OSCILLATION AND NONOSCILLATION OF
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DIFFERENTIAL EQUATIONS WITH INTE-
GRABLE COEFFICIENTS

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1. **Introduction**.

We are here concerned with the oscillatory behaviour of solutions of the following second order ordinary differential equation:

$$(1) \qquad x'' + a(t)x = 0, \qquad t \geq 0,$$

where $a(t)$ is a locally integrable function of t. We call equation (1) oscillatory if all solutions of (1) have arbitrarily large zeros on $[0, \infty)$, otherwise, we say equation (1) is non-oscillatory. As a consequence of Sturm's Separation Theorem [21], if one of the solutions of (1) is oscillatory, then all of them are. The same is true for the non-oscillation of (1).

The literature on second order linear oscillation is voluminous. The first such result is of course the classical theorem of Sturm which asserts that

$$(2) \qquad a(t) \geq a_o > 0 \implies \text{oscillation},$$

and

$$(3) \qquad a(t) \leq 0 \implies \text{non-oscillation},$$

where the inequalities in (2) and (3) are to be valid to all large t.[1] Using the classical Euler's equation as the basic

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[1] We shall assume such an understanding throughout all our later discussions.

comparison equation:

(4)
$$x'' + \frac{1}{4t^2} x = 0,$$

Kneser [9] showed that

(5)
$$t^2 a(t) \leq \frac{1}{4} \Longrightarrow \text{non-oscillation},$$

and

(6)
$$t^2 a(t) \geq \frac{1+\epsilon}{4} \Longrightarrow \text{oscillation}.$$

Fite [5] was first to give an integral criteria for oscillation, namely

(7)
$$a(t) \geq 0 \quad \text{and} \quad \int^{\infty} a(t)\,dt = \infty \Longrightarrow \text{oscillation}.$$

Condition (7) was complemented by results of Hille [8] which states that if

(8)
$$a(t) \geq 0 \quad \text{and} \quad \int^{\infty} a(t)\,dt < \infty,$$

then

(9)
$$tA(t) \leq \frac{1}{4} \Longrightarrow \text{non-oscillation},$$

and

(10)
$$tA(t) \geq \frac{1+\epsilon}{4} \Longrightarrow \text{oscillation},$$

where

(11)
$$A(t) = \int_{t}^{\infty} a(s)\,ds.$$

Later, Wintner [22][2] showed that Fite's condition remains valid

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[2]Wintner actually proved a stronger assumption than that of (12), cf. Remark.

without the additional assumption that a(t) be non-negative, i.e.

(12) $$\lim_{T \to \infty} \int_o^T a(s)\,ds = \infty \implies \text{oscillation.}^2$$

Wintner [23] also proved under the assumption that

(13) $$\lim_{T \to \infty} \int_o^T a(s)\,ds < \infty,$$

the following theorems:

(14) $$\int_o^\infty \exp(-k\int_o^t A(s)\,ds)\,dt < \infty, \quad k = 2 \implies \text{oscillation}$$

and

(15) $$A^2(t) \le \frac{a(t)}{4} \implies \text{non-oscillation.}$$

Assertion (14) remains valid remains valid with k = 4. This

improvement was due to Hartman [6]. On the other hand, Opial

[16] extended (15) to

(16) $$\int_t^\infty A^2(s)\,ds \le \frac{A(t)}{4} \implies \text{non-oscillation,}$$

and complemented (16) by

(17) $$\int_t^\infty A^2(s)\,ds \ge \frac{1+\epsilon}{4} A(t) \ge 0 \implies \text{oscillation.}$$

The significance of integral conditions, e.g. (7), (12),(14)

lies in the fact that no individual t-values occur explicitly

in the condition. In other words, a(t) and its repeated indefinite

integral A(t) can be of 'arbitrarily irregular growth'. However,

the integral conditions are merely criteria for osciallation

and gives no information concerning non-oscillation when the

conditions fail to hold. On the other hand, the integral com-

parison tests of Kneser (5),(6), Hille (9), (10), and Opial (16),

(17) are 'within ϵ' necessary and sufficient conditions for

oscillation or non-oscillation of equation (1).

We are primarily interested in equations with integrable

coefficients, namely those a(t) satisfying (13). In this

case, A(t) exists for all t. If A(t) $\geq$ 0 for large t,

then Opial's results, i.e. (16), (17), are applicable. However,

for most interesting examples with a(t) assuming negative values

for large t, its iterated integral A(t) will also assume

negative values for large t, e.g. a(t) = $\frac{\sin \beta t}{t}$, $\beta \neq 0$. There

seems to exist only a few results which cover equations with

coefficients of this sort. Earlier results by Wintner [23],

Hartman [6], and Moore [14], Opial [16], [17], and also comparison

theorems due to Wintner [24] and Hartman [7], (Exercise 7.9,

p. 369) are applicable to certain situations. Recently, a few

other papers appear which also treat this problem, e.g. Macki and

Wong [11], Willett [25], and Coles and Willett [4]. However, a

thorough study of the relationship between the integrability of

the coefficients a(t), (its iterated integral A(t)), and the

oscillation of equation (1) has not been made. The purpose of

this paper is to provide some ground work for generating oscilla-

tion criteria for equation (1) with integrable coefficients.

Consider the special example a(t) = $\frac{\sin \beta t}{t}$. Results

of Willett [25] show that for this special example, equation

(1) is oscillatory[3] if $|\beta| > \frac{1}{\sqrt{2}}$ and non-oscillatory if $|\beta| < \frac{1}{\sqrt{2}}$.

Similar to Opial's (16) of Wintner's result (15), Willett obtained

a further extension of Opial's result by integrating conditions

(16),(17) in some appropriate manner. Our results here will show that

this special equation is non-oscillatory when $|\beta| = \frac{1}{\sqrt{2}}$. Moreover,

our results are applicable to further perturbations of the boundary

situation $|\beta| = \frac{1}{\sqrt{2}}$, reminiscent of the Riemann-Weber [18] extension

of the Euler's equation:

$$(18) \qquad x'' + \left(\frac{1}{4t^2} + \frac{c}{4(t \log t)^2}\right)x = 0,$$

which states that (18) is oscillatory if $c > \frac{1}{4}$ and non-oscillatory

otherwise.

The basis for our extensions is the introduction of a new

Riccati integral equation which is useful in this study especially

which $A(t)$ is not non-negative for large t. It will be shown

that under suitable assumptions, one may iterate this new Riccati

integral equation again and obtain another one of the same type.

In section 2, we establish the equivalence of non-oscillation of

(1) and the existence of a solution of certain Riccati integral

equation. In section 3, we prove generalizations of results of

Opial [1] and Willett [25]. In section 4, we present some refine-

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[3]The fact that $|\beta| = \frac{1}{\sqrt{2}}$ is the dividing line for oscillation and non-oscillation of this special example was previously known to the author and J. J. Mahony by methods of asymptotic expansions. In fact, our 'computational' technique is applicable to a large number of special examples of this sort.

ments of comparison theorems due to Wintner [24] and Hartman [7]. We devote section 5 to a discussion of some relevant examples, most of which are so far not covered by the existing results. Finally, we indicate in section 6 how transformation of variables may be used to extend our results to a larger class of equations than those of equation (1). Portions of sections 2 and 3 was earlier announced in Wong [26].

2. Basic Theory.

For later discussions, it is convenient to introduce the notion of disconjugacy. We say a solution $x(t)$ of (1) is disconjugate on the interval $[t_o, \infty)$, where $t_o \geq 0$, if $x(t)$ has at most one zero in $[t_o, \infty)$. It is clear from Sturm's Separation Theorem that (1) is non-oscillatory if and only if it has a solution $x(t)$ which is disconjugate on $[t_o, \infty)$ for some t_o . The following lemma is basic for all later discussions and was due to Wintner [23], [24], the proof may be found in Hartman [7] and is hence omitted.

LEMMA. _The following statements are equivalent:_

(i) _Equation (1) is non-oscillatory,_

(ii) _There exists a function_ $\varphi \epsilon C^1[t_o, \infty)$, _where_ t_o _is some positive real number such that_

$$(19) \qquad \varphi'(t) + \varphi^2(t) + a(t) \leq 0,$$

(iii) _There exists a solution to the following Riccati integral equation:_

$$(20) \qquad v(t) = A(t) + \int_t^\infty v^2(s)\,ds..$$

We are now ready to prove the following three theorems which form the basis of the present theory.

THEOREM 1. The non-oscillation of (1) is equivalent to the existence of a solution to the following integral equation

$$(21) \qquad u(t) = \bar{A}(t) + \int_t^\infty Q_A(s,t)u^2(s)\,ds,$$

where

$$(22) \qquad \bar{A}(t) = \int_t^\infty A^2(s)Q_A(s,t)\,ds,$$

and

$$(23) \qquad Q_A(s,t) = \exp\!\left(2\int_t^s A(\tau)\,d\tau\right).$$

PROOF. Assume that (21) has a solution $u(t)$, then an easy computation shows that $v(t) = u(t) + A(t)$ defines a solution of (20). As a consequence of the above lemma, we conclude that (1) is non-oscillatory. Conversely, if (1) is non-oscillatory then equation (20) has a solution $v(t)$. Define $u(t) = \int_t^\infty v^2(s)\,ds$. One easily sees that $u(t)$ satisfies the differential equation:

$$(24) \qquad u'(t) = -(u(t) + A(t))^2.$$

Using $Q_A(t,0)$ as an integrating factor, we integrate (24) and obtain

$$(25)\quad u(t) = u(T)Q_A(T,t) + \int_t^T A^2(s)Q_A(s,t)\,ds + \int_t^T u^2(s)Q_A(s,t)\,ds.$$

Since all integrals on the right hand side of (25) have non-negative integrands, we infer from (25) that $\bar{A}(t)$ exists and

(26) $$\lim_{T \to \infty} u(T) Q_A(T,t) = L(t).$$

We now claim that the limit $L(t)$ in (26) must be zero for all t.
Suppose that $L(t) > 0$ for some t. Choose $t_1 \geq t$ such that
for $s \geq t_1$,

(27) $$u(s) Q_A(s,t) \geq \frac{1}{2} L(t) > 0.$$

Substituting (27) into the last integral in (25), one finds

(28) $$u(t) \geq \int_t^T u^2(s) Q_A(s,t) ds \geq \frac{1}{4}(L(t))^2 \int_t^T Q_A(t,s) ds, \quad t \geq t_1.$$

Letting T tend to infinity in (28), we obtain (14) which implies
that (1) is oscillatory, which is a desired contradiction. Since
t is arbitrary, this shows that $L(t) \equiv 0$. Now we may replace
T in (25) by ∞ and readily obtain (21).

Using (21) instead of (20), we obtain the following result
which complements the result of Hartman [6] in case $A(t)$ is
integrable.

THEOREM 2. If $A(t)$ is integrable, i.e.

(29) $$\lim_{T \to \infty} \int_0^T A(s) ds < \infty,$$

and $\overline{A}(t)$ satisfies

(30) $$\int_0^\infty \exp\left(-4 \int_0^t A(u) du\right) dt < \infty;$$

then (1) is oscillatory.

PROOF. Assume the contrary that (1) has a non-oscillatory
solution $x(t)$. Define $v(t) = \dfrac{x'(t)}{x(t)}$, then it follows from

Theorem 1 that the function $u(t) = \int_t^\infty v^2(s)\,ds$ satisfies (21).

Define $\rho(t) = \int_t^\infty u^2(s)Q_A(s,t)\,ds$, then one easily sees that ρ satisfies

$$\rho'(t) = -u^2(t) - 2A(t)\rho(t)$$

$$(31) \qquad = -\rho^2(t) - 2(\bar{A}(t) + A(t))\rho(t) - \bar{A}^2(t)$$

$$\leq -(4\bar{A}(t) + 2A(t))\rho(t).$$

Using (29) in (30), we obtain that $\int_o^\infty \rho(t)\,dt < \infty$. Observe that

$$\int_{t_o}^T \int_t^\infty u^2(s)Q_A(s,t)\,ds \geq \int_{t_o}^T (\int_{t_o}^s \exp(-2\int_o^t A(\sigma)\,d\sigma)\,dt)\,u^2(s)Q_A(s,0)\,ds$$

$$(32) \qquad\qquad \geq c_1 \int_{t_o}^T s\,u^2(s)Q_A(s,0)\,ds, \qquad c_1 > 0,$$

$$(33)\quad (\int_{t_o}^T u(s)\,ds)^2 \leq (\int_{t_o}^T s\,u^2(s)Q_A(s,0)\,ds)(\int_{t_o}^T \frac{1}{s}Q_A(0,s)\,ds),$$

and

$$(34)\quad (\int_{t_o}^T \frac{1}{s}Q_A(0,s)\,ds)^2 \leq (\int_{t_o}^T \exp(-4\int_o^s A(\sigma)\,d\sigma)\,ds)\int_{t_o}^T \frac{ds}{s^2} \leq c_2.$$

Using (32) and (34) in (33), we obtain that $\int_o^\infty u(s)\,ds \leq c_4$. From the definitions of $u(t)$ and $v(t)$, we have

$$(35)\qquad (\int_{t_o}^T v(s)\,ds)^2 \leq (\int_{t_o}^T s\,v^2(s)\,ds)\int_{t_o}^T \frac{ds}{s}$$

and

$$(36)\qquad (\int_{t_o}^T s\,v^2(s)\,ds) \leq \frac{1}{2}\int_{t_o}^T u(s)\,ds.$$

Using (35) and (36), we obtain the following estimate for the

non-oscillatory solution $x(t)$, namely

$$(37) \quad \log\left(\frac{x(t)}{x(t_o)}\right) = \int_{t_o}^{t} v(s)\,ds \leq c_5 (\log t)^{1/2}, \quad c_5 > 0$$

where c_5 as the previous c_i's is an appropriate positive

constant depending only on t_o. As a consequence of (37), we have

$$\frac{1}{x^2(t)} \geq c_6 \exp(-2c_5 (\log t)^{1/2}) \geq c_6 \exp(-\tfrac{1}{2} \log t), \quad c_6 > 0$$

thus

$$\int^{\infty} \frac{dt}{x^2(t)} \geq c_6 \int^{\infty} \exp(-\tfrac{1}{2} \log t)\,dt = \infty.$$

This contradicts the existence of nonprincipal solutions (Hartman

[7], p. 355) and completes the proof.

We remark that the above proof is a modification of Hartman's

proof of assertion (14) with $k = 4$. Here, we apply the technique

to equation (21) instead of (20). Now, we may use Theorem 2 in

place of (14) to obtain a further Riccati integral equation in

case $A(t)$ is integrable. The analogue of Theorem 1 is the

following:

THEOREM 3. If $A(t)$ is integrable then (1) is non-oscillatory

if and only if there exists a solution to the following integral

equation:

$$(38) \quad w(t) = \overline{\overline{A}}(t) + \int_{t}^{\infty} Q_{\overline{A}}(s,t) w^2(s)\,ds,$$

where

$$(39) \quad \overline{\overline{A}}(t) = \int_{t}^{\infty} \overline{A}^2(s) Q_{\overline{A}}(s,t)\,ds,$$

and

$$(40) \quad Q_{\overline{A}}(s,t) = \exp\left(2 \int_{t}^{s} (A(\tau) + \overline{A}(\tau))\,d\tau\right).$$

PROOF. Assume that (38) has a solution $w(t)$, then an easy computation shows that $v(t) = w(t) + A(t) + \overline{A}(t)$ is a solution of (20), hence by the above Lemma, (1) is non-oscillatory. Conversely, if (1) is non-oscillatory then by Theorem 1, equation (21) has a solution $u(t)$. Define $w(t) = \int_t^\infty Q_A(s,t)u^2(s)\,ds$. It is easy to see that $w(t)$ satisfies

$$(41) \qquad w'(t) = -w^2(t) - \overline{A}^2(t) - 2(A(t) + \overline{A}(t))w(t).$$

Using $Q_{\overline{A}}(t,0)$ as an integrating factor, we can integrate (41) as before and obtain

$$(42) \quad w(t) = w(T)Q_{\overline{A}}(T,t) + \int_t^T \overline{A}^2(s)Q_{\overline{A}}(s,t)\,ds + \int_t^T w^2(s)Q_{\overline{A}}(s,t)\,ds.$$

Proceeding as in the proof of Theorem 1, we need to show that

$$\lim_{T \to \infty} w(T)Q_{\overline{A}}(T,t) = \overline{C}(t) \equiv 0,$$

for all large t. Suppose that $\overline{C}(\overline{t}) > 0$ for some $\overline{t}$. Choose $t_2 \geq \overline{t}$ such that for $s \geq t_2$, we have

$$(43) \qquad w(s)Q_{\overline{A}}(s,\overline{t}) \geq \tfrac{1}{2}\overline{C}(\overline{t}) > 0.$$

Substituting (43) into the last integral in (42), we obtain

$$(44) \quad w(t) \geq \int_t^T w^2(s)Q_{\overline{A}}(s,t)\,ds \geq \tfrac{1}{4}(\overline{C}(\overline{t}))^2 \int_t^T Q_{\overline{A}}(t,s)\,ds,$$

for all $t \geq t_2$. Letting $T \to \infty$ in (44), we obtain (30), which by Theorem 2 contradicts the assumption that (1) is non-oscillatory. This completes the proof.

It is clear from the proofs of Theorems 1,2,3 that the process of generating higher-order iterated Riccati integral equations may be continued if we assume further that the iterated integrals $A(t)$, $\overline{A}(t),\ldots$ are integrable. As immediate consequences of Theorems

1 and 3, we obtain the following oscillation criteria which seems to be new.

COROLLARY 1. (i) If a(t) is integrable, then

$$(45) \qquad \int_0^\infty A^2(s) \exp\left(2 \int_0^s A(\tau)\,d\tau\right) ds = \infty \Rightarrow \text{oscillation},$$

and (ii) if A(t) is integrable then

$$(46) \qquad \int_0^\infty \bar{A}^2(s) \exp\left(2 \int_0^s \bar{A}(\tau)\,d\tau\right) ds = \infty \Rightarrow \text{oscillation}.$$

Example 1. As a nontrivial example, one may take

$a(t) = \dfrac{\alpha \sin \beta t}{(\ln t)^\lambda}$ which satisfies (45) for all $\alpha, \beta \neq 0$ and $\lambda > 0$. Previous result implies oscillation only for $\lambda < 1$, (cf. Macki and Wong [11].)

Example 2. Consider $a(t) = \dfrac{\alpha \sin \beta t}{t^\gamma}$. It is easy to check that in this case for all α and $\beta \neq 0$ (45) is satisfied if $0 < \gamma < \frac{1}{2}$ but (46) is satisfied if $0 < \gamma < 1$. No previous result in literature, with the exception of Willett [25], seems to cover this example.

In the following sections, we shall develop other oscillation and non-oscillation criteria when (45) and (46) fail to hold.

3. Main Results.

In this section, we shall improve the results of Opial [16] and Willett [25] mentioned in the introduction, based upon our results in section 2. The following theorem is a further elaboration of Opial's condition (16).

THEOREM 4. <u>If</u> <u>there</u> <u>exists</u> <u>a</u> <u>function</u> B(t) <u>such</u> <u>that</u>

(47) $\text{Max}(|A(t)|, |A(t) + \int_t^\infty (A(s) + B(s))^2 ds|) \leq |A(t) + B(t)|,$

<u>then</u> (1) <u>is</u> <u>non-oscillatory</u>.

PROOF. We shall use condition (47) to prove the existence of a solution of (20) by the Caratheodory's successive approximation technique. Let N be a positive integer. We define the sequence of functions $\{v_N(t)\}$ as follows:

(48) $v_N(t) = \begin{cases} A(t) + B(t), & t \geq N \\ A(t) + \int_{t+\frac{1}{N}}^\infty v_N^2(s)\,ds, & 0 \leq t \leq N \end{cases}$

Equation (48) defines $v_N(t)$ successively on the intervals $[N - \frac{k}{N}, N - \frac{k-1}{N}]$ for k = 1,2,..., and hence $v_N(t)$ is defined on $[0,\infty)$. For $N - \frac{1}{N} \leq t < \infty$, we have from (48) that

(49) $v_N(t) \leq |A(t) + \int_t^\infty (A(s) + B(s))^2 ds| \leq |A(t) + B(t)|$

and

(50) $v_N(t) \geq A(t) \geq -|A(t)| \geq -|A(t) + B(t)|.$

Using (49) and (50) and a simple induction, it is easy to show that for all $t \geq 0$, we have $|v_N(t)| \leq |A(t) + B(t)| \leq M$. We now wish to show that $\{v_N(t)\}$ forms an equi-oscillatory family of uniformly bounded functions on $[0,\infty)$, i.e. for each $\epsilon > 0$, there exist $\delta > 0, N_0 > 0$ such that for $|s - t| < \delta$, we must have $|v_N(t) - v_N(s)| < \epsilon$ for all $N > N_0$. Since A(t) → 0, for

each $\epsilon > 0$, we may choose $T \geq 0$ such that for all $t \geq T$, we have $|A(t)| \leq \frac{\epsilon}{4}$. On the finite interval $[0,T]$, $A(t)$ is continuous and hence uniformly continuous, therefore, there must exist $\delta_1 > 0$ such that for all $0 \leq s \leq t \leq T$, $|A(s) - A(t)| < \frac{\epsilon}{4}$ whenever $|s - t| < \delta_1$. From (48), we may estimate as follows:

$$|v_N(s) - v_N(t)| \leq |A(s) - A(t)| + \int_{s+\frac{1}{N}}^{t+\frac{1}{N}} v_N^2(s)\,ds$$

$$\leq |A(s) - A(t)| + M|s - t|.$$

Choose $\delta = \text{Min}(\delta_1, \frac{\epsilon}{2M})$, one readily sees that $|v_N(s) - v_N(t)| < \epsilon$, whenever $|s - t| < \delta$, $N \geq N_0$. Now an application of Arzela-Ascoli's Theorem shows that $\{v_N(t)\}$ contians a pointwise convergent sub-sequence $\{v_{N_k}(t)\}$, which converges uniformly on every compact subinterval of $[0,\infty)$. Denote its pointwise limit by $\tilde{v}(t)$. Note that condition (47) implies there exists a sequence $\{t_j\}$, $t_j \to \infty$ as $j \to \infty$ such that $|A(t_j) + B(t_j)| \to 0$. Write equation (48) in the form

$$(51) \qquad v_{N_k}(t) = \int_t^{t_j} a(s)\,ds + \int_{t+\frac{1}{N_k}}^{t_j} v_{N_k}^2(s)\,ds + \beta_k(t_j),$$

where

$$\beta_k(t_j) = A(t_j) + \int_{t_j}^{\infty} v_{N_k}^2(s)\,ds.$$

From (47) it follows that $|\beta_k(t_j)| \leq |A(t_j) + B(t_j)|$. Letting $k \to \infty$ in (51), we have

$$(52) \quad \lim_{k \to \infty} \inf \beta_k(t_j) \leq \tilde{v}(t) - \int_t^{t_j} a(s)\,ds + \int_t^{t_j} v^2(s)\,ds \leq \lim_{k \to \infty} \sup \beta_k(t_j).$$

Now, since $\beta_k(t_j) \to 0$ as $j \to \infty$, letting $j \to \infty$ in (52), we establish that $v(t)$ is a solution of (20).

COROLLARY 2. __If__ $A(t) \geq 0$ __and__ __there__ __exists__ __a__ __function__ $B(t)$ __such__ __that__

$$(53) \qquad \int_t^\infty (A(s) + B(s))^2 ds \leq B(t),$$

__then__ (1) __is__ __non-oscillatory__.

We note that Corollary 2 reduces to Opial's condition (16) by taking $B(t) \equiv A(t)$. In view of the Wintner's Lemma, we can present independently a short proof of Corollary 2, hence the result (16) of Opial.

PROOF. Define $\varphi(t) = A(t) + \int_t^\infty (A(s) + B(s))^2 ds \geq 0$.

A simple computation shows that $\varphi'(t) + a(t) + (A(t) + B(t))^2 = 0$. Condition (53) implies that $\varphi^2(t) \leq (A(t) + B(t))^2$, from which (19) readily follows. Hence (1) is non-oscillatory.

__Example 3__. Consider the example $A(t) = \dfrac{1 + \sin t}{kt}$. It is easily checked that in this case condition (16) fails for all $k > 0$. On the other hand by taking $B(t) = \dfrac{1}{kt}$, one can show that for large t, condition (53) is satisfied with $k > 4\frac{1}{2}$.

__Example 4__. Consider the example $A(t) = \dfrac{-(1+\sin t)}{kt}$, $k > 0$ for which neither Opial's condition (16) nor Corollary 2 is applicable. However, in this case we may take $B(t) = \dfrac{4}{kt}$, and easily checked that condition (47) is satisfied for $k > 5$.

In view of the other Riccati integral equations developed in the previous section, we can now apply similar techniques to (21) and (38) and obtain non-oscillation theorems. Note that in case of (21) and (38), $\overset{-}{A}(t)$ and $\overset{=}{A}(t)$ are non-negative, hence analogues of Theorem 4 take the same form as that of Corollary 2.

THEOREM 5. If there exists a function $\overset{-}{B}(t)$ such that

$$(54) \qquad \int_t^\infty (\overset{-}{A}(s) + \overset{-}{B}(s))^2 Q_A(s,t)\,ds \leq \overset{-}{B}(t),$$

then (1) is non-oscillatory.

PROOF. By Theorem 1, it is sufficient to show that (54) implies the existence of a solution of equation (21). Define $u_0(t) = A(t) + B(t)$ and inductively, $n = 1,2,3,\ldots$

$$(55) \qquad u_n(t) = A(t) + \int_t^\infty Q_A(s,t) u_{n-1}^2(s)\,ds.$$

Using (54) and (55), it is easy to show by induction that

$$0 \leq \overset{-}{A}(t) \leq u_n(t) \leq u_{n-1}(t) \leq \overset{-}{A}(t) + B(t).$$

Thus the sequence of functions $\{u_n(t)\}$ has a pointwise limiting function $\tilde{u}(t) = \lim_{n \to \infty} u_n(t)$. Since the integrand in (55) is non-negative, it follows from the monotone convergence theorem that $\tilde{u}(t)$ is a solution of (21).

Applying the same line of argument given in the proof of Theorem 5 to equation (38), we obtain

THEOREM 6. If $A(t)$ is integrable, and there exists a function $\overset{=}{B}(t)$ such that

(56) $$\int_t^\infty (\widetilde{\widetilde{A}}(s) + \widetilde{\widetilde{B}}(s))^2 Q_{\widetilde{A}}(s,t)\,ds \le \widetilde{\widetilde{B}}(t),$$

then (1) <u>is</u> <u>non-oscillatory</u>.

Taking $\widetilde{B}(t) \equiv \widetilde{A}(t)$ and $\widetilde{\widetilde{B}}(t) \equiv \widetilde{\widetilde{A}}(t)$ in the above two

theorems, we obtain the following

COROLLARY 3. (i) (Willett [25]). <u>If</u>

(57) $$\int_t^\infty \widetilde{A}^2(s) Q_A(s,t)\,ds \le \tfrac{1}{4}\,\widetilde{A}(t),$$

<u>then</u> (1) <u>is</u> <u>non-oscillatory;</u> <u>and</u> (ii) <u>if</u> A(t) <u>is</u> <u>integrable</u> <u>and</u>

(58) $$\int_t^\infty \widetilde{\widetilde{A}}^2(s) Q_{\widetilde{A}}(s,t)\,ds \le \tfrac{1}{4}\,\widetilde{\widetilde{A}}(t),$$

<u>then</u> (1) <u>is</u> <u>non-oscillatory</u>.

We also note that our proof differs slightly from that of Willett [25] and both proofs are much simpler than that of Opial's [16]. We shall now present a further generalization of the oscillation part of Opial's result (17).

THEOREM 7. <u>If</u> A(t) $\ge$ O <u>and</u> <u>there</u> <u>exists</u> <u>a</u> <u>non-negative</u> <u>function</u> C(t) $\not\equiv$ O <u>such</u> <u>that</u>

(59) $$\int_t^\infty A^2(s)\,ds \ge \tfrac{1}{4}\,A(t) + C(t),$$

<u>and</u>

(60) $$\int_t^\infty C(s)A(s)\,ds \ge \delta\,C(t),$$

<u>for</u> <u>some</u> $\delta \ge \tfrac{1}{4}$, <u>then</u> (1) <u>is</u> <u>oscillatory</u>.

PROOF. Assume that (1) is non-oscillatory, then by Wintner's Lemma, equation (20) must have a solution $v(t)$, which satisfies $v(t) \ge a_0 A(t) \ge O$, with $a_0 = 1$. Substituting this into (20), we find

(61) $\qquad v(t) \geq a_1 A(t) + b_1 C(t), \qquad a_1 = 1 + \frac{1}{4}, \ b_1 = 1.$

Substituting (61) again into (20), we obtain

$$v(t) \geq (1 + \frac{a_1^2}{4}) A(t) + 2a_1 b_1 \int_t^\infty C(s) A(s) \, ds + a_1^2 C(t)$$

(62)

$$= a_2 A(t) + b_2 C(t), \qquad a_2 = 1 + \frac{a_1^2}{4}, \ b_2 = a_1^2 + 2\delta a_1 b_1.$$

Using (62) and an easy induction, we can show in general that

(63) $\quad v(t) \geq a_n A(t) + b_n C(t), \ a_n = 1 + \frac{a_{n-1}^2}{4}, \ b_n = a_{n-1}^2 + 2\delta a_{n-1} b_{n-1}.$

From the recurrence relation given by (63) and the fact that

$a_2 \geq a_1$, $b_2 \geq b_1$, one readily sees that $a_n \geq a_{n-1}$, $b_n \geq b_{n-1}$.

Furthermore, it is easy to show that $\lim_{n \to \infty} a_n = 2$. Now, if

$\lim_{n \to \infty} b_n$ is finite, then we can show from (63) that $4\delta < 1$,

contrary to the given hypothesis. Since $C(t) \neq 0$, and $b_n \to \infty$

as $n \to \infty$, the desired contradiction follows from (63).

We observe that Opial's condition (17) follows from Theorem

7 by taking $C(t) \equiv \frac{\epsilon}{4} A(t)$ for some $\epsilon > 0$. Applying the same

argument to equations (21) and (38), we obtain respectively:

THEOREM 8. <u>If there exists a non-negative function</u> $\overline{C}(t) \neq 0$

<u>such that</u>

(64) $$\int_t^\infty \overline{A}^2(s) Q_A(s,t) \, ds \geq \frac{1}{4}\overline{A}(t) + \overline{C}(t),$$

<u>and</u>

(65) $$\int_t^\infty \overline{C}(s) \overline{A}(s) Q_A(s,t) \, ds \geq \delta \overline{C}(t),$$

<u>for some</u> $\delta \geq \frac{1}{4}$, <u>then</u> (1) <u>is oscillatory.</u>

THEOREM 9. _If_ $A(t)$ _is_ _integrable_ _and_ _there_ _exists_ _a_ _non-negative_ _function_ $\overline{\overline{C}}(t) \neq 0$ _such_ _that_

(66)
$$\int_t^\infty \overline{\overline{A}}^2(s)\, Q_{\overline{\overline{A}}}(s,t)\, ds \geq \frac{1}{4}\, \overline{\overline{A}}(t) + \overline{\overline{C}}(t),$$

and

(67)
$$\int_t^\infty \overline{\overline{C}}(s)\overline{\overline{A}}(s)\, Q_{\overline{\overline{A}}}(s,t)\, ds \geq \delta\overline{\overline{C}}(t),$$

for _some_ $\delta \geq \frac{1}{4}$, _then_ (1) _is_ _oscillatory_.

COROLLARY 4. (i) (Willett [25]). _If_ $\overline{A}(t) \neq 0$ _satisfies_

(68)
$$\int_t^\infty \overline{A}^2(s)\, Q_A(s,t)\, ds \geq \frac{1+\epsilon}{4}\overline{A}(t)$$

for _some_ $\epsilon > 0$, _then_ (1) _is_ _oscillatory_, (ii) _If_ $\overline{\overline{A}}(t) \neq 0$ _satisfies_

(69)
$$\int_t^\infty \overline{\overline{A}}^2(s)\, Q_{\overline{\overline{A}}}(s,t)\, ds \geq \frac{1+\epsilon}{4}\overline{\overline{A}}(t)$$

for _some_ $\epsilon > 0$, _then_ (1) _is_ _oscillatory_.

4. Comparison Theorems.

Consider another linear second order equation of the same form as that of (1)

(70)
$$y'' + b(t)y = 0,$$

we wish to discuss conditions on $a(t)$ and $b(t)$ such that non-oscillation of (1) would imply non-oscillation of (70), or vice versa. Indeed, Sturm's Comparison Theorem states the following: Suppose that for all large t,

(71)
$$a(t) \geq b(t),$$

then the oscillation of (70) implies that (1) is oscillatory. In case both $a(t)$ and $b(t)$ are integrable, hence $A(t)$ and

$$B(t) = \int_t^\infty b(s)ds$$ exist, Hille [8] showed that in this case if

(72) $a(t), b(t) \geq 0$, and $A(t) \geq B(t)$,

then, the same conclusion results. Hille's result was later improved by Wintner [24], who showed that condition (72) may be weakened to

(73) $A(t) \geq B(t) \geq 0$,

Later, Hartman [7] (p. 369, Exercise 7.9) indicated that Wintner's condition (73) may be improved to read

(74) $A(t) \geq |B(t)|$.

Hartman's result is particularly important in our discussion here, for it seems to be the first result of comparison type which is applicable when $B(t)$ is not necessarily non-negative. In particular, we have

Example 5. Consider $b(t) = \dfrac{\alpha \sin \beta t}{t^\gamma}$. Take $a(t) = \dfrac{1}{4t^2}$ in (74), we conclude from the result of Hartman that (70) is non-oscillatory for all α, β if $\gamma > 1$ and $\left|\dfrac{\alpha}{\beta}\right| < \dfrac{1}{4}$ if $\gamma = 1$.

On the other hand, Hartman result gives no information concerning the oscillation of (1) when $A(t)$ assumes negative values of large t. In view of the Riccati-integral equations (21), (38) introduced here, we can offer similar comparison theorem complementing that of Hartman's.

THEOREM 10. Suppose that $A(t)$ is integrable, and in addition we have

(75) $\bar{A}(t) \geq \bar{B}(t)$ and $A(t) + \bar{A}(t) \geq B(t) + \bar{B}(t)$,

then if (1) is non-oscillatory, (70) is also non-oscillatory.

PROOF. Suppose that (1) is non-oscillatory. By Theorem 1, equation (21) has a solution $u(t)$. Define $\rho(t) = \int_t^\infty u^2(s) Q_A(s,t) ds$, which satisfies:

(76) $\rho'(t) + \rho^2(t) + 2(A(t) + \bar{A}(t))\rho(t) + \bar{A}^2(t) = 0.$

Now define $z(t) = \exp(\int^t \rho(s) ds)$, which on account of (76) is a solution of the following second order equation:

(77) $z''(t) + 2(A(t) + \bar{A}(t))z'(t) + \bar{A}^2(t) z(t) = 0.$

Consider the second order equation with coefficients $B(t)$ and $\bar{B}(t)$ in the same form as (77)

(78) $r''(t) + 2(B(t) + \bar{B}(t))r'(t) + \bar{B}^2(t) r(t) = 0.$

Since $A(t)$ is integrable, non-oscillation of (1) implies on account of Theorem 2 that

(79) $\int_0^\infty \exp(-2\int_0^t A(u) du) dt = \infty.$

Note that $z(t)$ is non-oscillatory, this together with (79) show that (77) has a solution $z(t)$ satisfying $z(t) > 0$, $z'(t) \geq 0$ (Cf. Hartman [7], p. 352, Exercise 6.2). Using this fact concerning equation (77) and (75), we may apply another result of Hartman [7] (p. 363, Exercise 7.2) to conclude that equation (78) is non-oscillatory which in turn implies that there is a solution to the following Riccati differential equation:

$$\sigma'(t) + \sigma^2(t) + 2(B(t) + \tilde{B}(t))\sigma(t) + \tilde{B}^2(t) = 0. \tag{80}$$

Define $y(t) = \exp(\int_0^t (B(s) + \tilde{B}(s) + \sigma(s))ds)$. An easy computation shows that $y(t)$ is a non-oscillatory solution of (70). The proof is complete.

Example 6. Consider $a(t) = \dfrac{\alpha \cos \beta t}{t^\gamma}$, $0 < \gamma < 1$. Take $b(t) = \dfrac{\sin t}{t^\lambda}$ for some λ, $\gamma < \lambda < 1$. It is easily verified in this case that $A(t)$ is integrable and condition (75) is satisfied. Since equation (70) is oscillatory with this particular $b(t)$ from Example 2, we conclude that (1) is oscillatory.

We remark that (46) is also applicable to Example 6. However, Theorem 10 offers comparison between coefficients $a(t)$, $b(t)$, whose iterated integrals $A(t), B(t)$ may both be negative for large t.

5. **More Examples.**

We shall now consider more elaborate examples demonstrating various applications of the results established in the previous two sections.

Example 7. Consider $a(t) = \dfrac{\alpha \sin \beta t}{t}$. Using Corollary 4(i), it is easy to show that (1) is oscillatory if $|\frac{\alpha}{\beta}| > \dfrac{1}{\sqrt{2}}$ and similarly using Corollary 3(i), we obtain non-oscillation of (1) when $|\frac{\alpha}{\beta}| < \dfrac{1}{\sqrt{2}}$. (Details of these may be found in Willett [25]). We shall now show that Theorem 5 implies non-oscillation of (1) when $|\frac{\alpha}{\beta}| = \dfrac{1}{\sqrt{2}}$. Note that in this case, we have

(81) $A(t) = \frac{\alpha}{\beta} \frac{\cos \beta t}{t} + O(\frac{1}{t^2})$,

(82) $Q_A(s,t) = 1 + O(\frac{1}{t}) \leq 1 + \frac{L}{t}$, $L > 0$.

From (81) and (82), we obtain the following estimate for $\overline{A}(t)$:

(83) $\overline{A}(t) = \frac{\alpha^2}{2\beta^2} \cdot \frac{1}{t} + O(\frac{1}{t^2})$.

Let $K > 0$ be the constant such that $\overline{A}(t) \leq \frac{1}{4t} + \frac{K}{t^2}$. We choose $\overline{B}(t) = \frac{1}{4t} + \frac{M}{t^2}$, where M is to be determined in terms of K and L. Using (82), (83) and the estimates on A and $\overline{A}$, we can compute the integral in (54) and obtain

$$\int_t^\infty (A(s) + B(s))^2 Q_A(s,t) ds = \frac{1}{4t} + \frac{K+M+L/4}{2t^2} + O(\frac{1}{t^3}).$$

In order that (54) is satisfied, it is sufficient to require:

$M > K + \frac{L}{4}$.

Example 8. Consider $a(t) = \frac{\alpha \sin \beta t}{t} + \frac{\lambda}{t^2}$, with $\alpha, \beta, \lambda \neq 0$. Willett [25] shows that (1) is oscillatory if $\lambda > \frac{1}{4} - \frac{1}{2}(\frac{\alpha}{\beta})^2$ and non-oscillatory if $\lambda < \frac{1}{4} - \frac{1}{2}(\frac{\alpha}{\beta})^2$. Following a similar argument as outlined in Example 7, we find that (1) is non-oscillatory if $\lambda = \frac{1}{4} - \frac{1}{2}(\frac{\alpha}{\beta})^2$. We note that in this case $A(t)$ is not integrable and estimate (82) for $Q_A(s,t)$ does not hold.

Example 9. Consider $a(t) = \frac{\alpha \sin \beta t}{t} + \frac{\mu \cos \nu t}{t}$, with $\alpha, \beta, \mu, \nu \neq 0$. Applying Theorems 5 and 8 as before, we find that (1) is oscillatory if $(\frac{\alpha}{\beta})^2 + (\frac{\mu}{\nu})^2 > \frac{1}{4}$ and non-oscillatory if $(\frac{\alpha}{\beta})^2 + (\frac{\mu}{\nu})^2 \leq \frac{1}{4}$.

Example 10. Consider $a(t) = \frac{\alpha \sin \beta t}{t} + \frac{\mu \sin \nu t}{t^\sigma}$, with $|\frac{\alpha}{\beta}| = \frac{1}{\sqrt{2}}$, $\sigma > 0$, $\nu \neq \beta$. In this case, we find that (1) is

oscillatory if $\sigma \leq 1$ and non-oscillatory if $\sigma > 1$.

Similar to the role that equation (18) plays with regard to equation (4), we also consider the logarithmetic perturbation to the boundary case of Example 7, i.e. $\left|\frac{\alpha}{\beta}\right| = \frac{1}{\sqrt{2}}$.

Example 11. Consider $a(t) = \frac{\alpha \sin \beta t}{t} + \frac{\xi \sin \eta t}{t \log t}$, with $\left|\frac{\alpha}{\beta}\right| = \frac{1}{\sqrt{2}}$, $\xi, \eta \neq 0$. Case I, $\beta = \eta$, we have

$$(84) \qquad A(t) = \frac{\alpha \cos \beta t}{\beta t} + \frac{\xi \cos \beta t}{\beta t \log t} + O\left(\frac{1}{t^2}\right),$$

$$(85) \qquad \overline{A}(t) = \frac{1}{4t} + \frac{2\alpha\xi}{\beta^2 t \log t} + O\left(\frac{1}{t(\log t)^2}\right),$$

and

$$(86) \quad \int_t^\infty \overline{A}^2(s) Q_A(s,t)\,ds = \frac{1}{16t} + \frac{\alpha\xi}{\beta^2 t \log t} + O\left(\frac{1}{t(\log t)^2}\right).$$

Substituting (85) and (86) in (59) and (60), and applying Theorem 8, we conclude that (1) is oscillatory if $\alpha\xi > 0$, i.e. ξ has the same sign as α. As a direct consequence of Theorem 5, we also have that (1) is non-oscillatory provided that $\alpha\xi < 0$. In this example, (84) implies that $A(t)$ is integrable, so Theorems 6 and 9 are also applicable and indeed they produce the same conclusion. Note that Willett's results [25] in particular are not applicable here, and hence give no information for this case. Case II, $\beta \neq \eta$, we have

$$(87) \qquad \overline{A}(t) = \frac{1}{4t} + \frac{1}{2}\left(\frac{\xi}{\eta}\right)^2 \frac{1}{t(\log t)^2} + O\left(\frac{1}{t^2}\right)$$

and

$$(88) \quad \int_t^\infty \bar{A}^2(s) Q_A(s,t)\,ds = \frac{1}{16} + \frac{1}{4}\left(\frac{\xi}{\eta}\right)^2 \frac{1}{t(\log t)^2} + O\left(\frac{1}{t(\log t)^3}\right).$$

Using (87) and (88) and a direction application of Theorem 8, we condlude that (1) is oscillatory in this case.

Example 12. Consider $a(t) = \mu \sin \nu t^2$. Since $a(t)$ is integrable, our results are applicable. It can easily be computed that in this case

$$A(t) = \frac{\mu}{2\nu} \frac{\cos \nu t^2}{t} + O\left(\frac{1}{t^2}\right),$$

and

$$\bar{A}(t) = \frac{1}{2}\left(\frac{\mu}{2\nu}\right)^2 \frac{1}{t} + O\left(\frac{1}{t^2}\right).$$

Applying similar techniques as before, we obtain oscillation for $\left|\frac{\mu}{\nu}\right| > 1$ and nonoscillation for $\left|\frac{\mu}{\nu}\right| \le 1$.

We remark that in all examples discussed above the conclusions remain valid if sines and cosines are interchanged or the coefficient $A(t)$ has a perturbation term of order $O\left(\frac{1}{t^\lambda}\right)$, $\lambda > 1$. Also note that part of the results may be obtained by a combination of Theorem 10 and Theorems 5 through 9. It is believed that in case $A(t)$ is integrable, Theorems 6 and 9 will be useful in discussing perturbations of higher order, namely $O((t \log t \log|\log t|)^{-1})$.

6. Extensions and Remarks.

Using transformation of variables, one can easily translate results concerning equation (1) to the general Sturm-Louiville equation of second order:

$$(87) \qquad (p(t)x')' + q(t)x = 0, \qquad p(t) > 0.$$

Following Moore [13], we can transform equation (87) into the same form as equation (1), depending on the convergence or divergence of the integral $\int^{\infty} \frac{dt}{p(t)}$. In the first case, we introduce the change of variables

(88)
$$ s = \int_{0}^{t} \frac{d\tau}{p(\tau)} \ , \quad y(s) = x(t), $$

which transforms (87) into

(89)
$$ \frac{d^2 y}{ds^2} + p(t)q(t)y(s) = 0, \quad s \epsilon [s_0, \infty). $$

In the other case, we introduce the transformation

(90)
$$ s = (\int_{t}^{\infty} \frac{d\tau}{p(\tau)})^{-1}, \quad s^{-1}y(s) = x(t), $$

which tranxforms (89) into

(91)
$$ \frac{d^2 y}{ds^2} + \frac{p(t)q(t)}{s^4} y(s) = 0. \quad s \epsilon [s_0, \infty). $$

In both cases, equations (90) and (91) are of the same form of (1) to which results based upon (1) are readily applicable. Some results on asymptotic behaviour of equation (87) and in particular, oscillation or non-oscillation results, which have appeared in literature are for the most part simple consequences of trans-formations (88), (90), or some slightly more general transformations as given in [13]. This phenomenon occurs also in its nonlinear analogue, Willett [25] and Wong [27] listed a few references of this nature.

In order that results in this paper be applicable to equation (87), we require instead of condition (13) the equivalent condi-

tions on p(t) and q(t):

(92)
$$\int^{\infty} \frac{dt}{p(t)} = \infty, \qquad \int^{\infty} q(t) dt < \infty,$$

or

(93)
$$\int^{\infty} \frac{dt}{p(t)} < \infty, \qquad \int^{\infty} q(t) \left(\int_{t}^{\infty} \frac{d\tau}{p(\tau)}\right)^{2} dt < \infty.$$

Under assumptions (92) and (93), we can introduce functions $Q_p(t)$, and $\overline{Q}_p(t)$ for equation (87) which play the same role as that of $A(t)$ and $\overline{A}(t)$ for equation (1). In case when condition (92) holds, an easy computation yields the following formulae for Q_p and $\overline{Q}_p$:

$$Q_p(t) = \int_{t}^{\infty} q(\tau) d\tau,$$

$$\overline{Q}_p(t) = \int_{t}^{\infty} \frac{Q_p^2(\tau)}{p(\tau)} \exp\left(2 \int_{t}^{\tau} \frac{1}{p(\xi)} Q_p(\xi) d\xi\right) d\tau.$$

Similarly, when condition (93) is satisfied, we have the alternative formulae for Q_p and $\overline{Q}_p$.

$$Q_p(t) = \int_{t}^{\infty} q(\tau) \left(\int_{\tau}^{\infty} \frac{d\xi}{p(\xi)}\right)^{2} d\tau,$$

$$Q_p(t) = \int_{t}^{\infty} \frac{Q_p^2(\tau)}{p(\tau)} \exp\left(2 \int_{t}^{\tau} \frac{Q_p(\xi)}{p(\xi)} \left(\int_{\xi}^{\infty} \frac{d\eta}{p(\eta)}\right)^{-2} d\xi\right) \left(\int_{\tau}^{\infty} \frac{d\xi}{p(\xi)}\right)^{-2} d\tau.$$

It is rather important to point out that transformations (88) and (90) are just two of many possible 'oscillation invariant' transformations, i.e. $s \to \infty$ as $t \to \infty$ and $y(s)$ vanishes whenever $x(t)$ does. In many instances when oscillation or non-oscillation criteria fail to apply to equation (1) directly,

some appropriate oscillation invariant transformation may be introduced so that in the reduced equation (87) results on oscillation or nonoscillation may be applicable. In this regard, we discuss the following two examples.

Example 13. Consider the following equation with independent variable s:

$$(94) \qquad \frac{d^2 y}{ds^2} + (\cos s)y = 0, \quad s \geq 0.$$

Since cos s is not integrable on $[0,\infty)$, our results are not directly applicable to equation (94). However, by introducing the 'oscillation invariant' transformation:

$$s = \log t, \quad y(s) = tx(t),$$

we may transform equation (94) into the form (87) as follows:

$$(95) \qquad (t^3 x')' + (t + t \cos \log t)x = 0,$$

to which our results are applicable. In particular, an easy application of Corollary 1 (i) to equation (95) with the above-mentioned modification, replacing $Q_p, \bar{Q}_p$ for A and $\bar{A}$, yields oscillation of all solutions of (94).

Example 14. Consider the following equation with independent variable t:

$$(96) \qquad x'' + (\mu t \cos \nu t^2)x = 0, \quad \mu, \nu \neq 0.$$

Obviously, none of the above-mentioned results is applicable here. Now, using the following 'oscillation invariant' transformation:

$$2s = t^2, \quad y(s) = tx(t),$$

we can transform (96) into:

(97)
$$\frac{d}{ds}(s^{-1}y'(s)) + (\frac{2}{s^3} + \frac{\mu \cos \nu s}{s^{3/2}})y(s) = 0.$$

Since (97) is of the form (87), an easy verification of (92) shows that our results are applicable here. In fact, using any oscillation criteria above, we obtain the conclusion that (96) is oscillatory for all $\mu, \nu \neq 0$.

Finally, we close our discussion with a number of remarks relating our present work to others: (Other related results may be found in Reid [19], [20].)

Remark 1. For non-negative coefficients $a(t)$ other results are known. See for example, Nehari [15], Barrett [1]. We note that our results are best possible in the sense that in cases Euler's equation (4) and the Riemann-Weber equation (18) inequalities concerning $A(t), \bar{A}(t)$ become equalities.

Remark 2. Examples 13 and 14 belong to a class of so called 'wildly oscillating' coefficients. For these examples, the technique of averaging plays an important role, (Cf. Wintner [22], Hartman [6], Coles [2], Willett [25], Macki and Wong [11], and Coles and Willett [4].) The fact that some of these cases can be handled by some appropriate oscillation invariant transformations via results on integrable coefficient seems not yet fully understood. On the other hand, Macki and Wong [11] demonstrates an example with integrable coefficient which can be handled by averaging methods.

Remark 3. The behaviour of non-oscillatory solutions has been studied by a number of authors, under the assumption that a(t) must eventually be of one sign (see Levin [10], and other references listed there). In that case, specific information on the limiting behaviour of non-oscillatory solutions are known. In view of the present development it will be of interest to improve the above mentioned results by relaxing the sign restriction on a(t).

Remark 4. Marcus and Moore [12] have presented a detailed discussion of disconjugacy domains of solutions of (1) when a(t) is non-negative and almost periodic. We hope that the present development will lead to an improvement of their results when a(t) is not required to be non-negative.

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