MINIMUM GIRTH OF SPHERES

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Let X be a real normed linear space with norm || ||, and let £ be its unit ball, with the boundary d£. Assume dim X ^ 2. These notations and assumptions will be maintained throughout the paper. In [1] we defined the girth of £ to be 2m(X), where m(X) = inf{6(-p,p): p€dS} and 6 denotes the inner metric of d£ induced by the norm, or, equivalently, m(X) = inf{t(c): c a rectifiable curve in d£ with antipodal endpoints}. If dim X < «, then these infima are attained, and m(X) > 2 [1; Lemma 5.1, Theorem 5.5]. The purpose of this paper is to sharpen this inequality and to determine the best lower bound for all spaces of a given finite dimension.

The conclusion is best stated in terms of m^.(n) = min{m(X) : dim X = n}, n = 2, 3, . . ., a sequence of numbers introduced (and shown to exist) in [1].

In [2] it was shown that, if dim X = n and £ is a parallelotope (i.e., X is congruent to l_n^{∞}), then m(X) = $2(1+(n-1)^{-1})$; it was further shown that, if n is odd, there is a subspace Y of co-dimension 1 such that m(Y) is still $2(1+(n-1)^{-1})$. (We remark that exactly the same results obtain if X is taken to be congruent to l_n^1 instead of to l_n^{∞}, but we omit the proof.) From this and from the elementary fact that the sequence (m^.(n)) is non-increasing we obtain

1. Theorem ([2; Theorem 7]). $m^*(2n+1) \leq m^*(2n) \leq 2+n^{-1}$, $n = 1,2,\ldots$.

We propose to show that equality holds for all n, thus giving an affirmative answer to the query at the end of [2]. The fact that $m^*(3) = 3$ was proved independently in [3].

2. Theorem. If $\dim X = n$ is an odd integer, then $m \geq 2n(n-1)^{-1}$.

Proof. Consider any given rectifiable curve in $\partial\Sigma$ with antipodal endpoints; let ℓ be its length and $g: [0,1] \to \partial\Sigma$ its parametrization in terms of arc-length. Let $p_k = g(kn^{-1}\ell)$, $k = 0,\ldots,n$, so that $p_0 = -p_n$. Let A be a non-zero n-linear form on X (unique up to multiplication by a non-zero scalar) and define $\varphi: [0,1] \to R$ by $\varphi(t) = A(g((1-t)n^{-1}\ell), g((2-t)n^{-1}\ell),\ldots, g((n-t)n^{-1}\ell))$. Now φ is continuous, and, since n is odd,

$$\varphi(1) = A(p_0,p_1,\ldots,p_{n-1}) = A(-p_n,p_1,\ldots,p_{n-1}) = -A(p_1,p_2,\ldots,p_n) = -\varphi(0).$$

Therefore $\varphi(s) = 0$ for some s, $0 \leq s \leq 1$. Set $q_k = g((k-s)n^{-1}\ell)$, $k = 1,\ldots,n$. Then $0 = \varphi(s) = A(q_1,\ldots,q_n)$, so the q_k, $k = 1,\ldots,n$, are linearly dependent.

Set $q_0 = -q_n$, and $x_k = q_k - q_{k-1}$, $k = 1,\ldots,n$. Then

(1) $\qquad \left\| \sum_{i=1}^{j} x_i + \sum_{j+1} x_i \right\| = \|-2q_j\| = 2, \quad j = 0,\ldots,n,$

$\qquad n^{-1}\ell)-g(0)\| + \|g(\ell)-g((n-s)n^{-1}\ell)\| \leq$

(2) $\qquad (1-s)n^{-1}\ell + sn^{-1}\ell = n^{-1}\ell.$

$$|x_k| = |g((k-s)n^{-1}\ell) - g((k-1-s)n^{-1}\ell)| \leq n^{-1}\ell, \quad k = 2,\ldots,n.$$

Further, the x_k, k = l,...,n, are linearly dependent; i.e., there exist real numbers *a^,* k = l,...,n, not all 0, such that

$$(3) \qquad T^n_1 \, cux^\wedge_{KK} = 0.$$

We may assume without loss that

$$(4) \qquad \max_k |a_k| = 1$$

and that, say, |o.^{n}| = 1 for some h, 1 ^ h <£ n. Then

h-1 n n ' . ' h 1 . n n+1 ' . ' , h-1 I n n . ' h 1 n h+1 ^) | =

21 ex$_n$, I = 2. Setting j = h-1 or j = h and replacing <u>every</u> a_k by -<x$_k$ if necessary, we may consequently assume, without invalidating (2),(3), that

$$(5) \qquad -D^j_1 \, ot_v + E^n_{j+1} a_{,--}) \, 1 \quad \text{for some } j, \; 0 \pounds j \pounds n.$$

Combining (1) for that value of j with (3), (4), (2), (5), we obtain

$$2 = || -25^j_1 \, x_K + E^n_{j+1} \, 3c_K \, || = || \,|^j_1 -r(l+a_V)x^\wedge_{KK} + E^n_{j+1} d \, -a_K \,)_K x, \; k(XJ^j_1 (l+a_V)_K + E^n_{j+1} (l-a.)_K) n^{-x} \ell \leq$$

$$\mathbf{(n-1)n^{-1}\ell.}$$

Thus $I \geq 2n(n-l)^{""1}$; since the curve in 92 with antipodal endpoints was arbitrary, the conclusion follows.

<u>Remark</u>. Part of this theorem is a refinement of [4; Theorem 3.2]. For even dimension n, a similar proof (using the fact that any n+1 points are linearly dependent) shows that $m(X) \geq 2n^{\sim 1}(n+1)$,

but this will follow from Theorems 1 and 2.

We can now give the exact value of m^(n) for every n ^ 2:

3. <u>Theorem</u>. m^.(2n+l) = m^.(2n) = $2+n^{-1}$, n = 1,2,... .

<u>Proof</u>. By Theorem 2, m^(2n+l) $\geq 2+n^{-1}$; the conclusion follows from Theorem 1.

References,

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