LINEARIZING GEOMETRIC PROGRAMS

by

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Abstract

A geometric program concerns minimizing a function subject to constraint functions, all functions being of posynomial form. In this paper the posynomial functions are condensed to monomial form by use of the inequality reducing a weighted arithmetic mean to a weighted geometric mean. The geometric mean is a monomial and by a logarithmic transformation it becomes a linear function. This observation shows that the condensed program is equivalent to a linear program. Moreover by suitable choice of the weights it is found that the minimum of the condensed program is the same as the minimum of the original programs. This fact together with the duality theorem of linear programming proves that the maximum of the dual geometric program is equal to the minimum of the primal geometric program. With this result as a basis a new approach to the duality properties of geometric programs is carried through. In particular it is shown that a "duality gap" cannot occur in geometric programming.

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LINEARIZING GEOMETRIC PROGRAMS

1. Introduction

It was shown by Federowicz that by a simple transformation of variables a linear program can be expressed as a geometric program [1, p.265]. In this paper a converse question is treated. It is shown that a geometric program can define an associated family of linear programs. This linearization can be of use both analytically and computationally.

The linearization is achieved by use of the inequality reducing an arithmetic mean to a geometric mean. This operation performed on the polynomial functions defining the geometric program condenses them to monomial functions. But by the Federowicz transformation a monomial geometric program is equivalent to a linear program.

Linearization is employed in this paper as a theoretical tool. It enables the duality theory of linear programming to prove part of the duality theory of geometric programming. In this way it is shown that if the minimization of a primal geometric program yields a finite value then maximization of the dual program is feasible and yields the same value. This result is termed Theorem 4a.

To obtain a complete duality theory it is necessary to show that, conversely, if the maximization of the dual program yields a finite value, then the minimization of the primal program is feasible and yields the same value. This is termed Theorem 4b. The proof of Theorem 4b is given by a reduction [ft] ad absurdum[11] making essential use of Theorem 4a.

The duality theorems proved here are essentially equivalent to those proved previously [1,2]. However, the old proof and this new proof shed light on different facets of the problem.

It will be apparent from Section 4 to follow that the condensation method used to linearize geometric programs suggests various computational applications. However, these questions will not be pursued.

The proofs to follow depend on elementary inequalities and the duality theorem of linear programming. Otherwise the paper is self contained.

2. Definition of posynomial geometric programs.

The primal geometric program is denoted by the letter A and is stated as follows.

Primal Program A. *Seek the minimum value of a function* $g_o(t)$ *subject to the constraints*

(1) $t_1 > 0,\ t_2 > 0,\ \dots,\ t_m > 0$

and

(2) $g_1(t) \leq 1,\ g_2(t) \leq 1,\ \dots,\ g_p(t) \leq 1$.

Here

(3) $g_k(t) = \sum_{i=m_k}^{n_k} c_i t_1^{a_{i1}} t_2^{a_{i2}}, \dots, t_m^{a_{im}}, \quad k = 0,1,\dots,p$

where $n_p = n$ *and*

$$m_o = 1,\quad m_1 = n_o+1,\ m_2 = n_1+1,\ \dots,\ m_p = n_{p-1}+1 .$$

The exponents a_{ij} *are arbitrary real constants, but the coefficients* c_i *are positive constants.*

The functions $g_k(t)$ are termed *posynomials*. If there is a point t which satisfies the constraints (1) and (2) then program A is said to be *consistent*. If A is consistent let

(4) $M_A = \inf g_o(t)$

for points t which satisfy the constraints. Then M_A is termed the infimum of A. Program A is said to have a *finite infimum* if $M_A > 0$.

Associated with the preceding mimimization program is a maximization program termed a *dual geometric program*. The dual program is denoted by the letter B and is stated as follows.

Dual program B. Seek the maximum value of a^ product function

$$(5)\quad v(6) = \prod_{i=1}^{n} \left(\frac{c_i}{6_i}\right)^{6_i} \prod_{k=1}^{p} \lambda_k^{\lambda_k}$$

where

$$(6)\quad \lambda_k = \sum_{i=m_k}^{i=n_k} 6_i$$

with $n_p = n$ and

$m_1 = 1$, $m_2 = n_1 + 1$, $m_3 = n_2 + 1$, ... , $m_p = n_{p-1} + 1$.

The factors c_i are assumed to be positive constants and the vector variable $6 = (6_1, \ldots, 6_n)$ is subject to the linear constraints;

(7) $6_1 \geq 0$, $6_2 \geq 0$, ... , $6_n \geq 0$, (positivity)

(8) $\sum_{i=1}^{n} 6_i = 1$, (normality)

and

(9) $\sum_{i=1}^{n} 6_i a_{ij} = 0$, $j = 1,2,\ldots,m.$ (orthocronality)

Here the coefficients a_{ij} are real constants.

In evaluating the product function $v(6)$ it is to be understood that $x^{x} = x^{-x} = 1$ for $x = 0$. This will make $v(6)$ continuous over its domain of definition. Program B is said to be consistent if there is a point 6 which satisfies the constraints (7),(8), and (9). If program B is consistent let

(10) $M_B = \sup v(6)$

for points 6 which satisfy the constraints. Then M_B is termed the supremum of B. If $M_B < \infty$ then program B is said to have a finite supremum.

3. The geometric inequality.

The inequality stated in the following lemma is termed a *geometric inequality*.

Lemma 1. *If* $u_i > 0$ *and* $\delta_i \geq 0$ *for* $i = 1, \ldots, N$ *then*

$$(1) \quad \left(\sum_{i=1}^{N} u_i\right)^{\lambda} \geq \prod_{i=1}^{N} \left(\frac{u_i}{\delta_i}\right)^{\delta_i} \lambda^{\lambda}$$

where

$$(2a) \quad \lambda = \sum_{i=1}^{N} \delta_i \quad \text{and}$$

$$(2b) \quad (u_i/\delta_i)^{\delta_i} = 1 \quad \text{if} \quad \delta_i = 0 \; .$$

Moreover the inequality becomes an equality if, and only if,

$$(3) \quad \delta_j \, \Sigma_1^N u_i = u_j \Sigma_1^N \delta_i \; , \quad j = 1, 2, \ldots, N \, .$$

Proof: We consider three exclusive cases: (a) all δ_i are positive, (b) some but not all δ_i are positive, and (c) all δ_i are zero.

In the case (a) let "weights" ϵ_i be defined as

$$(4) \quad \epsilon_i = \delta_i/\lambda \quad \text{and} \quad U_i = u_i/\delta_i .$$

Then the classical inequality stating that the weighted arithmetic mean of $U_1, U_2, \ldots, U_N$ is not less than the weighted geometric mean can be written as follows

$$(5) \quad \Sigma_1^N \epsilon_i U_i \geq \pi_1^N \; U_i^{\epsilon_i} \; .$$

Substituting relations (4) into (5) is seen to prove inequality (1). Moreover for positive weights ϵ_i it is known that (5) is an equality if and only if

$$(6) \quad U_1 = U_2 = \ldots = U_N .$$

Clearly this is equivalent to (3). This proves Lemma 1 in case (a).

In case (b) it is seen that if $\delta_j = 0$ then $(u_j/\delta_j)^{\delta_j} = 1$ so the right side of relation (1) does not depend on u_j. However the left side increases with u_j. This fact together with the proof of case (a) shows that (1) is a strict inequality in case (b). Moreover (3) is not satisfied for $i = j$. Thus relation (1) and (3) both hold in case (b)*

In case (c) it follows that (1) becomes the equality $1 = 1$ and (3) becomes $0 = 0$. This proves case (c). Since the three cases are exhaustive the proof of Lemma 1 is complete.

The following result is termed "the main lemma" of geometric programming. It is a consequence of repeated application of the geometric inequality of Lemma 1.

<u>Theorem</u> 1. <u>If</u> t <u>satisfies the constraints of primal program</u> A <u>and</u> δ <u>satisfies the constraints of its dual program</u> B <u>then</u>

(7) $$g_0(t) \geq g_0(t)\prod_{k=1}^{p}[g_k(t)]^{\lambda_k} \geq v(\delta).$$

Moreover, under the same conditions $g_0(t) = v(\delta)$ if, and <u>only if</u>,

(8) $$g_0(t)\delta_i = u_i(t) \quad i = 1, \ldots, n_0 \quad \text{and}$$

(9) $$\delta_i = \lambda_k u_i(t) \qquad i = m_k, \ldots, n_k .$$

<u>Here</u> $k = 1, \ldots, p$ <u>and</u>

(10) $$u_i(t) = c_i t_1^{a_{i1}} \cdots t_m^{a_{im}} .$$

<u>Proof</u>. Then for $k = 0,1, \dots , p$

(11) $g_k = \Sigma_{m_k}^{n_k} u_i$

and by virtue of the geometric lemma

(12) $(g_k)^{\lambda_k} \geq \pi_{m_k}^{n_k} (\frac{u_i}{\delta_i})^{\delta_i} \lambda_k^{\lambda_k}$ $k = 0,1,\dots,p.$

Here $\lambda_o = 1$ and $m_o = 1$. Multiplying these $p + 1$ inequalities together gives

$$g_o \pi_1^p g_k^{\lambda_k} \geq \pi_1^n (\frac{u_i}{\delta_i})^{\delta_i} \pi_1^p \lambda_k^{\lambda_k} .$$

Then by the definition of the dual function $v(\delta)$

(13) $g_o \pi_1^p g_k^{\lambda_k} \geq v(\delta)\ t_1^{D_1}\ t_2^{D_2} \dots t_m^{D_m}$

where

(14) $D_j = \Sigma_1^n \delta_i a_{ij}$ $j = 1, \dots , m$.

Since δ_i satisfies the orthogonality condition it follows that $D_j = 0$. Thus

(15) $g_o \pi_1^p g_k^{\lambda_k} \geq v(\delta)$.

But t satisfies the constraints of A so

(16) $g_k \leq 1$ $k = 1, \dots , p$.

Thus (15) and (16) together prove inequality (1).

Now suppose that the condition (8) of Theorem 1 holds. Then (12) is an equality for $k = 0$ because condition (8) of Theorem 1 is the same as condition (3) of Lemma 1 when $\lambda_o = 1$.

If condition (9) of Theorem 1 holds and $A_k = 0$ then (12) is an equality because then condition (9) implies condition (3) of Lemma 1. If A_k ^ 0 then sum (9) and obtain

$$(17) \quad \Sigma_{m_k}^{n_k} \delta_i = \lambda_k \Sigma_{m_k}^{n_k} u_i$$

This shows that $g_k = 1$ and consequently (3) and (9) are equivalent. Thus (12) is seen to be an equality also in the case A_k ^ 0.

Since (12) is an equality for $k = 0,1, \ldots p$ it follows that (15) is an equality. However if $A_k = 0$ then $g_k = 1$ and if A^ / 0 we have seen that $g_k = 1$. Thus g^ = 1 for $k = 1, \ldots, p$ and this together with the equality (15) give $g_Q = v$. This proves the second part of Theorem. 1.

To prove the third part of Theorem 1 suppose that $g_o = v$. Then (15) and (16) show that actually $g^K = 1$ for $k = 1, \ldots, p$. Moreover all the inequalities (12) must be equalities. Then relation (3) of Lemma 1 holds and since $g^K = 1$ for $k = 1, \ldots, p$ it follows that (3) implies (9). Since $A^o = 1$ it follows that (3) implies (8). Thus it has been shown that if $g^o = v$ then (8) and (9) hold and the proof of Theorem 1 is complete.

~~Corollary~~ 1. ~~Suppose that~~ Sv^*) < 1 ~~f^{or} some value of~~ k ~~and for some point~~ t* ~~satisfying the primal constraints~~. ~~Then the inequality~~

(18) v(6) ^ c

<u>for some positive constant</u> c <u>implies that</u>

$$(19) \quad A_k £ \log[g_o(t^)c''^1] / \log[g_k(t^*)]^{-1}$$

for any 6 satisfying the dual constraint and (18) .

Proof† It is a direct consequence of inequality (7) that

(20) $g_o(t^*)\ [g_k(t^*)]^K\ ;\geq\ v(6)$.

This is equivalent to (19).

Corollary 2. Jj[g(t) = v(6) where t satisfies the primal constraints and 6 satisfies the dual constraints then

(21) M- $\overset{n}{\underset{1}{7T}}$ 6. $^{A.}$ ^{L} = $\overset{P}{\underset{1}{\Pi}}$ A- ^{K} \ $\overset{n}{\underset{1}{V}}$ C.- $^{A.\,1}$

for any A satisfying the dual constraints.

Proof; The equations (8) and (9) are raised to the powers A^ and multiplied. This gives

(22) $g_o^{\,A^{\wedge}}$ $\overset{n}{\underset{1}{ir}}$ 6_l• • $^{A.}$ * = $\underset{l}{IT}$ $A_K^{A.\,K}$ $\overset{n}{\underset{1}{7T}}$ u_x• $^{A.\,X}$

where A_Q = "Ej^0 $\overset{n}{6_{\pm}}$. But $g_Q(t) = M_A$, $A_Q = 1$, and

(23) $\overset{n}{\underset{1}{\Pi}}\ u_1^{A_i}\ =\ \overset{n}{\underset{1}{IT}}\ c_X^{A_i}$.

Thus (22) and (23) prove (21).

Relation (21) is due to Clarence Zener. It can be used to derive the "maximizing equations" [1, p.88]. The maximizing equations are of prime importance for computational work.

Corollary 3. JEf program A and program B are both consistent then

(24) $M_A > \wedge > 0$.

Proof; This follows directly from (7).

4. Condensed Programs.

Proof for duality theorems to follow is based on the notion of a condensed program. A condensed program is equivalent to a linear program. It is denoted by $\bar{A}$ and defined as follows.

Primal Program $\bar{A}$. Given program A and a set of non-negative weights $\epsilon_1, \ldots, \epsilon_n$ such that

(1) $\sum_{m_k}^{n_k} \epsilon_i = 1 \quad k = 0,1,\ldots,p.$

Then the condensed program is obtained by replacing $g_k(t)$ by

(2) $\bar{g}_k(t) = \pi_{m_k}^{n_k} \left(\frac{u_i}{\epsilon_i}\right)^{\epsilon_i}$

where the u_i are the terms of g_k. Thus

(3) $\bar{g}_k(t) = \bar{c}_k \, t_1^{\bar{a}_{k1}} \ldots t_m^{\bar{a}_{km}}$,

(4) $\bar{c}_k = \pi_{m_k}^{n_k} \left(\frac{c_i}{\epsilon_i}\right)^{\epsilon_i}$,

(5) $\bar{a}_{kj} = \sum_{m_k}^{n_k} \epsilon_i \, a_{ij} \qquad k = 0,1,\ldots,p.$

Note that it is a direct consequence of the geometric inequality that

(6) $g_k(t) \geq \bar{g}_k(t)$.

Thus if program A is consistent so also is program $\bar{A}$ and $M_A \geq M_{\bar{A}}$.

In a similar manner a partially condensed program can be defined when at least two terms are condensed into a single term by the geometric inequality. Condensed and partially condensed programs suggest approximation procedures for computation but these questions are not treated.

The posynomials of program $\bar{A}$ have only one term so the dual program $\bar{B}$ can be expressed in the following form.

Dual Program $\bar{B}$. The maximum of a product function

(7) $\bar{v}(\lambda) = \prod_{k=0}^{p} (\bar{c}_k)^{\lambda_k}$

is sought where $\lambda_0, \lambda_1, \ldots, \lambda_p$ are subject to the constraints

(8) $\lambda_k \geq 0 \quad k = 0,1, \ldots, p$

(9) $\lambda_0 = 1$

(10) $\sum_{k=0}^{p} \lambda_k \bar{a}_{kj} = 0 \quad j = 1, \ldots, m$.

It will now be shown that $\bar{A}$ and $\bar{B}$ are equivalent to linear programs.

Make the following one to one transformation:

(11) $z_j = \log t_j \quad j = 1, \ldots, m$.

(12) $G_k = \log \bar{g}_k \quad k = 0,1, \ldots, p$.

(13) $C_k = \log \bar{c}_k \quad k = 0, \ldots, p$.

(14) $v = \log \bar{v}$.

Then program $\bar{A}$ becomes equivalent to the following linear program.

Program $\bar{A}_L$. Seek the minimum of the linear function

(15) $G_0 = \sum \bar{a}_{0j} z_j + C_0$

subject to the constraints

(16) $G_k = \sum \bar{a}_{kj} z_j + C_k \leq 0 \quad k = 1,2, \ldots, p$.

Under the trasnformation, program $\bar{B}$ also becomes equivalent to a linear program.

Program $\bar{B}_L$. *Seek the maximum of the linear function*

(17) $v = c_o + \Sigma_1^p \lambda_k c_k$

subject to the constraints

(18) $\lambda_k \geq 0 \qquad k = 1, \ldots, p$

(19) $\bar{a}_{oj} + \Sigma_1^p \lambda_k \bar{a}_{kj} = 0 \qquad j = 1, \ldots, m .$

It is clear that $\bar{A}_L$ and $\bar{B}_L$ are a pair of dual linear programs.

Lemma 2. *Suppose condensed program* $\bar{A}$ *is consistent. Then* $M_{\bar{A}} > 0$ *if and only if the weights* ϵ_i *satisfy the relations*

(20a) $\epsilon_i \Sigma_{m_k}^{n_k} \delta_j' = \delta_i' \qquad m_k \leq i \leq n_k$

for $k = 0, 1, \ldots, p$ *and for some* δ' *satisfying the constraints of program* B. *Moreover* δ' *can be chosen so as to also satisfy*

(20b) $M_{\bar{A}} = v(\delta')$.

Proof. Suppose first that program $\bar{A}$ has the infimum $M_{\bar{A}} > 0$. It follows by the transformations (11), (12) and (13) that program $\bar{A}_L$ has the infimum $\log M_{\bar{A}}$. Then by the duality theorem of linear programming it follows that there is a point λ' where the program $\bar{B}_L$ has the maximum value $\log M_{\bar{A}}$. Thus by the transformation (14) program $\bar{B}$ has the maximum value $M_{\bar{A}}$. Thus λ' satisfies the constraints of program $\bar{B}$ and

(21) $M_{\bar{A}} = \prod_{k=0}^{p} (\bar{c}_k)^{\lambda_k'}$.

Now substitute in (21) the expression (4) for $\bar{c}_k$ obtaining

$$(22)\quad M_A = \prod_{k=0}^{p} \left[\prod_{i=m_k}^{n_k} \left(\frac{c_i}{\epsilon_i}\right)^{\epsilon_i} \right]^{\lambda_k} = \prod_{1}^{n} \left(\frac{c_i}{\epsilon_i}\right)^{\delta_i'}$$

where δ_i' is defined as

$$(23)\quad \delta_i' = \epsilon_i \lambda_k' \quad \text{for} \quad m_k \le i \le n_k \;.$$

Summing these relations gives

$$(24)\quad \sum \delta_i' = \lambda_k' \;.$$

Multiplying relation (5) by λ_k' and summing gives

$$(25)\quad \sum_0^p \lambda_k' \bar{a}_k = \sum_0^p \lambda_k' \sum \epsilon_i a_i = \sum^n \delta_i' a_i \;.$$

It follows from (24) and (25) together with (8), (9) and (10) that δ' satisfies the constraints of program B, moreover (23) and (24) prove (20a). Substituting (23) in (22) gives

$$(26)\quad M_A = \prod_{k=0}^{p} \prod_{m_k}^{n_k} \left(\frac{c_i \lambda_k'}{\delta_i'}\right)^{\delta_i'} \;.$$

This may be written in the form

$$(27)\quad M_A = \prod_{1}^{n} \left(\frac{c_i}{\delta_i'}\right)^{\delta_i'} \prod_{1}^{p} (\lambda_k')^{\lambda_k'} \;.$$

The right side is the dual function $v(\delta')$. This proves (20b).

Next suppose δ' is given to satisfy the constraints of program B. Let λ_k' be defined by (24). If $\lambda_k' \neq 0$ let ϵ_i be defined by (23). If $\lambda_k = 0$ let ϵ_i be arbitrary. Then relations (20a) are satisfied. Clearly (8) and (9) hold and (25) proves (10) so A satisfies the constraints of B. Since B and A are both consistent Corollary 3 gives $M_A > 0$. Q. E. D.

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5. Superconsistent programs.

Program A is defined to be superconsistent if

(1) $g_k(t^*) < 1 \quad k = 1,2, \ldots , p$

for at least one point t^*.

Theorem 2. Suppose program A is superconsistent and has a minimum $M_A > 0$ at a point t'. Then in the condensed program $\overline{A}$ let the weights be chosen for $m_k \leq i \leq n_k$ as

(2a) $\epsilon_i = u_i(t')/\Sigma_{m_k}^{n_k} u_i(t') \quad k = 0,1, \ldots , p$.

Then program $\overline{A}$ also has a minimum at t'. Moreover $M_{\overline{A}} = M_A$.

Proof. It follows from (2a) and the definition of $\overline{g}_k$ that

(2b) $\overline{g}_k(t') = g_k(t')$.

Suppose that, contrary to the statement of the theorem, there is a point t^o such that

(3) $\overline{g}_o(t^o) < M_A, \quad \overline{g}_k(t^o) \leq 1, \quad k = 1, \ldots, p$.

Also since program A is superconsistent there is a point t^* such that

(4) $1 > g_k(t^*) \geq \overline{g}_k(t^*), \quad k = 1, \ldots , p$.

Let $\alpha > 0$, $\beta > 0$, and $\alpha+\beta = 1$. Then let

(5) $t'' = (t^o)^\alpha (t^*)^\beta$.

Since $\overline{g}_k$ has only one term

(6) $\overline{g}_k(t'') = [\overline{g}_k(t^o)]^\alpha [\overline{g}_k(t^*)]^\beta$.

It follows from (3), (4) and (6) that if p is sufficiently small

(7) $i_Q(t'') < M_A$, $i_k(t'') < 1$, $k = 1, \ldots, p$

Next choose h_j so that

(8) $t''_j = t_j e^{h_j}$.

Then define t_j in terms of s as

(8) $t_j = t_j e^{s h_j}$ $0 \leq s \leq 1$.

Thus

(9) $\log \bar{g}_k(t) = \log \bar{g}_k(t^*) + s \sum I_k . h_j .$

Since $g_0(t') = M^A$ and $g_0(t'') < M^A$ this shows that

(10) $\sum a_{oj} h_j < 0$.

Likewise if $\bar{g}_k(t') = 1$ then $\bar{g}_k(t'') < 1$ and

(11) $\sum \bar{a}_{kj} h_j < 0$.

Now note that

(12) $\frac{dg_k}{ds} = \sum_{m_k}^{n_k} u_i \sum_1^m a_{ij} h_j$

Set s = 0 and employ the definition of p_i and $\bar{a}_{ij}$ to obtain

(13) $\frac{1}{g_k} \frac{dg_k}{ds}\Big|_{s=0} = \sum \ldots a_{ij} h_j = \sum \bar{a}_{kj} h_j .$

It now follows that if k = 0 or if $\bar{g}_k(t') = 1$ then

(14) $\frac{dg_k}{ds} < 0$.

Hence if (14) holds and s is small and positive

(15) $g_k(t) < g_k(f)$ •

Consequently

(16) $g_Q(t) < M_A$ and

(17) $g_k(t) < 1$.

But if g, $(t^{(})$ < 1 then (17) also holds if s is small and positive. Thus (17) holds for k = 1, ... , p. Thus (16) and (17) give a contradiction showing that the assumed relation (3) is false. This proves $M_A = M^A$.

Theorem 3. Suppose program A jjs superconsistent and has an infimum $M_A > 0$. Then the dual program B JLS consistent and has a maximum M_B at a point 6^f. Moreover $M_A = M_B$.

Proof: First suppose program A has a minimum point $t^!$# Hence $g_O(t^1) = M_A > 0$. Then Theorem 2 gives that $M_A = M_A$. Thus according to Lemma 2 there is a 6^f such that

(18) $g_o(t^f) = M_B = v(6<)$.

This together with inequality (7) of Theorem 1 shows that $v(6^f) = M^A$. Hence Theorem 3 is true if program A has a minimum.

If program A does not have a minimum point consider a modified program A_n in which the following 2m additional constraints ar6 imposed:

(19) $g_{p+j}(t) = ht_j \leq 1$ j = 1, ... , m ,

(20) $g_{p+nvfj}(t) = ht_j^{-1} \leq 1$ j = l, ... , m .

Then if h is a sufficiently small positive number it is clear that A_n also is superoonsistent. Moreover the additional constraints insure that the set of consistent points is compact. Consequently program A_n has a minimum point t^{1*} and by what has just been proved

(21) $g_Q(t'') s v_h(6»)$

for a point 6^n satisfying the constraints of program B_n. The dual function is

(22) $$v_{11}(6) = v(6) \prod_{n+1}^{n+2m} h^{6_x} .$$

The constraints of B_n are

(23) $6_i \geq 0 \quad i = 1, \ldots, n + 2m$

(24) $\sum^n F_1^\circ 6_\pm = 1$

(25) $j; S_i a_{ij} + 6_{n+j} - 6_{n+m+j} = 0 \quad j=1,2,\ldots,m.$

Now let $h \to 0$ then $g_o(t^n) \to M_r$. Then because of (21) we have $v_u(6^n) \to M_A > 0$- Since program A_n has a superconsistent point t^* it follows that Corollary 1 applies to each of the functions g_k so there is a constant $d \geq 1$ such that as $h \to 0$

(26) $A_k \leq d \quad k = 1,2, \ldots , p + 2m.$

Also $A_o =1$ so as $h \to 0$

(27) $6_i'' \leq d \quad i = 1,2, \ldots , n + 2m.$

For $k > p$ we see that $g_k(t^*) = ht^* \to 0$ as $h \to 0$. Thus it follows from Corollary 1 that

(28) $\delta_i'' \to 0$ for $i = n+1, \ldots, n + 2m$.

By virtue of relation (27) the points δ_i'' are confined to a compact region. Thus as $h \to 0$ there is at least one limit point δ_i'. Since $\delta_i' = 0$ for $i > n$ it is seen from (23), (24) and (25) that δ_i' satisfies the constraints of program B. Also if $h < 1$

(29) $v_h(\delta'') \leq v(\delta'')$.

Taking the limit of this relation gives

(30) $M_A \leq v(\delta')$

but Theorem 1 shows that $M_A \geq v(\delta')$. This is seen to complete the proof of Theorem 3.

6. Subconsistent programs.

In Theorem 4 to follow the condition of consistency is replaced by a weaker condition termed subconsistency. Program A is termed subconsistent if the program A^{θ} defined by the functions

(1) $g_k^{\theta}(t) = \theta g_k(t) \qquad k = 0,1, \ldots , p$

is consistent for any constant θ in the range $0 < \theta < 1$. Thus suppose program A is subconsistent and let $M_{A^{\theta}}$ be the infimum of program A^{θ}. Note that $M_{A^{\theta}}$ is an increasing function of θ so as $\theta \to 1$ let the subinfimum be defined as

(2) $\mathfrak{m}_A = \lim_{\theta \to 1} M_{A^{\theta}}$

with the understanding that $\mathfrak{m}_A$ can be $+\infty$ when the right side is unbounded. Finite $\mathfrak{m}_A$ means $0 < \mathfrak{m}_A < \infty$.

Theorem 4a. *If program A is subconsistent and has a finite subinfimum $\mathfrak{m}_A$ then program B is consistent and has a finite supremum. Moreover $\mathfrak{m}_A = M_B$.*

Proof. Clearly the programs A^{θ} are all superconsistent. Moreover if θ is close to 1 then $M_{A^{\theta}} > 0$. Thus by Theorem 3 we have

(3) $M_{A^{\theta}} = M_{B^{\theta}} = v_{\theta}(\delta^{\theta})$.

Where v_{θ} is the dual function for B^{θ}. Clearly

(4) $v_{\theta}(\delta^{\theta}) = v(\delta^{\theta})\theta^{\sum_1^n \delta_i^{\theta}} \leq v(\delta^{\theta})$.

If δ^{θ} satisfies the constraints of B^{θ} it also satisfies the constraints of B so (3) and (4) give

(5) $M_{A}^{p} \geq v(\theta^{\circ}) \leq M_B$.

Allowing θ to approach zero give

(6) $m_A \leq M_B$.

On the other hand if θ satisfies the constraints of program B if follows from Theorem 1 that

(7) $M_{A}^{\theta} \geq v(\theta) e^{TL\ \theta_x}$.

Allowing θ to approach zero with θ fixed gives

(8) $m_A \geq v(\theta)$.

Taking the supremum of the right side gives

(9) $m_A \geq M_B$ •

Then (6) and (9) together prove Theorem 4a.

Theorem 4b. If program B is consistent and has a finite supremum M_B then program A is subconsistent and has a finite subinfimum m_A. Moreover $m_A = M_B$.

Proof. If program A is not subconsistent then it follows that there is an integer $q \leq p$ such that the set of inequalites

(10) $g_1 \leq 1$, ... , $g_{q-1} \leq 1$

is subconsistent but the constraints (10) imply that there is a constant D such that

(11) $g_q > D > 1$.

Thus let the program A^q be the minimization of g_q subject to the constraints (10). Then the dual program B^q is the maximizing of the dual function

(12) $W(\Delta) = \pi_{m_1}^{n_q} \left(\frac{c_i}{\Delta_i}\right)^{\Delta_i} \Lambda_1^{\Lambda_1} \cdots \Lambda_q^{\Lambda_q}$

where $\Lambda_1 = \Sigma_{m_1}^{n_1} \Delta_i$ etc. The constraints are

(13) $\Sigma_{m_1}^{n_q} \Delta_i \, a_{ij} = 0 \quad j = 1, \ldots, m$

(14) $\Delta_i \geq 0$ and $\Lambda_q = 1.$

Then by virtue of Theorem 4a it is possible to choose Δ so that

(15) $W(\Delta) > D$.

Let N be a positive integer then it follows from the definition of W that

(16) $W(N\Delta) = [W(\Delta)]^N$

Let $\Delta_i = 0$ for $i < m_1$ and for $i > n_q$ then it is seen that

(17) $\delta_i^{(N)} = \delta_i + N\Delta_i$

satisfies the constraints of dual program B if δ_i does. Clearly

(18) $\frac{V(\delta + N\Delta)}{V(\delta)} = \frac{W(\delta + N\Delta)}{W(\delta)}$ and

(19) $W(\delta + N\Delta) = [W(\Delta + \delta/N)]^N$.

By continuity of the function W it follows that there is an N_o such that for $N > N_o$

(20) $W(\Delta + \delta/N) > D$.

This inequality together with identities (18) and (19) give

(21) $v(\delta + NA) \geq \frac{v(\delta)}{W(\delta)} D^{N}$.

Hence v is unbounded as $N \to \infty$. This contradiction proves that program A is subconsistent.

Program A is superconsistent and program B is consistent. Thus by Theorem 1

(22) $M_{A}^{A} \wedge M_{B}^{A} > 0$.

Then by Theorem 3

(23) $M_{A}^{\theta} = v(\delta^{\theta})\theta^{\sum^{n} \delta_{x}^{\theta}}$

where δ^{θ} is a maximizing vector for B^{θ}. Since $0 < 1$ the equality (23) gives the inequality

(24) $M_{A}^{\theta} \leq v(\delta^{\theta}) \leq M_{A}$.

This shows that $M_{A^{\theta}}^{n}$ has a finite limit as $\theta \to 1$. In other words A has a finite subinfimum. Theorem 4b then follows directly from Theorem 4a.

7. The question of a duality gap.

It is easy to construct examples of primal programs which are subconsistent but which are not consistent. On the other hand if program A is consistent it is also subconsistent and clearly

(1) $m_A \leq M_A$.

It is worth noting that there are examples in programming theory where the relation corresponding to (1) can be a strict inequality [3], [4]. This has been termed a "duality gap". That a duality gap can never appear in geometric programming is now to be shown.

Theorem 5. If program A is consistent then the infimum M_A and the subinfimum m_A are equal.

Proof. If A is consistent then relation (1) holds. As a consequence the subinfimum can not be infinite. Then by the definition of the subinfimum there must exist a sequence of points $t = T(r)$ such that as $r = 1, 2, \ldots, \infty$.

(2) $g_0(T) \to m_A$,

(3) $\limsup g_k(T) \leq 1 \quad k = 1, \ldots, p$.

Since the terms $u_i(T)$ are uniformly bounded it may be supposed that they have limits e_i for a suitable subsequence. Then the following lemma applies.

Lemma 3. Suppose the terms u_i have limits e_i as t runs through a sequence $t = T(r)$. Then there is another sequence $t = T'(r)$ such that as $r \to \infty$

(4) $u_i(T') \to e_i$ if $e_i \neq 0$,

(5) $u_i(T') \to 0$ if $e_i = 0$.

Proof. Linear functions $U_i(Z)$ are defined by

$$(6) \quad u_i = \log\left(\frac{u_i}{c_i}\right) = \sum_j a_{ij} Z_j$$

where $Z_j = \log T_j$. Thus

(7) $U_i(Z) \to E_i = \log(e_i/c_i)$ if $e_i \neq 0$

(8) $U_i(Z) \to -\infty$ if $e_i = 0$.

Let N be the set of functions $U_1, \ldots, U_n$. Let P be the subset corresponding to (7) and let Q be the subset corresponding to (8). Let p be a maximal linearly independent subset of P. Let q be a maximal subset of Q such that $M = p \cup q$ is linearly independent. Thus assigning values of U_j in the set M determines all U_i in the set N. Moreover for certain b_{ij}

(9) $U_i = \sum_p b_{ij} U_j$ U_i in P

(10) $U_i = \sum b_{ij} U_j + \sum b_{ij} U_j$ U_i in Q

Let $U_j' = E_j$ in p, $U_j' = U_j(Z)$ in q. Then (9) shows that

(11) $U_i' = E_i$ in P .

Also (7), (8), (9) and (10) show that $U_i' - U_i$ is bounded. Thus

(12) $U_i' \to -\infty$ in Q .

However the U variables are dependent on the Z variables so there is a sequence Z' such that $U_i' = U_i(Z')$ in N. Then by taking $T' = \exp(Z')$ it is seen that (11) and (12) prove (4) and (5).

Lemma 4. *Let* $G(t)$ *be a posynomial and let* $t_j' > 0$ *and* $t_j^* > 0$. *Let*

(13) $t_j'' = (t_j')^{\alpha}(t_j^*)^{\beta} \quad j = 1, \ldots ; m$

where α *and* β *are positive constants such that* $\alpha + \beta = 1$ *then*

(14) $G(t'') \leq [G(t')]^{\alpha}[G(t^*)]^{\beta}$.

Proof. Clearly

(15) $G(t'') = \Sigma[u_i(t')]^{\alpha}[u_i(t^*)]^{\beta}$.

Then apply Holder's inequality with the standard terminology $1/p+1/q = 1$. Take $p = 1/\alpha$ and $q = 1/\beta$ and the proof of (14) follows.

Returning to the proof of Theorem 5 let the sequence T define a new sequence T' as in Lemma 3. Then given an $\epsilon > 0$ there exists an r_o such that

(16) $g_o(T') < m_A + \epsilon/2, \quad r \geq r_o$.

Let t^* be a point which satisfies the constraints of program A and let $T'' = (T')^{\alpha}(t^*)^{\beta}$. Then by use of Lemma 4

(17) $g_o(T'') \leq [g_o(T')]^{\alpha}[g_o(t^*)]^{\beta}$.

It is clear therefore, that if β is sufficiently small

(18) $g_o(T'') \leq m_A + \epsilon , \quad r \geq r_o$.

Next consider the constraint function g_k. First suppose g_k contains no term u_i for which $e_i = 0$ then by Lemma 4

(19) $g_k(T'') \leq [g_k(T')]^{\alpha} [g_k(t^*)]^{\beta} \leq 1$.

If g_k contains terms for which $e_i = 0$ let

(20) $G_k(t) = g_k(t)$ - (terms for which $e_i = 0$).

Then by Lemma 4

(21) $G_k(T'') \leq [G_k(T')]^{\alpha}[G_k(t^*)]^{\beta}$.

But $G_k(T') \leq 1$ and $g_k(t^*) \leq 1$. But since some terms of g_k have been deleted $G_k(t^*) < 1$. Hence there exists a constant $h > 0$ such that

(22) $G_k(T'') \leq 1 - h$.

But $g_k(T'') - G_k(T'') \to 0$ as $r \to \infty$. Thus there is an r_k such that

(23) $g_k(T'') < 1$ for $r \geq r_k$.

Let $R = \max r_k$ for $k = 0,1, \ldots, p$. Then (19) and (23) show that T'' satisfies the constraints if $r \geq R$. Moreover (18) holds and since ϵ is arbitrary it follows that

(24) $M_A \leq m_A$.

Of course this implies $M_A = m_A$ and the proof is complete.

<u>Theorem</u> 6. <u>If program</u> A <u>is consistent and has a finite infimum</u> M_A <u>then program</u> B <u>is consistent and has a finite infimum</u> M_B. <u>Moreover</u> $M_A = M_B$.

<u>Proof</u>. This is a direct corollary of Theorem 4a and Theorem 5.

It is now seen that Theorems 4 and 6 form a rather complete duality theory for geometric programming.

References

[1] R. J. Duffin, E. L. Peterson, and C. Zener, Geometric Programming, John Wiley, New York, 1967.

[2] R. J. Duffin and E. L. Peterson, Duality theory for geometric programming, SIAM Jour. Appl. Math. 14 (1966) 1307-1349 .

[3] R. J. Duffin and L. A. Karlovitz, An infinite program with a, duality gap, Management Science JL9 (1965) 122-134.

[4] R. J. Duffin, An orthogonality theorem of Dines relating to moment problems and linear programming, Jour, of Combinational Theory 2, (1967) 1-26.