1) \binom{\delta}{\nu} \frac{f^{\nu-1}(z)}{d^{\nu}} = \frac{1}{z} - \sum_{\nu=0}^{\infty} c_{\nu 1} z^{\nu}.$$

Since $-c_{11} = a_2^2 - a_3 - \delta(1 - \delta)(2d^2)^{-1}$, (5.4) contains the inequality

$$(5.5) \qquad \left|a_2^2 - a_3 - \frac{\delta(1 - \delta)}{2d^2}\right| \leq 1,$$

where d is any value omitted by f in $|z| < 1$, and $\delta \in [0,1]$. Because of $f \neq \infty$, this contains the classical inquality $|a_2^2 - a_3| \leq 1$; if the latter inequality is combined with (5.5), we obtain $\delta(1 - \delta)(2|d|^2)^{-1} \leq 2$, i.e., for $\delta = \frac{1}{2}$, $|d| \geq \frac{1}{4}$.

Theorem 5.1 is a direct consequence of Lemma 2.1. Indeed, if we set

$$\sigma(w) = [1 - \tfrac{w}{d}]^{\delta} \sum_{\mu=1}^{n} \alpha_{\mu} P_{\mu}[\tfrac{1}{w}],$$

$|\sigma(w)|$ is single-valued and continuous in D. Thus, by (2.1),

$$\mathrm{Re}\{\frac{1}{2\pi i} \int_{C_{\rho}} \overline{\sigma(w)} \, d\sigma(w)\} \leq 0.$$

Passing to the z-plane and substituting the expansions (5.1) for the terms of $[f(z)]$, we obtain (5.2) on letting $\rho \to 1$ after the integration has been carried out.

It is not difficult to see that, for $\delta = \frac{1}{2}$, Theorem 5.1 is equivalent to the special case $\alpha_0 = 0$ of Theorem 4.1. To obtain a result which corresponds to the case $\alpha_0 \neq 0$, we have to use a function $\sigma_\zeta(w)$ which contains a logarithmic term (and reduces to (4.4) for $\delta = \frac{1}{2}$). It will be sufficient to consider the case $\zeta = 0$; the general case can then be treated by combining our result with the procedure leading to Theorem 5.1. In the proof we shall assume that $w = d$ and $w = \infty$ are actual boundary points of D (and not only points of $\bar{D}$); once the result is obtained, this restriction can be removed.

We consider the function

$$(5.6) \qquad \sigma(w) = \int_{\infty}^{w} \left(1 - \frac{t}{d}\right)^{-\delta} \frac{dt}{t}, \quad 0 < \delta \leq 1,$$

where the integration path (which may be assumed to be an analytic curve) remains within the domain Δ_ζ obtained by removing from $\bar{D}_\zeta$ a curve consisting of an analytic arc γ joining $w = d$ and $w = \infty$, and of the linear segment γ_ζ connecting $w = d$ with the point of C_ζ nearest to it. Any particular branch of the integrand in (5.6) is single-valued in $\Delta_\zeta + \gamma_\zeta$ and, if $R_1(w_0)$, $R_2(w_0)$ denote the limits of the integrand for $w \to w_0$ $(w_0 \in \gamma)$ we have $R_1(w_0) = R_2(w_0)e^{2\pi i\delta}$ (if the two sides of have been properly identified). In Δ_ζ, $\sigma(w)$ is likewise single-valued and, since the integration starts at $w = \infty$, we again have $\sigma_1(w_0) = \sigma_2(w_0)e^{2\pi i\delta}$ for the values taken by σ on opposite sides of the slit. As a result, the contributions of the two edges of γ to the integral

$$\int_{\partial\Delta_\zeta} \overline{\sigma(w)}\, d\sigma(w)$$

cancel out. To evaluate the contributions of the two edges of γ_ρ to this integral, we note that $\sigma_1(w_o) = \sigma_2(w_o) + c$, where c is a constant and σ_1, σ_2 are the limits of σ if w approaches a point w_o of γ_ρ from the two different sides γ_ρ^+ and γ_ρ^- of γ_ρ. This contribution is therefore

$$\int_{\gamma_\rho^+} \bar{\sigma}_1 d\sigma_1 - \int_{\gamma_\rho^-} \bar{\sigma}_2 d\sigma_2 = \int_{\gamma_\rho^+} (\bar{\sigma}_2 + \bar{c})\, d\sigma_2 - \int_{\gamma_\rho^-} \bar{\sigma}_2 d\sigma_2$$
$$= \bar{c} \int_{\gamma_\rho^+} d\sigma_2 = \bar{c}[\sigma_2]_{\gamma_\rho^+}.$$

Accordingly, we have

$$(5.7) \qquad \int_{\partial\Delta_\rho} \bar{\sigma}\, d\sigma = - \int_{C_\rho} \bar{\sigma}\, d\sigma + \bar{c}[\sigma_2]_{\gamma_\rho^+},$$

where the orientation of C_ρ is positive with respect to D_ρ. Hence, by Green's formula,

$$- \iint_{\bar{D}_\rho} |\sigma'|^2 dA = - \iint_{\Delta_\rho} |\sigma'|^2 dA = -\frac{1}{2i} \int_{\partial\Delta_\rho} \bar{\sigma} d\sigma = \frac{1}{2i} \int_{C_\rho} \bar{\sigma}\, d\sigma - \bar{c}[\sigma_2]_{\gamma_\rho^+}.$$

The function $\sigma_2(w)$ is continuous at $w = d$ and, since $d \in \partial D$, γ_ρ will be contained in an arbitrarily small neighborhood of d if ρ is sufficiently close to 1. It follows that $[\sigma_2]_{\gamma_\rho^+} \to 0$ for $\rho \to 1$ and, therefore,

$$(5.8) \qquad \lim_{\rho \to 1} \frac{1}{2i} \int_{C_\rho} \bar{\sigma}\, d\sigma = - \iint_{\bar{D}} |\sigma'|^2 dA \leq 0,$$

with equality if $\bar{D}$ has zero area.

We now set $w = f(z)$ and we note that, by (5.6) the function $\sigma[f(z)]$ is regular in $|z| < 1$ except for a logarithmic pole of residue 1 at $z = 0$. Accordingly, if we set $\sigma[f(z)] = U(z) + iV(z)$, $U(z)$ is single-valued in $|z| < 1$, and we have

$$(5.9)\qquad \int_{|z|=\rho} U\frac{\partial U}{\partial n}ds = \int_{|z|=\rho} UdV = [UV]_{|z|=\rho} - \int_{|z|=\rho} VdU$$

$$= \frac{1}{2}[UV]_{|z|=\rho} + \frac{1}{2}\int_{|z|=\rho}[UdV - VdU]$$

$$= \frac{1}{2}[UV]_{|z|=\rho} + \mathrm{Re}\{\frac{1}{2i}\int_{|z|=\rho}\overline{\sigma(f)}\,d\sigma(f)\}$$

$$= \frac{1}{2}[UV]_{|z|=\rho} + \mathrm{Re}\{\frac{1}{2i}\int_{C_\rho}\bar{\sigma}\,d\sigma\}.$$

The two terms in the last line depend, of course, on the point z_ρ on $|z| = \rho$ at which the integration starts and terminates. If we determine z_ρ by the requirement that $|\sigma(w)|$ should assume its minimum on C_ρ for $w_\rho = f(z_\rho)$, we have

$$[UV]_{|z|=\rho} = 2\pi U(w_\rho) = 2\pi \mathrm{Re}\{\sigma(w_\rho)\}$$

and this tends to zero as $\rho \to 1$ since $w = \infty$ is a boundary point of D and $\sigma(w) \to 0$ for $w \to \infty$. Hence, by (5.8) and (5.9),

$$(5.10)\qquad \lim_{\rho\to 1}\int_{|z|=\rho} U\frac{\partial U}{\partial n}ds \le 0,$$

with equality if the complement of D has zero area.

To express the left-hand side of (5.10) in terms of the coefficients of f, we write (5.6) in the form

$$\sigma(w) = \int_{\infty}^{d}(1 - \tfrac{t}{d})^{-\delta}\,\frac{dt}{t} - \int_{0}^{d}[(1 - \tfrac{t}{d})^{-\delta} - 1]\frac{dt}{t} + \log\frac{w}{d}$$

$$(5.11) \qquad + \int_{0}^{w}[(1 - \tfrac{t}{d})^{-\delta} - 1]\frac{dt}{t}$$

$$= -e^{\pm\pi i\delta}\int_{1}^{\infty}(t - 1)^{-\delta}\frac{dt}{t} - \int_{0}^{1}[(1 - t)^{-\delta} - 1]\frac{dt}{t} + \log\frac{w}{d}$$

$$+ \int_{0}^{w}[(1 - \tfrac{t}{d})^{-\delta} - 1]\frac{dt}{t}\ .$$

Choosing the branch of $(\ )^{\delta}$ for which $1^{\delta} = 1$ and noting that

$$\int_{1}^{\infty}(t - 1)^{-\delta}\,\frac{dt}{t} = \frac{\pi}{\sin\pi\delta}\ ,$$

we obtain

$$U(z) = \mathrm{Re}\{\sigma[f(z)]\} = -\gamma(\delta) - \log|d| + \log\left|\frac{f(z)}{z}\right|$$

$$(5.12) \qquad + \mathrm{Re}\left\{\int_{0}^{f(z)}[(1 - \tfrac{t}{d})^{-\delta} - 1]\frac{dt}{t}\right\} + \log|z|$$

$$= \mathrm{Re}\{H(z)\} + \log|z|,$$

where

$$(5.13) \qquad \gamma(\delta) = \pi\cot\pi\delta + \int_{0}^{1}[(1 - t)^{-\delta} - 1]\frac{dt}{t}$$

$$= \frac{1}{\delta} - \sum_{\nu=1}^{\infty}\left(\frac{1}{\nu} - \frac{1}{\nu+\delta}\right).$$

The last line in (5.12) is of the same form as the last line in (4.8) (with $\alpha_0 = 1$). Since the integral in (4.7) was found to have the value (4.9), we may thus conclude that (5.10) is equivalent to

$$(5.14)\qquad \lim_{\rho\to 1} \operatorname{Re}\Big\{\frac{1}{2i}\int_{|z|=\rho} \overline{H}dH + \frac{1}{i}\int_{|z|=\rho} H\frac{dz}{z}\Big\} + 2\pi \log\rho \le 0,$$

where H is defined in (5.12). If the coefficients $c_{\nu o}$ are defined by

$$(5.15)\qquad -\gamma(\delta) - \log|d| + \log\frac{f(z)}{z} + \int_0^{f(z)} [(1 - \frac{t}{d})^{-\delta} - 1]\frac{dt}{t} = \sum_{\nu=0}^{\infty} c_{\nu o} z^{\nu},$$

(5.14) is equivalent to

$$(5.16)\qquad \frac{1}{2}\sum_{\nu=1}^{\infty} \nu|c_{\nu o}|^2 \le \gamma(\delta) + \log|d|.$$

This inequality was obtained under the assumption that $w = d$ and $w = \infty$ are actual boundary points of D, but is is now easy to see that it remains valid if it is only assumed that these two points are in $\overline{D}$. Indeed, if (5.16) is true for a sequence of functions which converge uniformly to f in $|z| < r$ for any $r\epsilon(0,1)$, it is also true for f. Now it is well known that any $f\epsilon S$ can be approximated in this manner by a sequence of functions each of which maps $|z| < 1$ onto a domain bounded by an analytic slit T extending to $w = \infty$. If $d\epsilon\overline{D}$ and $d\notin T$ we add to T a rectilinear 'spike' T_d which connects d with one of the points of T which are nearest to d. The complement of $T + T_d$ is also simply-connected, and the functions mapping $|z| < 1$ onto these domains evidently also converge to f. Since all these functions are subject to the restriction under which (5.16) was proved, the result follows (after a trivial modification which allows for the fact that the approximating functions $f_n(z)$

do not necessarily satisfy $f_n'(0) = 1$).

This proves the following result.

Theorem 5.2. *Let* $f \in S_d$ *and let* $\delta \in [0,1]$. *If the* $c_{\nu 0}$ *are defined by the expansion* (5.15), *and* $\gamma(\delta)$ *is the constant* (5.13), *then* (5.16) *holds*. *Equality in* (5.16) *is possible only if* $\bar{D}$ *has zero area*.

It may be noted that, for $\delta = \frac{1}{2}$, the expression (5.13) has the value $\log 4$, and (5.16) reduces to (4.11).

The procedure leading to Theorem 5.2 can be generalized so as to lead to a result of the type of Theorem 4.1, in which the coefficients appearing in the inequalities are defined by a generating function. However, it is much easier to obtain and state an equivalent result in which these coefficients are defined by means of generalized Faber polynomials.

Theorem 5.3. *Let* $f \in S$ *and let* $\delta \in [0,1]$. *If the coefficients* $c_{\nu\mu}$ *are defined by the expansions* (5.15) (*where* $\gamma(\delta)$ *is the constant* (5.13)) *and* (5.1), *then*

$$(5.17) \qquad 2\alpha_0 \operatorname{Re}\{\sum_{\mu=1}^{n} c_{\nu\mu}\alpha_\mu\} + \sum_{\nu=1}^{\infty} \nu \left|\sum_{\mu=1}^{n} c_{\nu\mu}\alpha_\mu\right|^2 \le \sum_{\mu=1}^{n} \frac{|\alpha_\mu|^2}{\mu},$$

where α_0 *is real and* $\alpha_1, \ldots, \alpha_n$ *are arbitrary complex constants*. *Equality in* (5.17) *is possible only if* $\bar{D}$ *has zero area*.

The proof is essentially identical with that of Theorem 5.2, the main difference being that we now use the function

$$(5.18) \qquad \sigma(w) = \alpha_0\sigma_0(w) + \sum_{\mu=1}^{n}\alpha_\mu\sigma_\mu(w),$$

where $\sigma_0(w)$ is the function (5.6), i.e.,

$$(5.19) \qquad \sigma_0(w) = \int_\infty^w (1 - \frac{t}{d})^{-\delta} \frac{dt}{t},$$

and

$$(5.20) \qquad \sigma_\mu(w) = (1 - \frac{w}{d})^{\delta} P_\mu(\frac{1}{w}),$$

where the $P_\mu(t)$ are the generalized Faber polynomials defined in (5.1). It is easily verified that (5.18) has all the properties of the function (5.6) which were utilized in the proof of Theorem 5.2. Accordingly, the steps leading to inequality (5.14) will result in the analogous inequality

$$(5.21) \quad \lim_{\rho\to 1} \operatorname{Re}\{\frac{1}{2i} \int_{|z|=\rho} \bar{H} dH + \frac{\alpha_0}{i} \int_{|z|=\rho} H \frac{dz}{z}\} + 2\pi\alpha_0 \log \rho \le 0,$$

where H is now defined by

$$H(z) = \sigma[f(z)] - \alpha_0 \log z$$

and σ is the function (5.18). Using (5.18), (5.19), (5.20) and the expansions (5.15), (5.1) of the functions $\sigma_\mu(z)$, we find that

$$H(z) = \sum_{\mu=1}^{n} \frac{\alpha_\mu}{\mu z^\mu} - \sum_{\nu=1}^{\infty} (\sum_{\mu=0}^{n} c_{\nu\mu}\alpha_\mu) z^\nu.$$

Using this expansion of H in (5.21) and letting $\rho \to 1$ in the result, we obtain (5.17).

It may be noted that (5.17) contains the weaker inequality

$$\operatorname{Re}\{ \sum_{\nu,\mu=0}^{\infty} c_{\nu\mu}\alpha_\nu\alpha_\mu \} \le \sum_{\nu=1}^{n} \frac{|\alpha_\nu|^2}{\nu},$$

as is shown by the procedure used to obtain (4.3) from (4.2).

6. The procedures which led to the inequalities of the preceding section can be generalized so as to apply to functions of S which leave out more than one given value. While this is very easily accomplished in the case of Theorem 5.1, the arguments required to establish the corresponding generalizations of Theorems 5.2 and 5.3 are lengthy and laborious. We therefore content ourselves with the generalization of Theorem 5.1.

We first state our result for the case in which the function f is known to omit a finite number of given values.

Theorem 6.1. *Let* $f \in S$, *and let* $f(z) \neq \alpha_k$, $k = 1,\ldots,m$ *for* $|z| < 1$. *Let* $\delta_k > 0$, *and let* $\delta_1 + \ldots + \delta_m \leq 1$. *Let* $P_\mu(t)$ *be the polynomial of degree* μ *defined by* $P_\mu(0) = 0$ *and the requirement that*

$$(6.1) \qquad \prod_{k=1}^{m} \left[1 - \frac{f(z)}{\alpha_k}\right]^{\delta_k} P_\mu\left[\frac{1}{f(z)}\right] = \frac{1}{z^\mu} + \sum_{\nu=0}^{\infty} a_{\nu\mu} z^\nu$$

should contain no negative powers of z *except for* $z^{-\mu}$. *Then,*

$$(6.2) \qquad \sum_{\nu=1}^{\infty} \nu \left| \sum_{\mu=1}^{n} a_{\nu\mu} \beta_\mu \right|^2 \leq \sum_{\mu=1}^{n} \mu |\beta_\mu|^2,$$

where the $a_{\nu\mu}$ *are the coefficients of the expansion* (6.1), *and* $\beta_1,\ldots,\beta_n$ *are arbitrary complex numbers. Equality in* (6.2) *is possible only if* $\overline{D}$ *has zero area.*

The proof of this result is identical with that of Theorem 5.1, once it is observed that the function

$$\left| \prod_{k=1}^{n} \left[1 - \frac{w}{\alpha_k}\right]^{\delta_k} P_\mu\left[\frac{1}{w}\right] \right|$$

is single-valued and continuous in D.

As an example for the use of Theorem 6.1, we consider the case $\mu = 1$. Since $P_1(t) \equiv t$, (6.1) and (6.2) show that

$$\sum_{\nu=1}^{\infty} \nu |c_\nu|^2 \leq 1$$

if the c_ν are given by the expansion

$$\frac{1}{z} + \sum_{\nu=1}^{\infty} c_\nu z^\nu = \frac{1}{f(z)} e^{-\sum_{\nu=1}^{\infty} \frac{\gamma_\nu}{\nu} f^\nu(z)} \tag{6.3}$$

where

$$\gamma_\nu = \sum_{k=1}^{n} \frac{\delta_k}{\alpha_k^\nu} . \tag{6.4}$$

In particular, $|c_1| \leq 1$, i.e., by (6.3),

$$|a_2^2 - a_3 + \frac{1}{2}(\sum_{k=1}^{n} \frac{\delta_k}{\alpha_k})^2 - \sum_{k=1}^{n} \frac{\delta_k}{\alpha_k^2}| \leq 1,$$

which is the generalization of (5.5) to the case in which the function f is known to omit the values $\alpha_1, \ldots, \alpha_n$.

Thoerem 6.1 is easily generalized so as to apply to the case in which $f(z)$ is known to omit a continuum of values. Suppose $f(z) \neq \alpha(s)$ for $|z| < 1$, where s varies in the real interval $[0,1]$, and $\alpha(s)$ is a complex-valued continuous function of s. If the real function $p(s)$ is non-decreasing in $[0,1]$ and $0 \leq t_o < t_1 < \ldots < t_n \leq 1$, we may set $\alpha_k = \alpha(t_k)$ and $\delta_k = p(t_k) - p(t_{k-1})$ $(k = 1,\ldots,n)$ and apply Theorem 6.1. As (6.3) shows (in the case $\mu = 1$), the coefficients $a_{\nu\mu}$ in (6.1) are polynomials in the numbers γ_ν defined in (6.4) (and, of course, in the coefficients of f). If $p(1) - p(0) \leq 1$, it is therefore permissible to let $n \to \infty$, and we are led to the following result.

Theorem 6.2. *Let* $f \in S$ *and let* $f(z) \neq \alpha(s)$ *for* $|z| < 1$, *where* $\alpha(s)$ *is a continuous complex-valued function on* $[0,1]$. *Let the real function* $p(s)$ *be non-decreasing on* $[0,1]$ *and such that*

$$\int_0^1 dp(s) \leq 1. \tag{6.5}$$

Let $\phi(z)$ *be the analytic function defined by*

$$\phi(z) = \exp\{ \int_0^1 \log[1 - \frac{f(z)}{\alpha(s)}] dp(s) \}, \tag{6.6}$$

and let $P_\mu(t)$ *be the polynomial of degree* μ *defined by* $P_\mu(0) = 0$ *and the requirement that the Laurent expansion*

$$\phi(z) P_\mu[\frac{1}{f(z)}] = \frac{1}{z^\mu} + \sum_{\nu=0}^{\infty} a_{\nu\mu} z^\nu \tag{6.7}$$

should contain no negative powers of z *except for* $z^{-\mu}$. *Then*

$$\sum_{\nu=1}^{\infty} \nu \left| \sum_{\mu=1}^{m} a_{\nu\mu} \beta_\mu \right|^2 \leq \sum_{\mu=1}^{m} \mu |\beta_\mu|^2, \quad m = 1,2,\ldots, \tag{6.8}$$

where the $a_{\nu\mu}$ *are given by the expansion* (6.7), *and* $\beta_1, \ldots, \beta_n$ *are arbitrary complex parameters*.

By suitable choices of the function $p(s)$, Theorem 6.2 can be made to yield a large variety of inequalities. Moreover, it is possible to choose $p(s)$ in such a way that the resulting coefficient inequalities are valid for all functions of S (i.e., without reference to values omitted by f). We shall illustrate this procedure by two examples. In both cases we replace the function $f(z)$ of Theorem (6.2) by $F(z) = r^{-1}f(rz)$, where $0 < r < 1$, and we set $\alpha(s) = F(e^{2\pi is})$. Our first choice of $dp(s)$ is

$$(6.9) \qquad dp(s) = \frac{1}{2\pi i}\Big[\sum_{k=1}^{n} \frac{\gamma_k}{\zeta^k} + \gamma_0 + \sum_{k=1}^{n} \bar{\gamma}_k \zeta^k\Big]\frac{d\zeta}{\zeta}$$

where $\zeta = e^{2\pi is}$, $0 < \gamma_0 \leq 1$, and the complex parameters $\gamma_1, \ldots, \gamma_n$ are subject to the condition

$$(6.10) \qquad \gamma_0 + 2\mathrm{Re}\{\sum_{k=1}^{n} \gamma_k e^{ik\theta}\} \geq 0, \quad \theta \in [0, 2\pi].$$

This condition ensures that $dp(s) \geq 0$, and, because of $\gamma_0 \in [0,1]$, condition (6.5) is satisfied.

To compute the function (6.6), we note that

$$(6.11) \qquad \log[1 - \frac{F(z)}{F(\zeta)}] = \log \frac{F(z) - F(\zeta)}{z - \zeta} - \log \frac{F(\zeta)}{\zeta} + \log(1 - \frac{z}{\zeta}).$$

If $|z| < 1$ and $a_{\nu\mu}$ are the Grunsky coefficients given by

$$\log \frac{F(z) - F(\zeta)}{z - \zeta} = \sum_{\nu,\mu=0}^{\infty} a_{\nu\mu} z^{\nu} \zeta^{\mu},$$

we thus have

$$\log[1 - \frac{F(z)}{F(\zeta)}] = \sum_{\nu=1}^{\infty} \sum_{\mu=0}^{\infty} a_{\nu\mu} z^{\nu} \zeta^{\mu} - \sum_{\nu=1}^{\infty} \frac{z^{\nu}}{\nu \zeta^{\nu}}.$$

In view of (6.9) and the residue theorem, this leads to

$$\int_0^1 \log[1 - \frac{F(z)}{\alpha(s)}]\,dp(s) = \sum_{\nu=1}^{\infty} \Big(\sum_{\mu=0}^{n} a_{\nu\mu} \gamma_{\mu} - \frac{\bar{\gamma}_{\nu}}{\nu}\Big) z^{\nu}.$$

Applying Theorem 6.2 to the function $F(z)$ and letting, in the resulting inequalities (6.8), $r \to 1$, we obtain the following result.

<u>Theorem</u> 6.3. <u>Let</u> $f \in S$, <u>let</u> $\beta_1, \ldots, \beta_m$ <u>be arbitrary complex parameters, let</u> $\gamma_0 \in (0,1]$, <u>and let</u> $\gamma_1, \ldots, \gamma_n$ <u>be complex parameters subject to the condition</u> (6.10). <u>Let</u> $a_{\nu\mu}$ <u>be defined by the generating function</u> (1.2), <u>and let</u> $\phi(z)$ <u>be the analytic function</u>

$$\phi(z) = \exp\{\sum_{\nu=1}^{\infty}(\sum_{k=0}^{n} a_{\nu k}\gamma_k - \frac{\overline{\gamma_\nu}}{\nu}) z^\nu\}.$$

<u>Let</u> $P_\mu(t)$ <u>be the polynomial of degree</u> μ <u>defined by</u> $P_\mu(0) = 0$ <u>and the requirement that the Laurent expansion</u>

$$\phi(z) P_\mu[\frac{1}{f(z)}] = \frac{1}{z^\mu} + \sum_{\nu=0}^{\infty} b_{\nu\mu} z^\nu$$

<u>should contain no negative powers of</u> z <u>except for</u> $z^{-\mu}$. <u>Then</u>

$$(6.12) \qquad \sum_{\nu=1}^{\infty} \nu |\sum_{\mu=1}^{m} b_{\nu\mu}\beta_\mu|^2 \leq \sum_{\mu=1}^{m} \mu|\beta_\mu|^2, \quad m = 1,2,\ldots .$$

For $\gamma_0 \to 0$, the inequalities (6.12) reduce to the classical Grunsky inequalities. If $\gamma_0 \in (0,1]$, the parameters $\gamma_1, \ldots, \gamma_n$ which enter into (6.12) give additional flexibility to these inequalities. Conceivably, this may be of help in the treatment of specific coefficient problems such as the verification of the Bieberbach inequality.

In our second application of Theorem 6.2 we use the differential

$$(6.13) \qquad dp(s) = \frac{\gamma}{2\pi i}[\frac{1 - |\zeta|^2}{(\eta - \zeta)(1 - \bar{\zeta}\eta)} - \frac{1 - |\zeta|}{1 + |\zeta|}]\frac{d\eta}{\eta}, \quad \eta = e^{2\pi i s}, |\zeta| < 1,$$

where

$$(6.14) \qquad 0 < \gamma \leq \frac{1 + |\zeta|}{2|\zeta|} .$$

It is easily verified that $dp(s) \geq 0$. Moreover, since $\int_0^1 dp(s) = 2\gamma|\zeta|(1+|\zeta|)^{-1}$, (6.14) shows that condition (6.5) is satisfied. Taking into account the identity (6.11) (and the fact that $\alpha(s) = F(e^{2\pi is}) = F(\eta))$, it follows from (6.13) that

$$\int_0^1 \log[1 - \frac{F(z)}{\alpha(s)}]dp(s) = \gamma[\log \frac{F(z) - F(\zeta)}{z - \zeta} - \log \frac{F(\zeta)}{\zeta} + \log(1 - \bar{\zeta}\eta)].$$

Applying Theorem 6.2, and letting $r \to 1$, we obtain the following result.

<u>Theorem</u> 6.4. <u>Let</u> $f \in S$, <u>let</u> $|\zeta| < 1$, <u>and let</u> γ <u>satisfy</u> (6.14). <u>Let</u> $P_\mu(t)$ <u>be the</u> n-<u>th degree polynomial</u> (<u>with</u> $P_\mu(0) = 0$) <u>determined by the expansion</u>

$$(6.15)\quad P_\mu[\frac{1}{f(z)}\frac{f(z) - f(\zeta)}{z - \zeta} \cdot \frac{\zeta}{f(\zeta)}(1 - \bar{\zeta}z)]^\gamma = \frac{1}{z^\mu} + \sum_{\nu=0}^{\infty} b_{\nu\mu} z^\nu.$$

<u>Then</u>,

$$(6.16)\qquad \sum_{\nu=1}^{\infty} \nu \left|\sum_{\mu=1}^{n} b_{\nu\mu}\beta_\mu\right|^2 \leq \sum_{\mu=1}^{n} \mu|\beta_\mu|^2,$$

<u>where the</u> $b_{\nu\mu}$ <u>are the coefficients given by</u> (6.15).

For $\gamma \to 0$, the inequalities (6.16) again reduce to the classical Grunsky inequalities.

7. In this section we show how the inequalities of the preceding sections can be sharpened it if is known that the univalent function $f(z)$ is also bounded in $|z| < 1$, i.e., if $|f(z)| \leq M$, $1 < M < \infty$. As is customary, we shall renormalize these functions

by dividing them by M. Accordingly, we consider univalent functions

$$(7.1) \qquad f(z) = \sum_{n=1}^{\infty} a_n z^n$$

for which $|f(z)| \leq 1$ in $|z| < 1$. Evidently, the coefficient a_1 in (7.1) satisfies the inequality $0 < a_1 \leq 1$, and $a_1 = 1$ is possible only if $f(z) \equiv z$. This class of bounded univalent functions will be denoted by S_1.

Although the indicated sharpening of the preceding inequalities is possible in all cases, we shall confine ourselves to the discussion of a few examples in which the requisite computations are particularly simple.

It was shown in [12] that the coefficients of a function of S_1 can be characterized in the following manner: If $a_{\nu\mu}$ and $b_{\nu\mu}$ are defined by the generating functions

$$(7.2) \qquad \log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{\nu,\mu=0}^{\infty} a_{\nu\mu} z^{\nu} \zeta^{\mu}$$

and

$$(7.3) \qquad -\log[1 - \overline{f(\zeta)} f(z)] = \sum_{\nu,\mu=1}^{\infty} b_{\nu\mu} z^{\nu} \bar{\zeta}^{\mu},$$

where $f \in S_1$, then

$$(7.4) \qquad \left| \sum_{\nu,\mu=1}^{n} a_{\nu\mu} \alpha_\nu \alpha_\mu \right| + \left| \sum_{\nu,\mu=1}^{n} b_{\nu\mu} \alpha_\nu \bar{\alpha}_\mu \right| \leq \sum_{\nu=1}^{n} \frac{|\alpha_\nu|^2}{\nu}$$

for arbitrary complex $\alpha_1, \ldots, \alpha_n$. Conversely, if (7.4) holds for all natural numbers n, then $f \in S_1$.

We shall now derive a strengthened version of the necessary conditions (7.4). This result will be stronger than (7.4) in two respects: first, its relation to (7.4) will be similar to the relation of the 'strong' Grunsky inequalities (1.4) to the 'weak' inequalities (1.3); and, second, it contains, rather surprisingly, an additional arbitrary real parameter.

Theorem 7.1. Let $f \in S_1$, and let $a_{\nu\mu}$ and $b_{\nu\mu}$ be defined by (7.2) and (7.3), respectively. If α_0 is real and $\alpha_1, \ldots, \alpha_n$ are complex, then

$$(7.5) \qquad 2\alpha_0 \operatorname{Re}\{\sum_{\mu=0}^{n} a_{0\mu}\alpha_\mu\} + \sum_{\nu=1}^{\infty} \nu \left| \sum_{\mu=0}^{n} a_{\nu\mu}\alpha_\mu + \sum_{\mu=1}^{k} b_{\nu\mu}\bar{\alpha}_\mu \right|^2 \le \sum_{\nu=1}^{n} \frac{|\alpha_\nu|^2}{\nu} .$$

Equality in (7.5) is possible only if the complement of D with respect to the unit disk has zero area.

In the proof of Theorem 7.1 we shall assume that $f(z)$ is regular for $|z| \le 1$. As shown repeatedly in similar situations, this is sufficient for our purposes. We consider the function

$$(7.6) \qquad u_\zeta(w) = \log\left|\frac{w - \eta}{1 - \bar{\eta} w}\right|, \quad \eta = f(\zeta), \ |\zeta| = r < 1,$$

which is harmonic and single-valued in the complement $\bar{D}^*$ of D with respect to the closed unit disk, and has the value 0 on $|w| = 1$. The same statements are also true of the function

$$(7.7) \qquad U(w) = \int_{|\zeta|=r} u_\zeta(w)\, dm(\zeta),$$

where $dm(\zeta)$ is the real differential

$$(7.8) \qquad dm(\zeta) = \frac{1}{2\pi i}\left[\sum_{\mu=1}^{n} \frac{r^{\mu}\alpha_{\mu}}{\zeta^{\mu}} + \alpha_0 + \sum_{\mu=1}^{n} \frac{\bar{\alpha}_{\mu}\zeta^{\mu}}{r^{\mu}}\right]\frac{d\zeta}{\zeta}.$$

By Green's formula, we have therefore

$$\int_C U\frac{\partial U}{\partial n}\, ds = -\iint_{\overline{D^*}} (\text{grad } U)^2 dA + \int_{|w|=1} U\frac{\partial U}{\partial n}\, ds$$

$$= -\iint_{\overline{D^*}} (\text{grad } U)^2 dA .$$

Hence,

$$(7.9) \qquad \int_C U \frac{\partial U}{\partial n}\, ds \leq 0,$$

with equality only if $\overline{D^*}$ has zero area.

To express the left-hand side of (7.9) in terms of the coefficients of f we note that, with $w = f(z)$, (7.6) may be written in the form

$$u_{\zeta}[f(z)] = \text{Re}\{\log \frac{f(z) - f(\zeta)}{z - \zeta} - \log[1 - \overline{f(\zeta)}f(z) + \log(1 - \frac{\zeta}{z}) + \log z\}.$$

Substituting this in (7.7), and using (7.8) and the generating functions (7.2) and (7.3), we obtain

$$U[f(z)] = \text{Re}\{\sum_{\nu=0}^{\infty}(\sum_{\mu=0}^{n} a_{\nu\mu}\, r^{\mu}\alpha_{\mu} + \sum_{\mu=1}^{n} b_{\nu\mu}\, r^{-\mu}\bar{\alpha}_{\mu})z^{\nu} - \sum_{\mu=1}^{n}\frac{\alpha_{\mu}r^{\mu}}{\mu z^{\mu}}\} + \alpha_0\log|z|$$

(7.10)

$$= \text{Re}\{H(z)\} + \alpha_0 \log|z| = \text{Re}\{H(z)\}$$

(the last equality following from $|z| = 1$). If we use this expression in (7.9) and employ the procedure which led from (4.7) to (4.9), we find that (7.9) is equivalent to

$$\mathrm{Re}\{\frac{1}{2\pi i}\int_{|z|=1}\bar{H}dH + \frac{\alpha_0}{\pi i}\int_{|z|=1} H\frac{dz}{z}\} \leq 0.$$

Replacing H be its Laurent expansion (7.10) and letting $r \to 1$, we obtain (7.5).

To show that (7.5) implies (7.4), we employ the procedure used to show that (4.3) follows from (4.2). This leads to

$$\mathrm{Re}\{\sum_{\nu,\mu=0}^{n} a_{\nu\mu}\alpha_\nu\alpha_\mu + \sum_{\nu,\mu=1}^{n} b_{\nu\mu}\alpha_\nu\bar{\alpha}_\mu\} \leq \sum_{\nu=1}^{n}\frac{|\alpha_\nu|^2}{\nu}.$$

Since, by (7.3), $b_{\nu\mu} = \overline{b_{\mu\nu}}$, the second sum on the left-hand side is a Hermitian form. In [12], it was shown that this form is positive-definite. The last inequality may therefore also be written in the form

$$\mathrm{Re}\{\sum_{\nu,\mu=0}^{n} a_{\nu\mu}\alpha_\nu\alpha_\mu\} + |\sum_{\nu,\mu=1}^{n} b_{\nu\mu}\alpha_\nu\bar{\alpha}_\mu| \leq \sum_{\nu=1}^{n}\frac{|\alpha_\nu|^2}{\nu}.$$

If we set $\alpha_0 = 0$ and multiply the α_ν by a suitably chosen number of modulus 1, we obtain (7.4).

We note here the special case

$$2\mathrm{Re}\{a_{00}\} + \sum_{\nu=1}^{\infty}\nu|a_{\nu 0}|^2 \leq 0 \tag{7.11}$$

obtained by setting $\alpha_1 = \ldots = \alpha_n = 0$ in (7.5). If F is a function of S for which $|F(z)| \leq M$, and we set $f = M^{-1}F$, we have

$$\sum_{\nu=1}^{\infty} a_{\nu 0}z^\nu = \log\frac{F(z)}{z},\quad a_{00} = -\log M,$$

and (7.11) is therefore equivalent to the following result.

<u>Let</u> $F \in S$, <u>and let</u> $|F| \leq M$ <u>in</u> $|z| < 1$. <u>If</u>

$$\log \frac{F(z)}{z} = \sum_{\nu=1}^{\infty} c_\nu z^\nu,$$

<u>then</u>

$$\sum_{\nu=1}^{\infty} \nu |c_\nu|^2 \leq 2 \log M.$$

Our next result is the S_1-version of Theorem 3.1.

<u>Theorem</u> 7.2. <u>Let</u> $f \in S_1$, <u>let</u> $\delta > 0$, <u>and let</u> $P(t) = \sum_{k=1}^{n} A_k t^k$, <u>where the</u> A_k <u>are arbitrary complex constants</u>. <u>If the coefficients</u> b_ν <u>are defined by the expansion</u>

$$[\frac{f(z)}{z}]^\delta P[\frac{1}{f(z)}] = \sum_{\nu=-n}^{\infty} b_\nu z^\nu, \tag{7.13}$$

<u>then</u>

$$\sum_{\nu=-n}^{\infty} (\nu + \delta) |b_\nu|^2 + \sum_{k=1}^{n} (k - \delta) |A_k|^2 \leq 0. \tag{7.14}$$

<u>Equality in</u> (7.14) <u>is possible only if the intersection of</u> $\bar{D}$ <u>and</u> $|w| < 1$ <u>has zero area</u>.

To prove this result, we set

$$(w) = w^\delta P(\frac{1}{w})$$

and apply Lemma 2.2. If $\Gamma = C_\rho$ and Γ_1 is the circumference $|w| = 1$, we obtain

$$\begin{aligned} \text{Re}\{\frac{1}{2\pi i} \int_{C_\rho} \overline{w^\delta P(\frac{1}{w})}\, d(w^\delta P(\frac{1}{w}))\} \\ \leq \text{Re}\{\frac{1}{2\pi i} \int_{|w|=1} \overline{w^\delta P(\frac{1}{w})}\, d(w^\delta P(\frac{1}{w})). \end{aligned} \tag{7.15}$$

On $|w| = 1$, we may replace w^{-1} by $\overline{w}$, and the integral on the right-hand side has therefore the value

$$-\frac{1}{2\pi i}\int_{|w|=1} w^{-\delta}\Big(\sum_{k=1}^{n}\overline{A}_k w^k\Big)\Big(\sum_{k=1}^{n}(k-\delta)\frac{A_k}{w^k}\Big)w^{\delta-1}dw$$

i.e., $-\sum_{k=1}^{n}(k-\delta)|A_k|^2$. To compute the left-hand side of (7.15), we set $w = f(z)$ and transfer the integral to the z-plane. In view of (7.13) we obtain

$$\mathrm{Re}\Big\{\frac{1}{2\pi i}\int_{|z|=\rho}\Big(\overline{\sum_{\nu=-n}^{\infty} b_\nu z^{\nu+\delta}}\Big)d\Big(\sum_{\nu=-n}^{\infty} b_\nu z^{\nu+\delta}\Big)\Big\}$$
$$\leq -\sum_{k=1}^{n}(k-\delta)|A_k|^2 .$$

Computing the integral and letting $\rho \to 1$, we obtain (7.14).

It may be noted that in the preceding result δ may be any positive number, while in Theorem 3.1 this number had to be restricted to the interval $(0,1)$ in order to ensure the continuity of $|\sigma(w)|$ at $w = \infty$.

Our final result generalizes Theorem 4.1 to functions of the class S_1.

Theorem 7.3. *Let* $f \epsilon S_1$, *and let* $f(z) \neq d$ *in* $|z| < 1$, *where* $|d| < 1$. *Let the coefficients* $a_{\nu\mu}$ *and* $b_{\nu\mu}$ *be defined by the generating functions*

$$\text{(7.16)}\qquad \log\frac{f(z) - f(\zeta)}{(z-\zeta)[P(z)+P(\zeta)]^2} = \sum_{\nu,\mu=0}^{\infty} a_{\nu\mu} z^\nu \zeta^\mu ,$$

$$(7.17)\qquad \log \frac{[1 + \overline{P(\zeta)}P(z)]^2}{1 - \overline{f(\zeta)}f(z)} = \sum_{\nu,\mu=0}^{\infty} b_{\nu\mu} z^{\nu} \bar{\zeta}^{\mu},$$

<u>where</u>

$$(7.18)\qquad P(z) = \sqrt{\frac{d - f(z)}{1 - \bar{d}f(z)}},$$

<u>and the same branch of the square root is to be taken in</u> $P(z)$ <u>and</u> $P(\zeta)$. <u>Then,</u>

$$(7.19)\qquad 2\alpha_0 \mathrm{Re}\{\sum_{\mu=0}^{n}(a_{0\mu}\alpha_\mu + b_{0\mu}\bar{\alpha}_\mu)\} + \sum_{\nu=1}^{\infty}\nu\,|\sum_{\mu=0}^{n}(a_{\nu\mu}\alpha_\mu + b_{\nu\mu}\bar{\alpha}_\mu)|^2 \le \sum_{\mu=1}^{n}\frac{|\alpha_\mu|^2}{\mu}$$

<u>for any real</u> α_0 <u>and complex</u> $\alpha_1, \ldots, \alpha_n$. <u>Equality in</u> (7.19) <u>is possible only if the intersection of</u> $\bar{D}$ <u>and the unit disk has zero area</u>.

The proof of this result is very similar to that of Theorem 4.1, and it is therefore sufficient to call attention to those details in which the two proofs differ. The function (4.4) is now replaced by

$$(7.20)\qquad \sigma_\zeta(w) = \log\{(\frac{R(w) - R(z)}{1 - \overline{R(z)}R(w)})(\frac{1 + \overline{R(z)}R(w)}{R(w) + R(z)})\},\quad z = f(\zeta),$$

where

$$(7.21)\qquad R(w) = \sqrt{\frac{d - w}{1 - \bar{d}w}}.$$

Clearly, the function $u(w) = \mathrm{Re}\{\sigma_\zeta(w)\}$ vanishes on $|w| = 1$. If Δ_ρ is the intersection of $\bar{D}_\rho$ $(0 < |\zeta| < \rho < 1)$ and $|w| \le 1$ and γ is an arc in Δ_ρ which joins $w = d$ and a point on

$|w| = 1$, then $u(w)$ is single-valued in $\Delta_\rho - \gamma$, and the two values it takes at a point of γ have the sum zero. These properties are therefore shared by the function $U(w)$ defined by (4.5) and (4.6) (where $\sigma_\zeta(w)$ is now the function (7.20)). Accordingly, the function $U^2(w)$ is single-valued and subharmonic in Δ_ρ and vanishes on $|w| = 1$. The rest of the proof is identical with that of Theorem 4.1, except that now we use Lemma 2.4 rather than Lemma 2.3 (and the fact that $U(w) = 0$ for $|w| = 1$ is taken into account).

We note that (7.19) implies the weaker (but still sharp) inequality

$$\mathrm{Re}\{\sum_{\nu,\mu=0}^{\infty} a_{\nu\mu}\alpha_\nu\alpha_\mu + \sum_{\nu,\mu=0}^{\infty} b_{\nu\mu}\alpha_\nu\bar{\alpha}_\mu\} \le \sum_{\mu=1}^{n}\frac{|\alpha_\mu|^2}{\mu},$$

as can be seen by applying the procedure which led from (4.2) to (4.3).

For $\alpha_1 = \ldots = \alpha_n = 0$, (7.19) yields the special inequality

$$2\,\mathrm{Re}\{a_{00} + b_{00}\} + \sum_{\nu=1}^{\infty}\nu\,|a_{\nu 0} + b_{\nu 0}|^2 \le 0.$$

By (7.16) and (7.17), this is equivalent to the following statement.

<u>Let</u> $f \in S_1$ <u>and</u> $f \neq d$ $(|d| < 1)$. <u>If the coefficients</u> c_ν <u>are defined by</u>

$$(7.22)\quad \log\frac{f(z)}{z} + \log d - 2\log\frac{\sqrt{1-\bar{d}f(z)} + \sqrt{1-d^{-1}f(z)}}{\sqrt{1-\bar{d}f(z)} + |d|\sqrt{1-d^{-1}f(z)}} = \sum_{\nu=0}^{\infty} c_\nu z^\nu,$$

then

$$\frac{1}{2}\sum_{\nu=1}^{\infty} \nu |c_\nu|^2 \leq \log \frac{4|d|}{|a_1|(1+|d|)^2} . \tag{7.23}$$

In particular, this shows that $4|d| \geq |a_1|(1+|d|)^2$, i.e.,

$$|d| \geq \frac{|a_1|}{(1+\sqrt{1-|a_1|})^2} .$$

Equivalently, if $F \epsilon S$ and $|F| \leq M$, then a value γ not taken by F must satisfy

$$|\gamma| \geq \frac{1}{(1+\sqrt{1-\frac{1}{M}})^2} .$$

More generally, (7.23) shows that

$$|d| \geq \frac{B}{(1+\sqrt{1-B})^2} ,$$

where

$$B = |a_1| e^{\frac{1}{2}\sum_{\nu=1}^{\infty} \nu |c_\nu|^2} .$$

8. We conclude this paper with a few remarks concerning the precise relation between the contour integration method (used by Grunsky in his original derivation of the inequalities (1.3)) on the one hand, and the area method and its various extensions and generalizations, on the other hand. In order not to be led too far afield, we shall confine our remarks to the case in which the functions involved are single-valued.

Stripped to its essentials, the main assertion of the contour integration method can be summed up in the following statement [12].

Let D be a domain in the w-plane which is bounded by one or more analytic curves C. Let S(w) be a real function which is single-valued and harmonic in the w-plane except at a finite number of points of D, and let p(w) be a function which vanishes on C and is such that h(w) = S(w) - p(w) is harmonic in D. Then

$$\int_C h \frac{\partial p}{\partial n}\, ds \le 0. \tag{8.1}$$

The proof of (8.1) is very simple. If $\bar{D}$ is the complement of D with respect to the extended plane, and if $(\emptyset,\emptyset)_R$ denotes the Dirichlet integral of the function $\emptyset$ over the region R we have, by Green's formula,

$$-(S,S)_{\bar{D}} = \int_C S \frac{\partial S}{\partial n}\, ds, \tag{8.2}$$

and

$$(h,h)_D = \int_C h \frac{\partial h}{\partial n}\, ds.$$

Since

$$\int_C h \frac{\partial p}{\partial n}\, ds = \int_C S\left(\frac{\partial S}{\partial n} - \frac{\partial h}{\partial n}\right) ds$$

it follows that

$$\int_C h \frac{\partial p}{\partial n}\, ds = -(S,S)_{\bar{D}} - (h,h)_D. \tag{8.3}$$

Because of the non-negativity of the Dirichlet integrals, this establishes (8.1).

The area principle and its various generalizations, on the other hand, are expressed by the inequality

$$\int_C S \frac{\partial S}{\partial n} \, ds \leq 0, \tag{8.4}$$

which is a consequence of the identity (8.2). Since, by (8.2) and (8.3),

$$\int_C h \frac{\partial p}{\partial n} \, ds = \int_C S \frac{\partial S}{\partial n} \, ds - (h,h)_D, \tag{8.5}$$

it is clear that (8.4) implies (8.1). Hence, (8.4) is the stronger inequality, i.e., the inequalities obtained by means of the contour integration method are always contained in those deduced from the area principle (if the same singularity function $S(w)$ is used in both cases).

Novertheless, there are two aspects in which the contour integration method is superior. The first of these is its applicability to coefficient problems for univalent functions defined in multiply-connected domains, which is due to the fact that the integral in (8.1) can be evaluated by the residue theorem. Indeed, if σ and τ are the analytic functions for which $S = \operatorname{Re}\{\sigma\}$, $p = \operatorname{Re}\{\tau\}$, it follows from $p(w) = 0$ $(w \in C)$ that

$$\frac{\partial p}{\partial n} \, ds = \frac{1}{i} \, d\tau,$$

and (8.1) may therefore be replaced by

$$\operatorname{Re}\{\frac{1}{i} \int_C (\sigma - \tau) \, d\tau\}.$$

On the other hand, inequality (8.4) (as shown in Section 2) is equivalent to

$$\mathrm{Re}\{\frac{1}{i} \int_C \bar{\sigma}\, d\sigma\} \leq 0.$$

This expression is easily evaluated in terms of the coefficients of f if D is simply-connected, but it becomes completely unmanageable if D is of higher connectivity. It may be added that these remarks apply only if it is desired to obtain information involving the Taylor coefficients of the function. Inequalities for the coefficients of certain other expansions are accessible via the area method [8].

The other aspect which makes the contour integration method valuable is the information it provides concerning the cases in which there is equality in (8.1). If we relax the assumptions on C and permit C to consist of a finite number of analytic arcs, it is clear from (8.2) that there will be equality in (8.4) whenever $\bar{D}$ has the area zero, i.e., D is a slit-domain. This, of course, is also necessary in order to have equality in (8.1) but, as shown by (8.5), we now have the additional condition $(h,h)_D = 0$. Since D cannot have zero area, this will happen whenever h is constant in D. Because of $S - p = h = \gamma =$ const., this means that $S \equiv p + \gamma$. Since p is characterized by its singularities and by the fact that $p = 0$ on C, we finally reach the conclusion that there will be equality in (8.1) if, and only if, the boundary C of D consists of arcs along which the singularity function $S(w)$ takes constant values. This also shows, incidentally, that, regardless of the choice of $S(w)$, (8.1) is always a sharp inequality, even though it is weaker than (8.4). For example, the Grunsky inequality (1.3) is obtained from (8.1) by taking $S = \mathrm{Re}\{P(w^{-1})\}$, where $P(t)$ is a suitable polynomial of degree n. There will therefore be equality in (1.3) if, and only if, the function $f(z)$ maps $|z| < 1$ onto a domain bounded by slits along which $\mathrm{Re}\{P(w^{-1})\} = \mathrm{const.}$ Conceivably, this additional information may be of help in the treatment of the coefficient problem.

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