APPROXIMATE BOUNDARY CONTROLLABILITY OF THE HEAT EQUATION, II

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Report 69-3

This research was supported in part by N.S.F. Grant GP 7607 and AFOSR Grant AF-AFOSR-728-66.

In a previous note [1] the authors considered the problem of approximating a prescribed temperature state in a body, at time t, by adjusting the boundary temperature over a preceding interval of time. There, L_2-approximations were discussed, but we wish now to obtain analogous results for the C^m- topology. More precisely the following result was established. Let R be a bounded open subset of R^n with piecewise-smooth boundary ∂R having finite $(n-1)$-dimensional volume. For $-1 < a < b < 0$, let

$$B_{ab} = \partial R \times (a,b).$$

Let $\bar{u}$ be a continuous function in B_{ab} and let $U(x,t;\bar{u})$ denote the solution of the problem.

$$(1)\quad \begin{aligned} &U_t = \Delta U \quad \text{in} \quad R \times (-1,0] \\ &U(x,-1) = 0, \quad x \in R \\ &U(x,t) = \bar{u}(x,t) \quad \text{for} \quad (x,t) \in B_{ab} \\ &U(x,t) = 0 \qquad \text{for} \quad (x,t) \in (\partial R \times [0,1]) - B_{ab} \end{aligned}$$

<u>Theorem</u> 1 . <u>Suppose</u> $f \in L_2(R)$ <u>and</u> $\epsilon > 0$ <u>are given</u>. <u>Then for any</u> a <u>and</u> b, $-1 < a < b < 0$ <u>there exists a function</u> $\bar{u}$, <u>continuous in</u> B_{ab}, <u>such that</u>

$$\|U(\cdot,0;\bar{u})-f(\cdot)\|_{L_2(R)} < \epsilon.$$

In the present work we will show that it is also possible to obtain <u>uniform</u> approximations to the function f provided that it is sufficiently smooth. We show in fact that if f is k-times differentiable in $\overline{R}$ then it and all its derivatives up to order k can be approximated uniformly in R.

In the present application we will have to discuss derivatives of functions up to and including the boundary. Hence it is necessary to put some smoothness conditions on ∂R. In order not to have to concern ourselves with precise conditions in various situations we make the following hypothesis:

(A.1) ∂R is a C^{∞} surface.

Next we point out that if a function f is to be uniformly approximated by functions of the form $U(x,t;\overline{u})$ it is necessary that it satisfy certain compatability conditions on ∂R. Note first that any function of the form $U(x,t;\overline{u})$ is identically zero for $x \in \partial R$ and t near zero. Hence we must have

$$\frac{\partial^k}{\partial t^k} U(x,t;\overline{u}) \equiv 0 \quad \text{for} \quad x \in \partial R, \quad t = 1, \; k = 0,1,2,\ldots$$

But then it follows from the first of equations (1) that

(2) $\quad \Delta^k U(x,t;\overline{u}) \equiv 0 \quad \text{for} \quad x \in \partial R, \quad t = 1 \quad k = 0,1,2,\ldots$

Suppose now that $f \in C^M(\overline{R})$* and that f together with all derivatives up to order M can be uniformly approximated on $\overline{R}$ by functions of the form $U(x,t;\overline{u})$. Then it follows from (2) that f must satisfy the conditions,

(A.2) $\quad \Delta^j f \equiv 0 \quad \text{for} \quad x \in \partial R, \; j = 0,1,\ldots. \; [\frac{M}{2}]$

(*) That is $f \in C^M(R)$ and the derivatives of f up to order M are all uniformly continuous in R.

Our main result is that all functions f which satisfy conditions (A.2) can be uniformly approximated in the sense desired.

Theorem 2: Let R *be a domain in* R^n *satisfying* (A.1). *Let* $f \in C^M(\bar{R})$ *and suppose that* f *satisfies* (A.2). *Let* $\epsilon > 0$ *be given. Then for any* a *and* b, $-1 < a < b < 0$, *there exists a function* $\bar{u}$ *continuous in* B_{ab} *such that*

$$|f-U|_M = \sup_{\substack{x \in R \\ |\alpha| \leq M}} |D^\alpha f - D^\alpha U| < \epsilon \tag{3}$$

Here we use the standard notation in which α is a vector $(\alpha_1,\ldots,\alpha_n)$ with non-negative integer entries,

$$|\alpha| = \alpha_1 + \ldots. + \alpha_n$$

and

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots. \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

We shall prove theorem 2 presently. First, however, we wish to point out that we can modify our result so that it gives uniform approximations to *general* k-times differentiable functions. To accomplish this we must, of course, give up the requirement that the boundary functions $\bar{u}$ should vanish near $t = 0$. For $-1 < c < 0$ let us denote by B_c the set

$$B_c = \partial R \times (c,0].$$

Now define $U(x,t;\bar{u})$ as in (1) but with B_c replacing B_{ab}. Then we have the following result.

Theorem 3: Let R be a domain satisfying (A.1). Let $f \in C^M(\bar{R})$. Then for any c, $-1 < c < 0$, there exists a $\bar{u}$ continuous in B_c such that if $\hat{f}(x) = f(x) - U(x,0;\bar{u})$ then $\hat{f} \in C^M(\bar{R})$ and $\hat{f}$ satisfies (A.2).

Once we have theorems 2 and 3 we can approximate an arbitrary $f \in C^M(\bar{R})$ by an expression of the form $U(x,t;\bar{u}_1+\bar{u}_2)$ where $\bar{u}_1$ has support in B_{ab} and $\bar{u}_2$ has support in B_c, a,b,c arbitrary.

The proof of theorem 3 is easy and we give it here. Suppose that for the given function f we have,

$$\Delta^j f = \varphi_j \quad \text{for} \quad x \in \partial R \quad j = 0,1,\dots[\tfrac{M}{2}] \ .$$

Let $\bar{u}$ be a function defined on B_c such that

$$(4) \qquad \lim_{t\uparrow 0} \frac{\partial^j \bar{u}(x,t)}{\partial t^j} = \varphi_j(x) \quad x \in R, \quad j = 0,1,\dots.[\tfrac{M}{2}],$$

and consider the function $U(x,t;\bar{u})$. We have then

$$U(x,t;\bar{u}) = \bar{u}(x,t) \quad (x,t) \in B_c,$$

hence by (4)

$$(5) \qquad \lim_{t\uparrow 0} \frac{\partial^j U(x,t;\bar{u})}{\partial t^j} = \varphi_j(x) \quad x \in \partial R, \ j = 0,1,\dots[\tfrac{M}{2}].$$

On the other hand, $U(x,t;\bar{u})$ is a solution of the heat equation ; hence,

$$\Delta^j U(x,0;\bar{u}) = \frac{\partial^j}{\partial t^j} U(x,0;\bar{u}) = \varphi_j(x),$$

$$x \in R, \quad j = 0,1,\ldots[\tfrac{M}{2}],$$

by (5).

Proof of Theorem 2. For each integer m, let H_{2m} denote the Hilbert space composed of functions on R having $2m$ strong derivatives, in $L^2(R)$ and let $A_{m,\Delta}$ denote the closed subspace of H_{2m} which is generated by those functions in $C^{2m}(\bar{R})$ satisfying,

(6) $$\Delta^j f = 0 \quad \text{for} \quad x \in \partial R \quad \text{and} \quad j \leq m-1.$$

[That is, $A_{m,\Delta}$ is generated by functions all of whose normal derivatives of even order $\leq 2m-2$ are zero.] Now on $A_{m,\Delta}$ the bilinear form $(\cdot,\cdot)_{m,\Delta}$ defined by

(7) $$(u,v)_{m,\Delta} = \int_R \Delta^m u \, \Delta^m v \, dx$$

defines a norm $\|\ \|_{m,\Delta}$ which is equivalent to the H_{2m}-norm defined by

$$\|u\|_{2m}^2 = \sum_{|\alpha|\leq 2m} \|D^\alpha u\|_{L^2(R)}^2 .$$

That is, one has the following result.

Lemma 1. There exists a constant K depending only on m and R such that for any $u \in A_{m,\Delta}$,

$$\|u\|_{2m} \leq K\|u\|_{m,\Delta}$$

Proof: To see that $\|\ \|_{m,\Delta}$ is a norm, notice that if

$f \in C^{2m}(\overline{R})$ satisfies (6) and in addition $\|f\|_{m,\Delta} = 0$ then, in particular, we have

$$\Delta^m f = \Delta(\Delta^{m-1} f) \equiv 0 \quad \text{in} \quad R, \quad \Delta^{m-1} f = 0 \quad \text{for} \quad x \in \partial R.$$

Thus it follows that $\Delta^{m-1} f \equiv 0$ in R. Proceeding successively we deduce from (6) that $f \equiv 0$ in R. Moreover, for any $f \in C^r(\overline{R})$ which satisfies $f = 0$ for $x \in \partial R$ one has ([2], page 195)

$$(8) \qquad \|f\|_r \leq C\|\Delta f\|_{r-2},$$

where C depends only on R and r. Hence beginning with $r = 2m$ and proceeding successively, we deduce that for all $f \in C^{2m}(\overline{R})$ which satisfy (6) one has

$$(9) \quad \|f\|_{2m} \leq C\|\Delta f\|_{2m-2} \leq C'\|\Delta^2 f\|_{2m-4} \leq \cdots \leq K\|\Delta^m f\|_0 = K\|f\|_{m,\Delta},$$

where K depends only on m and R. This concludes the argument.

Now recall that, according to Sobolev's lemma, if $2m > M + \frac{n}{2}$ then the following inequality holds for all functions in H_{2m},

$$(10) \qquad |f|_M \leq C\|f\|_{2m},$$

where the norm on the left is the C^M-norm defined in (3) and where C depends only on M,m and R. Thus lemma 1 yields the following result.

<u>Corollary</u>. <u>If</u> $u \in A_{m,\Delta}$ <u>and</u> $2m > M + \frac{n}{2}$ <u>then</u>

(11) $$|u|_M \leq C\|u\|_{m,\Delta},$$

<u>where</u> C <u>depends</u> <u>only</u> <u>on</u> M,m <u>and</u> R.

The proof of theorem 2 now rests on the following two lemmas.

<u>Lemma</u> 2 <u>For any</u> a <u>and</u> b <u>the functions</u> $U(x,0;\bar{u})$, <u>for</u> $\bar{u}$ <u>continuous in</u> B_{ab}, <u>are dense in</u> $A_{m,\Delta}$.

<u>Lemma</u> 3 <u>Let</u> M <u>be given and let</u> m <u>be any integer with</u> $m > M + [\frac{n}{2}]$. <u>Let</u> $f \in C^M(\bar{R})$ <u>and satisfy</u> (A.2) <u>and let</u> $\epsilon > 0$ <u>be given</u>. <u>Then there exists an</u> $f_1 \in C^{2m}(\bar{R})$ <u>which satisfies</u> (6) <u>and is such that</u> $|f-f_1|_M < \epsilon$.

With these two lemmas in hand the proof of theorem 2 is immediate. Given $f \in C^M(\bar{R})$ and the numbers a,b and $\epsilon > 0$, lemma (3) states that we can find, for any $m > M + [\frac{n}{2}]$, an $f_1 \in C^{2m}(\bar{R})$ satisfying (6) and such that $|f-f_1|_M < \frac{\epsilon}{2}$. Then by lemma (2) we can find a function $\bar{u}$ continuous in B_{ab} such that

$$\|f_1(\cdot) - U(\cdot,0;\bar{u})\|_m < \epsilon/2C.$$

Since f_1 and $U(\cdot,\cdot,\bar{u})$ are both in $C^{2m}(\bar{R})$ and satisfy (6) it follows then by (11) that

$$|f_1(\cdot) - U(\cdot,0;\bar{u})|_M < \frac{\epsilon}{2};$$

hence

$$|f(\cdot) - U(\cdot,0;\bar{u})|_M < \epsilon.$$

Proof of Lemma 2.

Let us now outline the proof of lemma 2. This closely parallels that of [1] and proceeds as follows. The functions $U(x,t;\bar{u})$ at $t = 0$ can be written in the form,

$$(12) \qquad U(x,0;\bar{u}) = \int_{-1}^{0} \int_{\partial R} \bar{u}(y,\tau) G_\nu(x,y,-\tau)\,d\tau$$

Here $G(x,y,t-\tau)$ is the Green's function for the heat equation in R and G_ν denotes the normal derivative with respect to the variable y. For $(y,\tau) \in B_{ab}$ it is easy to see that $G_\nu(\cdot,y,-\tau)$, considered as a function of x, belongs to $C^{2m}(\bar{R})$ and satisfies (6), for any m. In [1] we exploited the fact that these functions spanned $L_2(R)$. The key here is that they also span $A_{m,\Delta}$.

Lemma 4. The functions $\{G_\nu(\cdot,y,-\tau)\}$ for $(y,\tau) \in B_{ab}$ span the space $A_{m,\Delta}$.

We assume the validity of lemma 4 for the moment and complete the proof of lemma 2. The f of lemma 2 belongs to $A_{m,\Delta}$. Hence given any ϵ we can find points $(y_1,\tau_1),\ldots(y_n,\tau_n)$ in B_{ab} and numbers $c_1,\ldots,c_n$ such that,

$$(13) \qquad \|f(\cdot) - \Sigma_1^n c_i G_\nu(\cdot,y_i,\tau_i)\|_{m,\Delta} < \frac{\epsilon}{2}.$$

Now the function $G_\nu(x,y,-\tau)$ has derivatives of all orders which are continuous in y and τ. Hence given any $\epsilon' > 0$ we can find δ such that

(14) $$|\Delta^m G_\nu(x,y,-\tau) - \Delta^m G_\nu(x,y_i,-\tau_i)| < \epsilon'$$

for $x \in \bar{R}$ and $|y-y_i| < \delta$, $|\tau-\tau_i| < \delta$. Let us choose functions $\theta_i(y,\tau)$, with supports in $|y-y_i| < \delta, |\tau-\tau_i| < \delta$ respectively such that

(15) $$\int_{-1}^{0}\int_{\partial R} \theta_i \, dyd\tau = 1$$

and consider the function

$$U(x,t;\ \Sigma_1^N c_i\theta_i).$$

We have by (14) and (15)

(16) $$|\Delta^m U(x,0;\ \Sigma_1^N c_i\theta_i) - \Sigma_1^N c_i \Delta^m G_\nu(x,y_i,-\tau_i) \int_{-1}^{0}\int_{R} \theta_i dyd\tau|$$

$$=|\sum_1^n c_i \int_{-1}^{0}\int_{\partial R} \theta_i(y,\tau)[\Delta^m G_\nu(x,y,-\tau)-\Delta^m G_\nu(x,y,-\tau_i)]|$$

$$\leq \epsilon' \sum_1^n |c_i|.$$

If we choose ϵ' according to the inequality,

$$\epsilon' \leq (\sum_1^n |c_i|)^{-1} A^{-1/2}\epsilon/2$$

where A denotes the n-volume of R then it follows from (15),(16) and Schwarz's inequality that,

$$\|f(\cdot) - U(\cdot,0;\ \Sigma_1^N c_i\theta_i)\|_{m,\Delta} < \epsilon.$$

Thus the proof of lemma 2 is reduced to that of lemma 4 .

In order to prove lemma 4 we need to make use of the eigenfunctions of the Laplacian for R. These are functions a_k satisfying the conditions,

$$(17) \qquad \Delta a_k = -\lambda_k a_k,\ a_k = 0 \ \text{ on } \ \partial R,\ \|a_k\|_{L_2(R)} = 1$$

The a_k's form an orthonormal basis for $L_2(R)$. The $\{\lambda_k\}$ are positive and if we let $\{\mu_j\}$ be the distinct λ_k-values, ordered by increasing magnitude, then $\{\mu_j/j^2\}$ is bounded away from 0 and ∞.

Lemma 5. The functions $\{\lambda_k^{-m} a_k\}$ form an orthonormal basis for $A_{m,\Delta}$.

Proof: It follows from (A.1) that the a_k are infinitely differentiable in $\overline{R}$ (see [2]pp.190,201). By (17) we see that $\Delta^j a_k = 0$ on ∂R for any j. Hence $\lambda_k^{-m} a_k \in C^{2m}(\overline{R}) \cap A_{m,\Delta}$. We have, by (17),

$$(\lambda_k^{-m} a_k, \lambda_\ell^{-m} a_\ell)_{m,\Delta} = \int_R (\lambda_k^{-m} \Delta^m a_k \lambda_\ell^{-m} \Delta^m a_\ell)\, dx$$

$$= \int_R a_k a_\ell\, dx = \delta_{k\ell}.$$

Hence the set $\{\lambda_k^{-m} a_k\}$ is orthonormal. Suppose $\varphi \in C^{2m}(\overline{R}) \cap A_{m,\Delta}$ is orthogonal to $\lambda_k^{-m} a_k$ for all k. Then

$$0 = (\lambda_k^{-m} a_k, \varphi)_{m,\Delta} = \int_R \lambda_k^{-m} \Delta^m a_k \Delta^m \varphi\, dx$$

$$= \pm \int_R a_k \Delta^m \varphi\, dx$$

Since the $\{a_k\}$ span $L_2(R)$, it follows that $\Delta^m \varphi = 0$ in R and this together with $\Delta^j \varphi = 0$ on ∂R, $j = 0,1,\ldots,m-1$ implies that $\varphi = 0$.

We can now complete the proof of lemma 2. Suppose the span of $\mathfrak{m} = \{G_\nu(\cdot,y,-\tau):(y,\tau) \in B_{ab}\}$ were not dense in $A_{m,\Delta}$. We could then find a non-zero $\varphi \in \mathfrak{m}^\perp$; that is a function $\varphi \in A_{m,\Delta}$ such that

$$0 = (\varphi, G_\nu(\cdot,y,-\tau))_{m,\Delta} \text{ for each } \nu. \tag{18}$$

We can expand φ in the form

$$\varphi = \Sigma\, \beta_k \lambda_k^{-m} a_k, \quad \beta_k = (\varphi, \lambda_k^{-m} a_k)_{m,\Delta} .$$

Hence (18) becomes

$$0 = \sum_k \beta_k \Gamma_k(y,\tau) \tag{19}$$

where $\Gamma_k(y,\tau)$ denotes the k-th Fourier coefficient of $G_\nu(\cdot,y,-\tau)$ with respect to $\{\lambda_k^{-m} a_k\}$. Now $G(x,y,-\tau)$ has the expansion

$$G(x,y,-\tau) = \sum_k a_k(x) a_k(y) e^{\lambda_k \tau} .$$

It follows that

$$\Delta^m G_\nu(x,y,-\tau) = \Sigma_k \Delta^m a_k(x) b_k(y) e^{\lambda_k \tau} = \Sigma_k (-\lambda_k)^m a_k(x) b_k(y) e^{\lambda_k \tau}$$

where $b_k(y) = \frac{\partial a_k}{\partial \nu}(y)$. Hence we have

$$\Gamma_k(y,\tau) = (-1)^m \lambda_k^{2m} b_k(y) e^{\lambda_k \tau}$$

and equation (19) becomes,

(20) $$0 = \sum_k \pm \beta_k \lambda_k^{2m} b_k(y) e^{\lambda_k \tau}.$$

Now the series (20) converges absolutely and uniformly for $(y,\tau) \in B_{ab}$. To see this note that the inequalities $\ell k^2 < \lambda_k < Lk^2$ imply the existence of a constant J such that $\lambda_k^{2m} e^{\lambda_k \tau} \leq J e^{\lambda_k b/2}$. Thus (20) is majorized by the series $\Sigma |\beta_k| \, |b_k(y)| e^{\lambda_k b/2}$ for $(y,\tau) \in B_{ab}$. This converges uniformly in y since $b_k(y) e^{\lambda_k b/4}$ are the Fourier coefficients of $G_\nu(\cdot, y, -b/4)$ for the set $\{a_k\}$ and the functions $G_\nu(\cdot, y, -\frac{b}{4})$ are uniformly bounded in $L_2(R)$, thus ensuring that $|b_k(y)| e^{\lambda_k b/2} \leq C e^{\lambda_k b/4}$ for some fixed C. This provides a convergent series dominating (20) since the β_k are the coefficients of φ in the set $\{\lambda_k^{-m} a_k\}$.

The remainder of the proof is exactly as in [1]. We collect terms in (20) in which the eigenvalues are the same. Thus, if $\{\mu_j\}$ are the distinct eigenvalues, (20) yields

(21) $$0 = \Sigma_j \, \gamma_j e^{\mu_j \tau} \qquad a \leq \tau \leq b < 0$$

where

(22) $$\gamma_j = \Sigma_{k \in K_j} \pm \mu_j^{2m} \beta_k b_k(y).$$

Here $K_j = \{k: \lambda_k = \mu_j\}$ (note that each K_j is finite). From (21) we obtain (see [1])

$$0 = \Sigma_{k \in K_j} \beta_k b_k(y) \quad \text{for} \quad y \in \partial R.$$

But then $\Sigma_{k\in K_j} \beta_k a_k(y)$ has both Dirichlet and Neumann data zero. Hence it is identically zero (see [1]) and by the independence of $\{a_k\}$ it follows that the β_k's are all zero.

Proof of Lemma 3. Finally we give a proof of lemma 3. Let $f \in C^M(\overline{R})$ and satisfy $\Delta^j f = 0$ on ∂R for $j = 0,1,\ldots[\frac{M}{2}]$. Now we can find a function Φ satisfying the conditions

(23) $$\Delta^{2m}\Phi = 0 \quad \text{in} \quad R$$

(24) $$\Delta^j \Phi = 0 \quad \text{on} \quad \partial R, \; j \leq m-1$$

(25) $$\frac{\partial^k \Phi}{\partial \nu^k} = \frac{\partial^k f}{\partial \nu^k} \quad \text{on} \quad \partial R \quad \text{for} \quad k \quad \text{odd and} \quad k \leq M,$$

where ν denotes the normal to ∂R. Conditions (24) and (25) constitute a part of the Dirichlet data for equation (23). The remaining part is a set of values of normal derivatives for Φ of orders greater than or equal to M and less than or equal to $2m$. These latter can be specified arbitrarily as C^∞ functions and Φ thus determined as a function in $C^{2m}(\overline{R}) \subset C^M(\overline{R})$.

Let $F = f-\Phi$. Then $F \in C^M(\overline{R})$ and moreover we have

$$\Delta^j F = 0 \quad \text{on} \quad \partial R \quad j \leq [\tfrac{M}{2}],$$

$$\frac{\partial^k F}{\partial \nu^k} = 0 \quad \text{on} \quad \partial R \quad \text{for} \quad k \quad \text{odd and} \quad k \leq M.$$

From these conditions it is easy to see that $F \in C_0^M(\overline{R})$; that is, F and all its derivatives up to order M vanish on ∂R. Hence if we write $f = \Phi+F$ we see that lemma 3 will be proved if we can show that it is possible to approximate functions in

$C_o^M(\overline{R})$ by functions in $C^{2m}(\overline{R})$ which satisfy (6). We establish, in fact, the stronger result that the set $C_o^\infty(R)$ (C^∞ functions of compact support) is dense in C_o^M.

<u>Lemma</u> 6. $C_o^\infty(R)$ <u>is a dense subset of</u> $C_o^M(\overline{R})$.

<u>Proof</u>: The smoothness assumption on ∂R implies that for each $x \in \partial R$ there exists a ball N^x containing x which is such that

(a) the center z_x of N^x lies in R

(b) none of the segments $\overline{z_x y}$, $y \in \overline{N^x \cap \partial R}$, is tangent to (i.e., supports) ∂R at y.

Hence, by compactness of ∂R there exists a finite collection $\{N^{x_i}\ i = 1,\dots,p\}$ covering ∂R and there is some $\theta_o \in (0,\pi/2)$ such that the segments

$$\overline{z_{x_i} y} \qquad (y \in \overline{N^{x_i} \cap \partial D}, \quad i = 1,\dots,p)$$

all make an angle with ∂R at y exceeding θ_o. Moreover, for some $\epsilon > 0$ the set $\bigcup_1^p N^{x_i}$ contains the closed ϵ-neighborhood $\mathcal{G}_\epsilon$ of ∂R. Let $R_\epsilon = R - \mathcal{G}_\epsilon$. Then the sets

$$R_\epsilon,\ \{N^{x_i}\}_{1 \le i \le p} \tag{26}$$

form an open covering for $\overline{R}$. Let $\{\varphi_j\}_{0 \le j \le p}$ denote a corresponding C^∞ partition of unity for $\overline{R}$:

$$\operatorname{supp} \varphi_o \subset R_\epsilon, \quad \operatorname{supp} \varphi_i \subset N^{x_i} \quad i = 1,\dots,p, \tag{27}$$

$$\sum_{j=0}^{p} \varphi_j(x) \equiv 1, \quad x \in \overline{R}. \tag{28}$$

Given an $f \in C_o^M(\bar{R})$ (extended as zero outside R) let $\delta > 0$ be prescribed and examine the functions f_j defined by

(29) $$f_j = f\varphi_j \quad j = 0,\ldots,p.$$

Now f_o is in $C^M(\bar{R})$ and has its support in R_ϵ, hence inside R. It follows ([3],p. 1642) that f_o can be approximated arbitrarily closely in the norm $|\ |_M$ by functions belonging to $C_o^\infty(R)$. Select $\tilde{f}_o$ satisfying

(30) $$|f_o-\tilde{f}_o|_M < \delta/2, \quad \tilde{f}_o \in C_o^\infty(R).$$

Next we observe that by (27)

$$f_i \in C_o^M(N^{x_i} \cap \bar{R}) \quad i = 1,\ldots,p.$$

Define the family of functions $\{f_i^t\}$, $t \in (1,\infty)$, by

(31) $$f_i^t(x) = f_i(tx + (1-t)z_{x_i}) \text{ for } t \in (1,\infty).$$

It follows by our construction of the $\{N^{x_j}\}$ that the functions $\{f_i^t\}$ satisfy

(32) $$f_i^t \in C_o^M(\bar{R}) \text{ for } t \in (1,\infty).$$

Moreover it is easily verified that

(33) $$|f_i^t-f_i|_M \to 0 \text{ as } t \to 1, \quad i = 1,\ldots,p.$$

Hence there exist numbers $t_i \in (1,\infty)$ such that

(34) $$|f_i^{t_i}-f_i|_M < \delta/2^{i+2} \quad i = 1,\ldots,p.$$

In addition, for $t \in (1,\infty)$ f_i^t has its support inside R and hence as above each $f_i^{t_i}$ can be approximated arbitrarily closely in the norm $|\ |_M$ by functions belonging to $C_o^\infty(R)$. Select $\{\tilde{f}_i\}_{1\leq i\leq p}$ such that

(35) $$|f_i^{t_i}-\tilde{f}_i|_M < \delta/2^{i+2}, \quad \tilde{f}_i \in C_o^\infty(R), \quad i = 1,\ldots,p.$$

By construction, the function $\tilde{f}$ defined by

(36) $$\tilde{f} = \sum_{j=0}^{p} \tilde{f}_j$$

belongs to $C_o^\infty(R)$ and by equations (33),(34),(35) and (36) we have

$$|f-\tilde{f}|_M \leq \sum_{j=0}^{p} |f_j-\tilde{f}_j|_M < \delta.$$

Since $\delta > 0$ was chosen arbitrarily, this concludes the argument.

ACKNOWLEDGEMENT.

This research was supported in part by the National Science Foundation under Grant GP 7607 and by the Air Force Office of Scientific Research under Grant AF-AFOSR-728-66.

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