CHARACTERISTIC VALUES ASSOCIATED WITH A CLASS OF NON-LINEAR BOUNDARY VALUE PROBLEMS

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Report 68-27

August, 1968

*This research was supported by NSF GP-7662.

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This note is a sequel to [1] and provides the answer to a question left open there. Let $P(x), F(t,x)$ be real valued functions, continuous on $[0,1]$ and $\pi_+ \times [0,1]$ respectively, and assume that for some $\epsilon > 0$, F satisfies, for each $x \in [0,1]$,

$$(1.1) \qquad 0 < \eta_1^{-\epsilon}F(\eta_1,x) \le \eta_2^{-\epsilon}F(\eta_2,x), \qquad 0 < \eta_1 < \eta_2.$$

Let

$$(1.2) \qquad G(\eta,x) = \int_0^{\eta} F(s,x)\,ds, \qquad 0 \le \eta < \infty,\ 0 \le x \le 1,$$

and for $y \in C[0,1]$ let,

$$(1.3) \qquad H(y) = \int_0^1 [y^2(x)F(y^2(x),x) - G(y^2(x),x)]\,dx.$$

A function y is <u>admissible</u> if it belongs to $C_0^2[0,1] = \{u \in C^2[0,1] \mid u(0) = u(1) = 0\}$, does not vanish identically, and satisfies,

$$(1.4) \qquad \int_0^1 y^2(x)\,[P(x) + F(y^2(x),x)]\,dx \ge \int_0^1 (y'(x))^2\,dx.$$

An <u>admissible set</u> is a subset of $C_0^2[0,1]$ consisting entirely of admissible functions. We denote by $\mathfrak{A}_m$ the class of subsets of $C_0^2[0,1]$ which are compact, symmetric, and admissible and have genus (see def. in [1]), $\ge m$. Then, as defined in [1], the <u>characteristic values</u> of the boundary value problem, (1.5) $$y'' + P(x)y + yF(y^2,x) = 0, \quad y(0) = y(1) = 0,$$

are the numbers,

(1.6) $$\lambda_m = \inf_{B \in \mathcal{F}_m} \max_{y \in B} H(y) .$$

The purpose of this note is to show that the characteristic numbers for (1.5) defined by (1.6) are the same as those defined in [2]. Assuming the results of [1], it amounts to the same thing to show that to each non-vanishing characteristic number λ_m there corresponds a non-trivial solution y of (1.5) which has precisely $m - 1$ zeros in $[0,1]$ and for which,

(1.7) $$H(y) = \lambda_m .$$

For $y \in C_0^1[0,1]$, let $\mu_m = M_m(y)$ denote the mth eigenvalue of the linear problem,

(1.8) $$v'' + \mu v(P(x) + F(y^2(x),x)) = 0, \quad v(0) = v(1) = 0.$$

Lemma 1. Let $y \in C_0^2[0,1]$ ~~and assume that~~,

(1.9) $$M_m(y) \le 1,$$

<u>then</u>

(1.10) $$H(y) \ge \lambda_m .$$

As Theorem 4 of [1] we proved the special case of the above assertion in which y is a solution of (1.5) with precisely $m - 1$ zeros in $(0,1)$. Exactly the same argument works if we assume only (1.9).

Lemma 2. <u>Let</u> $B \in \mathcal{G}_m$, <u>then there is at least one point</u> $y \in B$ <u>for which</u> (1.9) <u>holds</u>.

<u>Proof</u>. We observe first that the eigenvalues of (1.8) depend continuously on $y \in C_0^2[0,1]$. Secondly, if we define,

$$v_k = v_k(\cdot, y),$$

to be the k<u>th</u> eigenfunction of (1.8) normalized by

$$\int_0^1 v_k^2(x,y)(P(x) + F(y^2(x),x))dx = 1, \quad v_k'(0,y) > 0,$$

then $y \to v_k(\cdot,y)$ is continuous as a map of $C_0^2[0,1]\setminus\{0\}$ into itself. The Fourier coefficients a_k of y with respect to $\{v_k(\cdot,y)\}$ are computed by,

(1.11) $$a_k = \int_0^1 y(x)v_k(x,y)(P(x) + F(y^2(x),x))dx,$$

and if a_k vanishes for $k = 1,\dots,m-1$, then

(1.12) $$\int_0^1 y^2(x)(P(x) + F(y^2(x),x))dx \le (\lambda_m(y))^{-1}\int_0^1 (y'(x))^2dx.$$

Consider the mapping from B to R^{m-1} given by,

$$y \to (a_1,\dots,a_{m-1}),$$

where the a_k are given by (1.11). This mapping is odd and continuous, thus since $B \in \mathcal{G}_m$, there exists a $y \in B$ for which $a_1 = \dots = a_{m-1} = 0$, and for which consequently (1.12) holds. Together (1.4) and (1.12) imply (1.9), and this completes the proof.

Theorem. If. m is. a positive integer and if $X_m > 0$ then there exists a solution $y = y_m$ of (1.5) with precisely $m - 1$ zeros in (0,1) and such that

$$H(y_m) = X_m .$$

Proof. The proof of Theorem 2 of [1] shows that there exists a set $B \in \&_m$ such that,

$$\max_{y \in B} H(y) = X_m, \quad (1.12)$$

and with the additional property that $H(y) = X_m$ for $y \in B$ only if y is a solution of (1.5). (Indeed, in the notation of [1], take B = 0(0(N^)), c = X^) . By Lemma 2, there is a $y \in B$ for which (1.9) holds, and by Lemma 1 and (1.12), $H(y) = X_m$, so that y is a solution of (1.5). Since the non-zero characteristic values are simple, (Theorem 3 of [1]), we have $X_{m+1} > X_m$, and thus by Lemma 1, with m replaced by m + 1, we have $i_m(y) = 1$, and y has precisely $m - 1$ zeros in (0,1).

Corollary. The characteristic values of (1.5), as defined by (1.6), are the same as those defined in [2].

Proof. See the Corollary to Theorem 4 in [1].

References

1. C. V. Coffman, A minimum-maximum principle for a class of non-linear integral equations, Carnegie-Mellon University Report 68-26.

2. Z. Nehari, Characteristic values associated with a class of nonlinear second order differential equations, Acta Math. 105. (1961), 141-175.