STABLE MAPS AND SCHWARTZ MAPS

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I) Introduction

In the present paper M and N will denote two von Neumann Algebras where $N \subset M$. If A is any von Neumann Algebra, A' will denote the commutant of A. N^c will denote the relative commutant of N in M. i.e. $N^c = N' \cap M$. $U(N)$ will denote all unitary operators of N. Let G be a group of unitaries of M. Let ϕ be a linear map of M into M. ϕ is called G-stable if $\phi(U X U^{-1}) = \phi(X)$ for all X in M and all U in G. $S(G,M)$ will denote all Schwartz maps which are G-stable. The purpose of this paper is to study the existence and properties of G-stable expectations. The main results contained here are:

Theorem 1: Let Tr be a faithful, semi-finite trace on M. Let L be a von Neumann subalgebra of M such that Tr restricted to L is semi-finite. Then there exists a normal, faithful, $U(L^c)$ stable expectation ϕ of M on L such that $Tr(A \phi(X)) = Tr(A X)$ for all X in M and all A in L for which $Tr |A| < \infty$.

Theorem 2: Suppose M has a faithful, normal, semi-finite trace, call it Tr. Suppose $S(G,M)$ is sufficiently large, then there exists a faithful, normal $U(N^{cc})$ stable expectation of M on N^c.

As corollary to the above theorem, it follows that with the hypothesis of Theorem 2, N if finite, N^c can not be purely infinite. Moreover if M is of type I so is N^c. Another corollary to Theorem 2 is that if $S(G, L(h))$ has sufficiently many maps, then the von Neumann algebra N generated by G is atomic.

Next a notion of equivalence of two unitary groups will be defined. Two groups of unitary operators are equivalent if they generate the same von Neumann algebra.

Theorem 3: Assume $S(G,L(h))$ contains a normal map, then G is equivalent to a countable direct sum of finite groups. The next result is a sort of converse to Theorem 3.

Theorem 4: If G has the property (F), then G is a countable direct sum of finite groups and $S(G,M)$ has sufficiently many maps.

A corollary to Theorem 3 is that if N is a finite atomic von Neumann algebra, then N is generated by a direct sum of finite unitary groups.

Next uniqueness of expectations of certain type will be considered. The main result of this section is:

Theorem 5: Assume that

(1) $N^c \subset N$

(2) N is finite

(3) M is semi-finite

Then there exists at most one normal expectation ϕ of M on N.

2) Preliminaries

Definition: Let ϕ be a map of M into N which preserves the identity. Assume that ϕ is a positive linear map and that $\phi(AX) = A\phi(X)$ for all A in N and X in M. ϕ will then be called an expectation of M in N.

It is trivial to see that ϕ is onto N and that ϕ is a bounded map. The notion of expectations in von Neumann algebras was studied in [2], [7], and [9].

Definition: Let ϕ be an expectation of M on N, ϕ is called normal if $\phi(\text{Sup } A_\alpha) = \text{Sup } \phi(A_\alpha)$ for any increasing net of uniformly bounded self adjoint operators.

ϕ is called faithful if given a positive operator A such that $\phi(A) = 0$ then $A = 0$.

Let ϕ_α be a set of expectations of M onto N. The set ϕ_α is called complete if given a positive operator A such that $\phi_\alpha(A) = 0$ then $A = 0$.

Definition: Let G be a subgroup of $U(M)$. By a Schwartz map relative to (G,M) one means a linear map of M into itself such that

(1) $P(X) = U\,P(X)\,U^{-1}$ for all U in G and all X in M

(2) $P(X)$ is in $C_G[X]$ where $C_G[X]$ denotes the weak closure of the convex hull generated by elements of the type $U\,X\,U^{-1}$ as U ranges over G.

For more information on Schwartz maps see [6].

$S(G,M)$ will denote all Schwartz maps relative to (G,M) which are G-stable, i.e. $P(X) = P(V\,X\,V^{-1})$ for all V in G. $S(G,M)$ will be called sufficient if for any positive operator X in M such that $P(X) = 0$ for all P in $S(g,M)$ then $X = 0$.

Definition: A group G is said to be amenable as a discrete group if there exists a finitely additive probability measure μ on the field of all subsets of G such that $\mu(xE) = \mu(E)$. For more information on amenable groups see [4] and [5].

3) Stable Maps and Schwartz Maps

Lemma 1: If there exists a complete set of U-stable expectations of M on N then U is in N^c.

Proof: Let V be a unitary of N. Let ϕ_α be a complete set of U-stable expectations, then $\phi_\alpha(UVU^{-1}V^{-1}) = \phi_\alpha(VU^{-1}V^{-1}U) = V\phi_\alpha(U^{-1}V^{-1}U)$. This by Stability. Also $V\phi_\alpha(V^{-1}) = VV^{-1} = I$.

Let $W = UVU^{-1}V^{-1}$. Then $\phi_\alpha[(W - I)^*(W - I)] = 0$. By completeness $W = I$ or $UV = VU$. So U is in N^c.

<u>Lemma 2</u>: A normal $U(N)$ stable expectation of M on N^c is faithful.

Proof: Let ϕ be the expectation. Let $I = \{A/A \in M, \phi(A * A) = 0\}$ clearly as $\phi[(XA)^* (XA)] \leq ||X||^2 \phi(A^*A) = 0$ and $(A + B)^* (A + B) \leq 2(A^* A + B^* B)$. I is a left ideal.

Now to show that I is ultra-weakly closed. The ultra weak closure of I coincides with its ultra-strong closure. Let X_α be a net in I converging ultra strongly to X, then $(X_\alpha - X)^* (X_\alpha - X)$ converges ultra weakly to 0. Hence $\phi [(X_\alpha - X)^* (X_\alpha - X)]$ converges to 0 ultra-weakly (normality). As $\phi(X_\alpha - X)^* \phi(X_\alpha - X) \leq \phi (X_\alpha - X)^* (X_\alpha - X)$ it follows that $\phi(X_\alpha)$ converges to $\phi(X)$. $X^* X_\alpha$ and $X_\alpha X^*$ have the same limit so $\phi(X^* X) = 0$. Hence I is a left ultra-weakly closed ideal. So there exists a unique projection E in M such that $I = \{T/TE = T\}$. $U T U^{-1} \in I$ for all $U \in U(N)$ by stability. So $U E U^{-1} = E$. So $E \in N^c$. So $E = \phi(E) = 0$. So if $\phi(X^* X) = 0$ then $X = XE = 0$, so ϕ is faithful.

Now let G be a subgroup of $U(M)$. Let N be the von Neumann algebra generated by G.

<u>Lemma 3</u>: A Schwartz map relative to (G,M) is an expectation onto N^c

Proof: Let P be the Schwartz map. As $P(X)$ commutes with all unitaries of G, $P(X)$ is in N^c. Now if A is in N^c, $C_G[A]$ reduces to the element A. So $P(A) = A$. So $P^2 = P$. N^c is hence the range of P and $P(I) = I$. Now to show that $||P|| \leq 1$. Let $T = \sum_{i=1}^{n} \alpha_i U_i AU_i^{-1}$ where $\alpha_i \geq 0$ and $\Sigma \alpha_i = 1$. Then

$||T|| \leq ||A||$. Because $P(A)$ is in $C_G[A]$ this means that there exists a net T_α of the same form as T such that T_α converges strongly to $P(A)$. Let X be a vector of norm one. $||T_\alpha X||$ converges to $||P(A) X||$ but $||T_\alpha X|| \leq ||A||$. So $||P(A)|| \leq ||A||$. By a result of J. Tomiyama [7], this implies that P is an expectation.

Lemma 4: If G is amenable, $S(G,M)$ is non void.

Proof: Let λ be a mean. Let ξ and η be 2 vectors. Considering U as the variable, $\lambda(U^{-1} X U \xi, \eta)$ is a bounded hermition form. By Riez Lemma there exists an operator E_λ such that $\lambda(U^{-1}XU\xi, \eta) = (E_\lambda(X)\xi, \eta)$. It was shown in [1] that E_λ is in $S(G,M)$.

Lemma 5: Let M be finite and countable decomposable, let G be any subgroup of $U(N)$, then $S(G,M)$ is non void. (In particular if N is any von Neumann subalgebra of M, then $S(U(N),M)$ is non void).

Proof: Let Tr be a faithful, normal, finite trace on M[3]. By finiteness there exists a faithful, normal expectation ϕ of M and N^c such that $Tr(XB) = Tr(\phi(X)B)$ for all X in M and all B in N^c. Hence $\phi(V X V^{-1}) = \phi(X)$ for all X in M and all V in $U(N^{cc}) \supset U(N)$. Now to show $\phi(X)$ is in $C_G[X]$. $C_G[X]$ intersects N^c [3]. Let T be in $C_G[X] \cap N^c$ then by normality $T = \phi(T) = \phi(X)$. Hence ϕ is in $S(G,M)$.

Let G be a subgroup of $U(M)$. Let N be the von Neumann algebra generated by G.

Lemma 6: If $S(G,M)$ contains a normal map ϕ, then $S(G,M)$ reduces to ϕ and so does $S(U(N) M)$. Moreover $C_G[X]$ intersects N^c in just one point.

Proof: Let T be in $C_G[X]$, by normality $\phi(T) = \phi(X)$. Now let T be in $C_G[X] \cap N^c$. Then $T = \phi(T)$ by Lemma 3. So T is the unique point in $C_G[X] \cap N^c$. By normality ϕ is $U(N)$ stable, so $S(U(N),M) = \phi$.

Lemma 7: Let Tr be a faithful, normal, semi-finite trace on M. Let G be a subgroup of $U(M)$ and N the von Neumann algebra generated by G. Suppose $S(G,M)$ is sufficient, then the restriction of Tr to N^c is semi-finite.

Proof: In this proof the notation of [3] will be used. Let $\mathfrak{m}$ be the ideal whose positive part consists of positive operators A such that $\mathrm{Tr}\, A < \infty$. Consider $\mathfrak{m}^{1/2}$. If A is in $\mathfrak{m}^{1/2}$, $C_G[A] \subset \mathfrak{m}^{1/2}$ and $C_G[A] \cap N^c$ is non void [3]. Let S be a positive operator in N^c, $S \neq 0$. To show that there exists $S_1 \neq 0$, $S_1 \leq S$ where S_1 is a positive operator of $N^c \cap M$. Let A be in $\mathfrak{m}$ such that $0 \leq A \leq I$. Let P_α be in $S(G,M)$. Then $S \geq \sqrt{S}\, P_\alpha(A)\sqrt{S} = P_\alpha(\sqrt{S}\, A\sqrt{S})$. A can be picked such that $\sqrt{S}\, A\, \sqrt{S} \neq 0$ or else $A\sqrt{S} = 0$ for all A positive in $\mathfrak{m}$. By semi-finiteness there would exist a net A_α converging weakly to I so $I\sqrt{S} = 0$. So $S = 0$, a contradiction. Pick A then so that $\sqrt{S}\, A\sqrt{S} \neq 0$. Let $H = \sqrt{S}\, A\sqrt{S}$ then H is in $\mathfrak{m}^{1/2}$. $P_\alpha(\sqrt{H})$ is in ${}^{1/2} \cap N^c$. So $[P_\alpha(\sqrt{H})]^2$ is in $\mathfrak{m} \cap N^c$. So $(P_\alpha(\sqrt{H}))^2 \leq P_\alpha(H) \leq S$. By sufficiency, there exists an α_o such that $P_{\alpha_o}(\sqrt{H}) \neq 0$. Choose $S_1 = (P_{\alpha_o}(\sqrt{H}))^2$.

Theorem 1: Let Tr be a faithful, semi-finite trace of M. Let N be a von Neumann subalgebra of M and assume that the restriction of Tr to N is semi-finite, then there exists a normal, faithful $U(N^c)$-stable expectation ϕ of M on N such that $\mathrm{Tr}(A\,\phi(X)) = \mathrm{Tr}(A\,X)$

for all X in M and all A in N such that $Tr|A| < \infty$.

Proof: Using the notations of the above lemma let A and B be in $\mathfrak{m}^{1/2} \cap N$ (that intersection is non void), define $(A,B) = Tr(AB^*)$. Choose X positive in M and define $A,B = Tr(A B^* X)$. [,] is a bounded hermitian form respectively to (,). Let k be the completion of $\mathfrak{m}^{1/2}$ under (,). By Riez lemma there exists an operator $\phi(X)$ in L(k) such that $[A,B] = (\phi(X)\ (A),B)$. Now: Let R_c denote the right multiplication by C, where C is in $\mathfrak{m}^{1/2}$. $(R_c\ \phi(X)\ (A),B) = (\phi(X)\ (A),BC^*) = [A,BC^*] = Tr(ACB^*X)\ (\phi(X)\ R_c(A),B) = [R_c(A)\ B] = [AC,B] = Tr(ACB^*X)$ so $R_c\ \phi(X) = \phi(X)R_c$.

By the commutation theorem [3] this implies that $\phi(X)\ (A)$ is a left multiplication by an element of N. Call that element $\phi(X)$. Then $Tr(AB^*X) = Tr(\phi(X)AB^*) = Tr(AB^*\phi(X))$ for all A and B in $\mathfrak{m}^{1/2} \cap N$ and all X positive in M. ϕ can then be extended in the obvious fashion to all of M. As Tr is faithful, normal, it is easy to see that ϕ is faithful, normal, and $U(N^c)$ stable. For example to check that ϕ is $U(N^c)$ stable; let V be in $U(N^c)$, let A be in N, then:
$Tr(A\ \phi\ (V\ X\ V^{-1})) = Tr(A\ V\ X\ V^{-1}) = Tr(V\ A\ X\ V^{-1}) = Tr(A\ X) = Tr(A\ \phi\ (X))$. So $Tr[A(\phi V\ X\ V^{-1}) \cdot \phi(X))] = 0$ for all A in $N \cap \mathfrak{m}$. Since Tr is semi-finite on N, let P_α be a family of orthogonal projections of N such that $Tr\ P_\alpha < \infty$ and $\Sigma\ P_\alpha = I$. Make $A = (\phi(V\ X\ V^{-1}) - \phi(X))^*\ P_\alpha$. One has $P_\alpha(\phi\ (V\ X\ V^{-1}) - \phi(X)) = 0$ for all α, i.e. $\phi(V\ X\ V^{-1}) = \phi(X)$.

<u>Theorem 2</u>: Suppose M has a faithful, normal, semi-finite trace Tr. Suppose S(G,M) is sufficient, then there exists a faithful, normal $U(N^{cc})$ stable expectation of M on N^c. (N is the algebra generated by G).

Proof: By Lemma 7 the restriction of Tr to N^c is semi-finite. By Theorem 1 there exists a normal, faithful, $U(N^{cc})$ stable expectation ϕ of M on N^c such that $Tr(A X) = Tr(A \phi(X))$ for all A in N^c such that $Tr|A| < \infty$. Now ϕ is in S(G,M). Indeed ϕ is G-stable and if P is in S(G,M) then $\phi(P(X)) = P(X)$ (as ϕ is the identity on N^c). By normality $\phi(P(X)) = \phi(X)$. So $P = \phi$. Hence ϕ is a normal, faithful, $U(N^{cc})$ stable expectation of M on N^c by Lemma 3.

The above theorem says that if there is a sufficient number of G-stable expectations of M on N^c, there is a faithful, normal one which in fact is more than G-stable it is $U(N^{cc})$ stable.

Corollary 1: With the above hypothesis N^{cc} if finite.

Proof: By the above theorem $s(U(N^{cc}),M)$ is non void. Let P be in $S(U(N^{cc}),M)$. Let A be in N^{cc}, let C(A) be the norm closure of the convex hull K_A of points of the form $U A U^{-1}$ as U ranges over $U(N^{cc})$. Consider $C(A) \cap Z$ where Z is the center of N^{cc}. $C(A) \cap Z$ is non void [3]. By [3] it is sufficient to show that $C(A) \cap Z$ reduces to one point. P is constant on K_A hence on C(A). Let T_1 and T_2 be in $C(A) \cap Z$, then $T_1 = P(T_1) = P(T_2) = T_2$, so N^{cc} is finite. In particular N is finite.

Corollary 2: With the above hypothesis N^c can not be pure infinite.

Proof: In [7] J. Tomiyama proved that if π is an expectation from a semi-finite algebra M onto a purely infinite subalgebra A, then π is always singular, i.e. π is not normal. Since there exists a normal expectation from M on N^c, N^c is not purely infinite.

Corollary 3: With the above hypothesis if M is of type I, so is N^c.

Proof: In [7] it has been shown that if there exists an expectation from M of type I to a subalgebra of type II, that expectation is not normal. By the above corollary N^c has no part of type III and hence no part II or III are present, so N^c is of type I.

Let G be a subgroup of $U(M)$. Let N be generated by G.

Corollary 4: Let M be a countably decomposable von Neumann algebra and consider the following conditions:

(1) N is finite and there exists a faithful, normal expectation ϕ of M on N

(2) There exists a faithful, normal state ρ of M such that $\rho(U X U^{-1}) = \rho(X)$ for all U in G

(3) There exists a faithful, normal expectation ψ of M on N^c such that $\psi(V X V^{-1}) = \psi(X)$ for all V in $U(N)$

(4) $S(G,M)$ is sufficient and M has a faithful, semi-finite normal trace Tr.

Then (1) and (2) are equivalent. If $S(G,M)$ is non void, (2) and (3) are equivalent. Finally (4) always implies (3).

Proof: Assume (1), then there exists a faithful, normal finite trace λ on N. Let $r(X) = \lambda[\phi(X)]$. Clearly r is faithful, normal and bounded. Let U be in G, then $r(U X U^{-1}) = \lambda\phi(U X U^{-1}) = \lambda\phi(X) = r(X)$. Normalizing r, (2) is established.

Assume (2). By a classical Hilbert algebra argument one can show that there exists an expectation ϕ such that $\rho(AX) = \rho(A\,\phi(X))$ for all A in N and all X in M. ϕ will satisfy (1).

Assume now (2) together with the fact that $S(G,M)$ is non void. Let P be in $S(G,M)$. ρ is constant on $C_G[A]$. Hence $\rho(A) = \rho(P(A))$. This shows that P is faithful, normal and satisfies $P(V A V^{-1}) = P(A)$, for all V in $U(N)$. For example to check that $P(A) = P(V A V^{-1})$:

Let B be any element of N^c.

$\rho(B V A V^{-1}) = \rho(P(B V A V^{-1})) = \rho(B P (V A V^{-1}))$

$\rho(B V A V^{-1}) = \rho(V B A V^{-1}) = \rho(B A) = \rho(B P(A))$

Choose $B = (P(V A V^{-1}) - P(A))^*$, by faithfulness of ρ

$P(V A V^{-1}) = P(A)$.

Assume now (3). By countable decomposability there exists a faithful, normal state σ of M (get a maximal set of orthogonal projections P_n of M where each P_n is the projection on $[M'x_n]$, and let $\sigma = \Sigma W_{x_n}$ (Notation of 3). Let $\rho(X) = \sigma \psi(X)$. ρ in the state of (2).

Finally to show that (4) implies (3). By Theorem 2 there exists a faithful, normal expectation of M on N^c, call it Ψ such that $Tr(X A) = Tr(\Psi (X) A)$ for all A in $\mathfrak{m} \cap N^c$. As above one shows that $\Psi (V X V^{-1}) = \Psi(X)$.

Corollary 5: If $S(G,L(h))$ is sufficient, N, the algebra generated by G, is atomic.

Proof: By Theorem 2 there exists a faithful, normal, expectation of $L(h)$ on N' which is $U(N)$ stable. By Corollary 3, N' is of Type I, hence so is N[3]. Also N is finite by Corollary 1. Let Z be the center of N. Any projection of N dominates an abelian projection in N, call it $P \neq 0$. If Q is a projection of N such that $Q \leq P$, then $Q = PC$ where C is a projection of Z. Since Z is atomic [3], Q and hence P dominate a minimal projection. So N is atomic.

Remarks: The following statements are trivial to see:

(1) If $S(G,L(h))$ is sufficient then there exists a normal expectation ϕ from $L(h)$ to N' such that $\phi(U X U^{-1}) = \phi(X)$ for all U in G, this is part of Corollary 4.

(2) Assuming $S(G,L(h))$ contains a normal map, then $S(G,L(h))$ is sufficient. Let π be a normal map, then π is faithful. Indeed: by normality $\pi(U X U^{-1}) = \pi(X)$ for all U in $U(N)$. Assume that P is a projection such that $\pi(P) = 0$. Let $Q = \text{Sup } U P U^{-1}$ as $U \in U(N)$. Then $Q = V Q V^{-1}$ for all V in $U(N)$, so Q is in N'. Hence $Q = \pi(Q) = 0$. So $P = 0$.

Definition: Two groups of unitaries are equivalent if they generate the same von Neumann algebra.

Theorem 3: Assume $S(G,L(h))$ contains a normal map π, then G is equivalent to a countable direct sum of finite groups.

Proof: By Lemma 6, $S(G,L(h)) = \{\pi\}$. By the above remark π is faithful and by normality π is in $S(U(N),L(h))$. By Corollary 1 N is finite. Let Z be the center of N. By Corollary 5 Z is atomic. Pick a maximal set of orthogonal minimal projections, C_n of Z such that $N = \oplus N_{c_n}$. N_{c_n} is a factor of Type I_n. N_{c_n} is isomorphic to $n \times n$ matrices, so N_{c_n} is generated by a finite group K_n of unitaries. Let $K = \oplus K_n$ (all components are the identity except a finite number). The algebra generated by K contains all N_{c_n}, so it could contains N. Each K_n is a subgroup of $U(N)$. So the algebra generated by K is N.

Let M be a von Neumann algebra and let G be a subgroup of $U(M)$.

Definition: G will satisfy condition (F) if

(1) There exists orthogonal projections C_α of N' (N is the algebra generated by G) such that $I = \Sigma C_\alpha$ and $|GC_\alpha| < \infty$.

(2) For every U in G, $UC_\alpha = C_\alpha$ for all but a finite number of α.

Theorem 4: If G has property (F), then G is a countable direct sum of finite groups, and $S(G,M)$ is sufficient.

Proof: Define a map π_α on G by $\pi_\alpha(U) = UC_\alpha$. π_α is clearly a homomorphism of G and $\pi_\alpha(G)$ is finite. Also the intersection of all kernels of π_α is I. Let $F_\alpha = \pi_\alpha(G)$, then by definition of condition (F), $G = \oplus F_\alpha$. As each F_α is finite G is amenable since it is locally finite. So $S(G,M)$ is non void by Lemma 4. Now let A be a positive operator in M, let P be in $S(G,M)$ and suppose $P(A) = 0$ for all P in $S(G,M)$. If $A \neq 0$ $C_\alpha A C_\alpha \neq 0$ for some C_α, call α_o such an α. $C_{\alpha_o} P(A) C_{\alpha_o} = P(C_{\alpha_o} A C_{\alpha_o}) \in C_G[C_{\alpha_o} A C_{\alpha_o}]$. Let H be all elements of G where the α_o component is the identity. Then $G = HF_{\alpha_o}$. Let U be in G, then U is uniquely written as $U = VW$ where V is in H and W in F_{α_o}.

$U C_{\alpha_o} A C_{\alpha_o} U^{-1} = W C_{\alpha_o} A C_{\alpha_o} W^{-1}$ but there is only a finite number of $U C_{\alpha_o} A C_{\alpha_o} U^{-1}$. Hence $C_G[C_{\alpha_o} A C_{\alpha_o}]$ is the convex hull of $W C_{\alpha_o} A C_{\alpha_o} W^{-1}$ as W ranges in F_{α_o}.

So $0 = P(C_{\alpha_o} A C_{\alpha_o}) = \sum_{i=1}^{n} \alpha_i W_i C_{\alpha_o} A C_{\alpha_o} W_i^{-1}$. So $C_{\alpha_o} A C_{\alpha_o} = 0$, a contradiction. So $A = 0$ and $S(G,M)$ is sufficient.

Remark: While proving Theorem 3 it has been shown that if N if a finite atomic von Neumann algebra, then N is generated by a direct sum of finite groups K_n.

4) Uniqueness Properties

Lemma 8: If there exists only one faithful, normal expectation ϕ of M on N then $N^c \subset N$.

Proof: Let $\epsilon > 0$. Let H be a positive operator in N^c such that $H \geq \epsilon I > 0$ $\phi(H)$ is in N $\phi(H) \geq \epsilon I$. Let X be in N then $X\phi(H) = \phi(XH) = \phi(HX) = \phi(H)X$. So $\phi(H)$ is in N', so in $N \cap N' = 2_n$. Define $\pi(X) = \phi(H)^{-1} \phi(\frac{XH + HX}{2})$. Clearly π is another expectation of M on N, by uniqueness $\pi = \phi$. So $\phi(HX + XH) = 2\phi(H)\phi(X)$ for all H in N^c, positive and such that $H \geq \epsilon I$. In particular let $X = H$, then $\phi(H)^2 = \phi(H^2)$. This holds for any self adjoint operator in N^c which is positive. Let H be any self adjoint operator in N^c, pick $C > 0$ such that $CI + H \geq \epsilon I$, then $[\phi(CI + H)]^2 = \phi(CI + H)^2$ so $\phi(H)^2 = \phi(H)^2$. Let P be a projection in N^c, then $(P - \phi(P)^2 \geq 0$. So $\phi(P - \phi(P))^2 = (\phi(P) - \phi(P))^2 = 0$. By faithfulness $P = \phi(P)$, i.e. P is in N so $N^c \subset N$.

Lemma 9: Let N be normal in M (i.e. $N^{cc} = N$). A necessary and sufficient condition for at most one faithful, normal expectation to exist from M to N is that $N^c \subset N$.

Proof: The necessary condition was shown in Lemma 1. Now to show the sufficient condition: As $N^c \subset N$ N^c is the center of N in particular N^c is abelian. Hence, $U(N^c)$ is amenable, so $S(U(N^c),M)$ is non void by Lemma 4. Let P be in $S(U(N^c),M)$ then P is an expectation on $N^{cc} \cap M$ by Lemma 3. So P is an expectation on N.

Let $\phi(P(X)) = \phi(X)$. Also $\phi(P(X)) = P(X)$ so $\phi = P$. This shows that there exists at most one normal expectation.

Theorem 5: Assume the following conditions:

(1) $N^c \subset N$

(2) N is finite

(3) M is semi-finite

Then there exists at most one normal expectation ϕ.

Proof: N^c is the center of N, by finiteness the map $\natural$ (notation of [3]) is defined from N to N^c. If X is in M, define $\Psi(X) = (\phi(X))^{\natural}$. Ψ is a normal map. $S(U(N^c) M)$ is non void. Let P be in $S(U(N^c),M)$. If X is in $\mathfrak{m}^{1/2}$, $C_{U(N)}[X] \cap \mathfrak{m}^{1/2}$ and $C_{U(N)}[X]$ intersects N^c. Let T be in $C_{U(N)}[X] \cap N^c$. Ψ is invariant under $U(N)$, so $T = \Psi(T) = \Psi(X)$. So $N^c \cap C_{U(N)}[X] = \{\Psi(X)\}$. If ϕ_1 is another normal expectation of M on N. then define $\Psi_1(X) = [\phi_1(X)]^{\natural}$. Also $N^c \cap C_{U(N)}[X] = \Psi_1(X)$, so $\Psi = \Psi_1$ on $\mathfrak{m}^{1/2}$, hence on M.

Let λ be any normal finite trace on N. Then: $\lambda\phi(X) = \lambda\Psi(X) = \lambda\Psi_1(X) = \lambda\phi_1(X)$. Since the λ form a complete set $\phi = \phi_1$.

In conclusion consider the following problem. Let N be a von Neumann algebra. Suppose there exists sufficiently many expectations of M on N. Is N relatively semi-finite? An answer to that problem was given when the expectations are of a certain type (Lemma 7).

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