REPRESENTATION OF NONLINEAR TRANSFORMATIONS AND FUNCTIONALS ON L^p SPACES

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Introduction. Recently the author in collaboration with A. D. Martin [A] and K. Sundaresan [B] has obtained a characterization of certain classes of nonlinear functionals defined on spaces of measurable functions (see also K. Sundaresan [C]). The functionals in question had the form

$$(1) \qquad F(x) = \int_T (\varphi \circ x)\, d\mu = \int_T \varphi(x(t))\, d\mu(t)$$

with a continuous "representing function" $\varphi : R \to R$, or

$$(2) \qquad F(x,y) = \int_{S \times T} \varphi \circ (x,y)\, d\mu \otimes \nu = \int_{S \times T} \varphi(x(s), y(t))\, d\mu(s)\, d\nu(t)$$

with a separately continuous representing function $\varphi : R^2 \to R$. There are direct applications of this work to the theory of generalized random processes in probability (see Gelfand-Vilenkin [D]) and to the theory of fading memory in continuum mechanics [E]. However the main motivation for these studies was an interest in possible application to the functional analytic study of nonlinear differential equations. From the standpoint of this latter application it would also be desirable to characterize the broader class of functionals having the form

$$(3) \qquad F(x) = \int_T \varphi(x(t), t)\, d\mu(t),$$

where the representing function $\varphi : R \times T \to R$ satisfies "Caratheodory conditions". This can be readily understood if we recall that the existence theory for

$$\dot{x}(t) = \varphi(x(t),t)$$

with φ a function satisfying Caratheodory conditions is very close to that for

$$\dot{x}(t) = \varphi(x(t))$$

with $\varphi: R \to R$ continuous (see e.g. [F]).

In the present paper we obtain an abstract characterization for functionals having the form (3), a characterization which is of the kind obtained earlier for functionals having the form (1). In addition we characterize corresponding transformations from $L^p(T)$ to $C(S)$ where $C(S)$ is the space of continuous functions on a compact Hausdorff space. Our proofs utilize some results appearing in Krasnoselkii's important summary [G] of work on transformations of the type $x \to \varphi \circ x$. For some work on a problem analogous to ours for functionals on the space of continuous functions on a compact metric space see [H].

END OF INTRODUCTION

Throughout this paper $T = (T, \Sigma, \mu)$ is a measure space, R is the real line with Lebesgue measure, and $M(T)$ denotes the space of real valued measurable functions on T.

__Definition__. A real valued function $\varphi: R \times T \to R$ is said to be of __Caratheodory type__ for T and we write $\varphi \in Car(T)$ if it satisfies the following conditions,

(1) $\varphi(\cdot,t): R \to R$ is continuous for almost all $t \in T$,

(2) $\varphi(c,\cdot): T \to R$ is measurable for all $c \in R$.

One can extend this definition in an obvious way to functions $\varphi: R^m \times T \to R^n$. We remark that Car(T) is a subspace of the vector space M(R×T).

If x is a real valued measurable function on T and φ is in Car(T) then the function $\varphi \circ x$ defined by

$$(\varphi \circ x)(t) = \varphi(x(t),t),$$

is also a measurable functon on T. This is obviously true when x is a simple function. In the general case x is the limit almost everywhere of a sequence of simple functions x_n, so that by continuity of φ in its first argument, $\varphi \circ x$ being the pointwise limit of the measurable functions $\varphi \circ x_n$ is measurable. Thus for each $\varphi \in$ Car(T), the mapping $x \to \varphi \circ x$ is a mapping of M(T) into itself. It is useful to single out certain subspaces of the vector space CAR(T) in terms of their mapping properties.

<u>Definition</u>. Given the numbers p and q, $1 \leq p,q \leq \infty$, a function φ of Caratheodory type for T is said to be in the <u>Caratheodory (p,q)-class</u>, and we write $\varphi \in \mathrm{Car}^{p,q}(T)$ if φ maps $L^p(T)$ into $L^q(T)$. That is, φ is in $\mathrm{Car}^{p,q}(T)$ if

$$\varphi \circ x \in L^q(T) \quad \text{for all} \quad x \in L^p(T).$$

<u>Remark</u>. For the case of a non-atomic σ-finite measure space it is known ([G], p.27) that φ is in $\mathrm{Car}^{p,q}(T)$, $1 \leq p,q < \infty$, if and only if

$$|\varphi(x,t)| \leq a(t) + b|x|^{p/q}$$

for some $a \in L^q(T)$.

<u>Theorem 1</u>. <u>Let</u> $T = (T, \Sigma, \mu)$ <u>be a finite or σ-finite measure space</u>. <u>Let</u> F <u>be a real valued functional on</u> $L^\infty(T)$ <u>which satisfies</u>:

(i) $F(x+y) = F(x) + F(y)$ when $xy = 0$ a.e.,

(ii) F is uniformly continuous on each bounded subset of $L^\infty(T)$,

(iii) $F(x_n) \to F(x)$ whenever $\{x_n\}_{n \geq 1}$ converges boundedly a.e. to $x \in L^\infty(T)$.

<u>Then there exists a function</u> $\varphi \in Car^{\infty,1}(T)$ <u>such that</u>

$$(*) \quad F(x) = \int_T (\varphi \circ x) d\mu = \int_T \varphi(x(t),t)\, d\mu(t).$$

<u>Moreover</u> φ <u>can be taken to satisfy</u>

$$(a) \qquad \varphi(0,\cdot) = 0 \quad \text{a.e.,}$$

<u>and is then unique up to sets of the form</u> $R \times N$ <u>with</u> N <u>a null set in</u> T.

<u>Conversely, for every</u> $\varphi \in Car^{\infty,1}(T)$ <u>satisfying</u> (a), (*) <u>defines a functional satisfying</u> (i), (ii), and (iii).

<u>Remarks</u>: 1. The final statement of the theorem is valid for any $\varphi \in Car^{\infty,1}(T)$ satisfying

$$(a') \quad \int_T (\varphi \circ 0) d\mu = 0.$$

Moreover condition (i) on F can be modified in such a way that this result applies to <u>all</u> $\varphi \in Car^{\infty,1}(T)$. Namely we could replace (i) by

$$(i') \qquad F(x+y) - F(x) - F(y) = \text{const.} \qquad \text{when } xy = 0 \text{ a.e..}$$

(If we denote the constant in (i') by k then the functional $F_1(x) = F(x) + k$ satisfies (i), (ii) and (iii).)

2. Unlike the results in [A] and [B] the present characterization does not require a hypothesis concerning the non-atomic nature or almost non-atomic nature of T. The same holds true for Theorem 2 to follow.

<u>Proof of the theorem</u>:

It follows from (i) and (iii) that for each real number h the real valued set function a_h defined by

$$a_h(S) = F(h\chi_S)$$

is countably additive and absolutely continuous relative to μ. Hence by the Radon-Nikodym theorem there corresponds to each h a function $\varphi_h \in L^1(T)$, unique up to a null set, such that

$$F(h\chi_S) = \int_S \varphi_h \, d\mu.$$

The functions φ_h with h rational will be utilized below in constructing the function φ occurring in (*). This construction applies the following lemma whose proof will be deferred until later.

Lemma. *Given any $\eta > 0$ there is a measurable set $S_\eta = \bigcup_{i=1}^{\infty} S_{\eta,i}$ such that*

(1) $\mu(T-S_\eta) < \eta$, $\mu(S_{\eta,i}) < \infty$ $\quad i = 1,2,\dots$,

(2) on $S_{\eta,i}$ *there exists for each pair of numbers* $M, \varepsilon > 0$ *a* $\delta = \delta_i(\varepsilon,M) > 0$ *such that for rational* h *and* h' *we have*

$$h,h' \in [-M,M] \quad \text{and} \quad |h-h'| < \delta \Longrightarrow \sup_{t \in S_{\eta,i}} |\varphi_h(t) - \varphi_{h'}(t)| \leq \varepsilon.$$

Now select a sequence $\eta_m \to 0$ and define a function $\varphi: R \times T \to R$ as follows:

$$(1) \qquad \varphi(c,t) = \begin{cases} \lim\limits_{\substack{h \to c \\ (h \text{ rational})}} \varphi_h(t) & \text{for } t \in S = \bigcup_{m=1}^{\infty} S_{\eta_m} \\ 0 & \text{for } t \in T - S . \end{cases}$$

It follows from the lemma that this defines φ unambiguously and that $\varphi(\cdot,t)$ is continuous for each $t \in T$. Moreover since $T-S$ is a null set, for each $c \in R$ $\varphi(c,\cdot)$ is the almost everywhere pointwise limit of a sequence of measurable functions φ_h and is therefore measurable. Thus φ is of Caratheodory type for T. Further, since for c rational we have

$$\varphi(c,t) = \varphi_c(t) \qquad \text{a.e.},$$

it is clear that $\varphi(c,\cdot) \in L^1(T)$ for c rational and that φ satisfies (a). It remains to be shown that (*) holds.

Suppose $x \in L^{\infty}(T)$ is a simple function with rational values, i.e.

$$x = \sum_{k=1}^{N} c_k \chi_{T_k} \qquad c_k \text{ rational, } \{T_k\} \text{ disjoint.}$$

Then, using (i),

$$F(x) = \sum_{k=1}^{N} F(c_k \chi_{T_k}) = \sum_{k=1}^{N} \int_{T_k} \varphi_{c_k} \, d\mu$$

$$= \int_T \varphi \circ (\Sigma c_k \chi_{T_k}) \, d\mu$$

$$= \int_T (\varphi \circ x) \, d\mu .$$

Thus (*) holds in this special case.

Now each $x \in L^{\infty}(T)$ is the limit a.e. as well as in norm of a sequence x_n of simple functions with rational values,

$$x_n \rightarrow x \quad \text{a.e. and in } L^{\infty}(T).$$

Since $\varphi \in Car(T)$ it follows that

$$(2) \qquad \varphi \circ x_n \longrightarrow \varphi \circ x \quad \text{a.e.}$$

In addition, the sequence $\varphi \circ x_n \in L^1(T)$ is uniformly absolutely continuous, i.e.

$$(3) \qquad \int_R |\varphi \circ x_n| d\mu \rightarrow 0 \quad \text{as} \quad \mu(R) \rightarrow 0, \text{ uniformly in } n.$$

Otherwise there would exist for some $\varepsilon > 0$ a sequence of sets $R_m \subset T$ with $\mu(R_m) < 3^{-m}$ and a corresponding sequence $\varphi \circ x_{n_m}$ such that

$$\int_{R_m} |\varphi \circ x_{n_m}| d\mu > \varepsilon.$$

It follows that each R_m possesses a subset R_m' satisfying

$$\left| \int_{R_m'} \varphi \circ x_{n_m} d\mu \right| > \varepsilon/2 .$$

Now the functions $y_m = x_{n_m} \chi_{R_m'}$ form a bounded set in $L^{\infty}(T)$ since the x_n form such a set, and hence

$$y_m \longrightarrow 0 \qquad \text{boundedly a.e.}$$

Moreover y_m being a rational valued simple function implies

$$F(y_m) = \int_T (\varphi \circ y_m) d\mu = \int_{R'_m} (\varphi \circ x_{n_m}) d\mu .$$

However by the construction of R'_m this implies that the $F(y_m)$ do not converge to zero, contradicting property (iii).

Furthermore the sequence $\varphi \circ x_n$ has the property that for each $\varepsilon > 0$ there exists a set R_ε such that $\mu(R_\varepsilon) < \infty$ and

(4) $$\int_{T-R_\epsilon} |\varphi \circ x_n| d\mu < \varepsilon \qquad \text{for all } n.$$

Otherwise, for some $\varepsilon > 0$ there would exist an expanding sequence of sets R_m with $\mu(R_m) < \infty$ and $\bigcup_1^\infty R_m = T$ and a corresponding sequence $\varphi \circ x_{n_m}$ such that

$$\int_{T-R_m} |\varphi \circ x_{n_m}| d\mu > \varepsilon .$$

Thus for some $R''_m \subset T - R_m$

$$\left| \int_{R''_m} (\varphi \circ x_{n_m}) d\mu \right| > \varepsilon/2.$$

The functions $y_m = x_{n_m} \chi_{R''_m}$ satisfy

$$y_m \rightarrow 0 \qquad \text{boundedly a.e.}$$

while the formula

$$F(y_m) = \int_T (\varphi \circ y_m) d\mu = \int_{R''_m} (\varphi \circ x_{n_m}) d\mu$$

implies that the $F(y_m)$ do not converge to zero, contradicting (iii).

Since the sequence $\varphi \circ x_n$ in $L^1(T)$ satisfies (2), (3), and and (4) it follows by Vitalli's convergence theorem (see [J], p.150) that $\varphi \circ x$ belongs to $L^1(T)$ and that $\varphi \circ x_n \to \varphi \circ x$ in $L^1(T)$, whereby

$$F(x) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \int_T \varphi \circ x_n \, d\mu$$

$$= \int_T \varphi \circ x \, d\mu.$$

Thus $\varphi \in Car^{\infty,1}(T)$ and (*) holds. The uniqueness of φ follows from the fact that by (a)

$$F(c\chi_S) = \int_T \chi_S \, \varphi(c,t)\, d\mu = \int_S \varphi_c \, d\mu.$$

Considering only rational c we see that this condition determines $\varphi(c,\cdot)$ up to a null set, and hence determines $\varphi \in Car(T)$ up to sets of the form $R \times N$ as claimed.

For the converse let φ be a function in $Car^{\infty,1}(T)$ which satisfies condition (a). Then the functional F defined by (*) obviously satisfies (i). We proceed to show that (ii) holds. Otherwise there would exist numbers $A, a > 0$ such that corresponding to each positive integer n there is a pair of functions $x_n, y_n \in L^{\infty}(T)$ satisfying

$$\|x_n\|_\infty,\ \|y_n\|_\infty \le A,\quad \|x_n - y_n\|_\infty < 1/n \tag{5}$$

$$\|\varphi\circ x_n - \varphi\circ y_n\|_1 > a.$$

Consider first the case in which $\mu(T)$ is finite and set $S_1 = T$. We select a subsequence of x_n, y_n as follows. By the absolute continuity of the indefinite integral of $\varphi\circ x_1 - \varphi\circ y_1$ there exists an $\varepsilon_1 > 0$ such that

$$\int_S |\varphi\circ x_1 - \varphi\circ y_1|\, d\mu < a/3 \quad \text{whenever} \quad \mu(S) < 2\varepsilon_1.$$

Obviously $\varepsilon_1 < \frac{1}{2}\mu(T)$. Since $\varphi(\cdot, t)$ is continuous for almost all $t \in T$, it is uniformly continuous on the set $[-A, A] \subset R$ for such t. Thus for each ε,

$T = \bigcup_{n=1}^{\infty} \{t \mid c_1, c_2 \in [-A, A], |c_1 - c_2| \le \frac{1}{n} \Rightarrow |\varphi(c_1, t) - \varphi(c_2, t)| \le \varepsilon\} \cup N$ where $\mu(N) = 0$. Hence by selecting n_2 sufficiently large one can find a measurable set T_2 satisfying

$$|(\varphi\circ x_{n_2})(t) - (\varphi\circ y_{n_2})(t)| \le \frac{a}{3\mu(T)} \quad \text{for } t \in T_2, \text{ and}$$

$$\mu(T - T_2) < \varepsilon_1.$$

By (5) this implies that with $S_2 = T - T_2$,

$$\int_{S_2} |\varphi\circ x_{n_2} - \varphi\circ y_{n_2}|\, d\mu > 2a/3, \quad \mu(S_2) < \varepsilon_1.$$

Again, since the indefinite integral of $\varphi\circ x_{n_2} - \varphi\circ y_{n_2}$ is absolutely ctontinuous there exists an $\varepsilon_2 > 0$ such that

$$\int_S |\varphi\circ x_{n_2} - \varphi\circ y_{n_2}|\, d\mu < a/3 \quad \text{wherever} \quad \mu(S) < 2\varepsilon_2.$$

Obviously $2\varepsilon_2 < \frac{1}{2}\mu(S_2)$. Again by the uniform continuity of $\varphi(\cdot,t)$ on $[-A,A]$ for almost all t, there exists an n_3 sufficiently large and a corresponding set T_3 such that

$$|(\varphi\circ x_{n_3})(t) - (\varphi\circ y_{n_3})(t)| < \frac{a}{3\mu(T)} \quad \text{for } t\in T_3$$

$$\mu(T - T_3) < \varepsilon_2.$$

By (5) this implies that with $S_3 = T - T_3$,

$$\int_{S_3} |\varphi\circ x_{n_3} - \varphi\circ y_{n_3}|d\mu > 2a/3, \quad \mu(S_3) < \varepsilon_2.$$

Proceeding in this fashion we obtain a subsequence x_{n_k}, y_{n_k} and a corresponding sequence of sets S_k satisfying

$$\int_{S_k} |\varphi\circ x_{n_k} - \varphi\circ y_{n_k}|d\mu > 2a/3, \quad \int_{S_{k+1}} |\varphi\circ x_{n_k} - \varphi\circ y_{n_k}|d\mu < \frac{a}{3},$$

$$\mu(S_k) < \varepsilon_{k-1} < \mu(S_{k-1})/2.$$

Now define $R_k = S_k - \bigcup_{\ell=k+1}^{\infty} S_\ell$. The sets R_k are disjoint. Moreover

$$\mu(\bigcup_{\ell=k+1}^{\infty} S_\ell) < 2\mu(S_{k+1}) < 2\varepsilon_k$$

so that, recalling how the ε_j are defined, we have

$$\int_{R_k} |\varphi\circ x_{n_k} - \varphi\circ y_{n_k}|d\mu > a/3.$$

We now define

$$x = \sum_{\ell=1}^{\infty} x_{n_k}\chi_{R_k} \quad , \quad y = \sum_{k=1}^{\infty} y_{n_k}\chi_{R_k} \quad .$$

By construction $x, y \in L^{\infty}(T)$, so that $\varphi\circ x$, $\varphi\circ y, \in L^{1}(T)$, and

$$\int_{R_k} |\varphi\circ x - \varphi\circ y| = \int_{R_k} |\varphi\circ x_{n_k} - \varphi\circ y_{n_k}| d\mu > a/3, \quad k = 1,2,\dots .$$

Since the R_k are disjoint, this is a contradiction.

Consider now the case $\mu(T) = \infty$ and assume that (5) holds. We will construct sequences of functions $\{x_{n_k}\}, \{y_{n_k}\}$ and a sequence of disjoint sets $\{R_k\}$ such that

$$(6) \qquad \mu(R_k) < \infty, \quad \int_{R_k} |\varphi\circ x_{n_k} - \varphi\circ y_{n_k}| d\mu > a/2.$$

The procedure is again inductive. Let R_1 be a set of finite measure such that

$$\int_{R_1} |\varphi\circ x_1 - \varphi\circ y_1| d\mu > a/2.$$

This is possible by (5). Then, by the result in the preceding paragraph, for n_2 sufficiently large

$$\int_{R_1} |\varphi\circ x_{n_2} - \varphi\circ y_{n_2}| d\mu < a/2.$$

Hence there exists a set $R_2 \subset T - R_1$ such that $\mu(R_2) < \infty$ and

$$\int_{R_2} |\varphi\circ x_{n_2} - \varphi\circ y_{n_2}| d\mu > a/2.$$

Again since $\mu(R_1 \cup R_2) < \infty$ we have by our earlier result that for n_3 sufficiently large

$$\int_{R_1 \cup R_2} |\varphi \circ x_{n_3} - \varphi \circ y_{n_3}| d\mu < a/2.$$

Hence there exists a set $R_3 \subset T - (R_1 \cup R_2)$ such that $\mu(R_3) < \infty$ and

$$\int_{R_3} |\varphi \circ x_{n_3} - \varphi \circ y_{n_3}| d\mu > a/2.$$

Proceeding in this fashion we arrive at sequences of functions $\{x_{n_k}\}, \{y_{n_k}\}$ and of disjoint sets $\{R_k\}$ for which (6) holds. Defining

$$x = \sum_{k=1}^{\infty} x_{n_k} \chi_{R_k} \quad , \quad y = \sum_{k=1}^{\infty} y_{n_k} \chi_{R_k}$$

we again find that

$$\int_{R_k} |\varphi \circ x - \varphi \circ y| d\mu > a/2 \qquad k = 1,2,\ldots ,$$

contradicting the fact that $\varphi \circ x, \varphi \circ y \in L^1(T)$.

There remains the proof of (iii). Let x_n be a sequence such that

$$x_n \to x \quad \text{a.e.,} \qquad \|x_n\|_\infty, \|x\|_\infty \leq A.$$

Since $\varphi \in \mathrm{Car}(T)$ it follows that

(7) $$\varphi \circ x_n \longrightarrow \varphi \circ x \qquad \text{a.e.,}$$

while by (ii)

$$\|\varphi \circ x_n\|_1, \ \|\varphi \circ x\|_1 \leq M = M(A).$$

We will show that (iii) holds by proving that

$$\varphi \circ x_n \longrightarrow \varphi \circ x \quad \text{weakly.}$$

Now the sequence $\varphi \circ x_n \in L^1(T)$ is bounded and therefore weakly precompact. Suppose $\varphi \circ x_n$ does not converge weakly to $\varphi \circ x$. Then there exists a subsequence $\varphi \circ x_{n_k}$ no subsequence of which converges weakly to $\varphi \circ x$. Moreover by extracting a further subsequence we may suppose without loss of generality that

(8) $$\varphi \circ x_{n_k} \longrightarrow z \quad \text{weakly} \qquad \text{for some } z \neq \varphi \circ x.$$

Of course, by (7),

(9) $$\varphi \circ x_{n_k} \longrightarrow \varphi \circ x_n \qquad \text{a.e.}$$

Now by a theorem of Banach and Saks ([J], p.462), (8) implies that there is a subsequence $\varphi \circ x_{n_j'}$ of $\varphi \circ x_{n_k}$ such that

(10) $$\frac{1}{m} \sum_{j=1}^{m} \varphi \circ x_{n_j'} \longrightarrow z \quad \text{in measure.}$$

However (9) and (10) imply that $z = \varphi \circ x$, contradicting (8). Q.E.D.

Proof of the lemma: In the sequel we restrict the symbols h and r to denote rational numbers. Consider first the case of a finite measure space. To begin with we show that, with $M > 0$ given, for each integer n the contracting sequence of sets

$$T - S^M_{n,j} = \{t \,|\, |\varphi_h(t) - \varphi_{h'}(t)| > 1/n \quad \text{for some } h,h' \in [-M,M] \text{ with } |h-h'| < \tfrac{1}{j}\}$$
$$j = 1,2,\ldots$$

converges to a null set. Otherwise for some fixed $c > 0$,

$$\mu(T - S^M_{n,j}) \geq c \qquad j = 1,2,\ldots .$$

Now

$$T - S^M_{n,j} \subset \bigcup_{h\in[-M,M]} \bigcup_{r\in[-1/j,1/j]} B_{h,r} = \bigcup_{h\in[-M,M]} B^{(j)}_h ,$$

where

$$B_{h,r} = \{t \,|\, |\varphi_h(t) - \varphi_{h+r}(t)| > \tfrac{1}{n}\}, \quad B^{(j)}_h = \bigcup_{r\in[-1/j,1/j]} B_{h,r}.$$

Enumerating the rationals in $[-M,M]$ and $[-1/j,1/j]$ as $h_1,h_2,\ldots$ and $r_1,r_2,\ldots$, respectively, define the sets $C^{(j)}_{h_k}$ and C_{h_k,r_ℓ} as follows

$$C^{(j)}_{h_k} = B^{(j)}_{h_k} - \bigcup_{i=1}^{k-1} B^{(j)}_{h_i} \qquad k = 1,2,\ldots ,$$

$$C_{h_k,r_\ell} = B_{h_k,r_\ell} - \bigcup_{i=1}^{\ell-1} B_{h_k,r_i} \qquad \ell = 1,2,\ldots .$$

For each j define the functions x_j and y_j by,

(1) $$x_j = \sum_{k=1}^{\infty} h_k \chi_{C_{h_k}^{(j)}}$$

(2) $$y_j = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} (h_k + r_\ell) \chi_{C_{h_k, r_\ell}} .$$

By construction x_j, y_j are in $L^\infty(T)$ and satisfy

(3) $$\|x_j\|_\infty, \ \|y_j\|_\infty \le M + 1$$

(4) $$\|x_j - y_j\|_\infty \le 1/j .$$

Moreover

$$\sum_{k=1}^{N} h_k \chi_{C_{h_k}^{(j)}} \longrightarrow x_j \qquad \text{boundedly a.e., and}$$

$$\sum_{k=1}^{N} \sum_{\ell=1}^{N} (h_k + r_\ell) \chi_{C_{h_k, r_\ell}} \longrightarrow y_j \qquad \text{boundedly a.e. .}$$

Hence by (i) and (iii) and the definition of φ_h

$$F(x_j) - F(y_j) = \sum_{k=1}^{\infty} F(h_k \chi_{C_{h_k}^{(j)}}) - \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} F((h_k + r_\ell) \chi_{C_{h_k, r_\ell}})$$

$$= \int_T \sum_{k=1}^{\infty} \Big[\varphi_{h_k} \chi_{C_{h_k}^{(j)}} - \sum_{\ell=1}^{\infty} \varphi_{h_k + r_\ell} \chi_{C_{h_k, r_\ell}} \Big] d\mu > \frac{1}{n} c$$

$$j = 1, 2, \ldots ,$$

contradicting (ii).

It follows from the above that with M given there exists for each $\eta > 0$ a set S_η^M satisfying

(5) for each $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, M) > 0$ such that $h, h' \in [-M, M]$ and $|h-h'| < \delta \Rightarrow |\varphi_h(t) - \varphi_{h'}(t)| \leq \varepsilon$ for $t \in S_\eta^M$,

(6) $\mu(T - S_\eta^M) < \eta$.

For by the preceding paragraph one can select for each integer n an index j_n such that

$$\mu(T - S^M_{n,j_n}) < \eta/2^n \qquad n = 1,2,\ldots .$$

Then the set S_η^M defined by

$$S_\eta^M = \bigcap_{n=1}^{\infty} S_{n,j_n} \tag{7}$$

satisfies (5) and (6).

In addition, the set S_η defined by

$$S_\eta = \bigcap_{M=1}^{\infty} S^M_{\eta/2^M} \tag{8}$$

is readily seen to satisfy (5) and (6) for all M. Thus the lemma is proved in case T is a finite measure space.

Now suppose that $\mu(T) = \infty$. By hypothesis,

$$T = \bigcup_{i=1}^{\infty} T_i \qquad \text{with } \mu(T_i) < \infty .$$

Using the result established in the preceding paragraphs we construct sets $S_{\eta,i} \subset T_i$ $i = 1,2,\ldots$ by defining

$$S_{\eta,i} = S_{\eta/2^i} \qquad \text{(relative to the measure space } T_i).$$

It is then clear that the set $S_\eta \subset T$ which is defined by

$$S_\eta = \bigcup_{i=1}^{\infty} S_{\eta,i}$$

satisfies all the requirements stated in the lemma. Q.E.D.

Corollary. With T nonatomic let F be a real valued functional on $L^\infty(T)$ [or $M(T)$] which satisfies the conditions

(i) $F(x+y) = F(x) + F(y)$ when $xy = 0$ a.e.,

(ii) F is uniformly continuous on each bounded subset of $L^\infty(T)$,

(iii)' $F(x_n) \to F(x)$ whenever $\{x_n\}_{n \geq 1}$ converges a.e. to $x \in L^\infty(T)$ [or $M(T)$] .

Then there exists a function φ in Car(T) such that

(*) $$F(x) = \int_T (\varphi \circ x)\, d\mu \quad \text{for } x \in L^\infty(T) \text{ [or } M(T)],$$

and $F: L^\infty(T) \to R$ [or $M(T) \to R$] is bounded. In fact,

(b) $R_\varphi \subset L^1(T)$ is bounded.

Moreover, φ can be taken to satisfy (a) and is then unique in the *same sense as in Theorem* 1.

Conversely, for every $\varphi \in Car^{\infty,1}(T)$ *which satisfies conditions* (a) *and* (b), *the functional defined by* (*) *satisfies* (i), (ii) *and* (iii)'.

Proof: Observe that the functional $F_1 = F \mid L^{\infty}(T)$ satisfies (i), (ii) and (iii) of Theorem 1 and hence is given by

$$(1) \qquad F_1(x) = \int_T (\varphi \circ x)\, d\mu \qquad \text{for} \quad x \in L^{\infty}(T),$$

for some $\varphi \in Car^{\infty,1}(T)$.

We show that $R_{\varphi} \subset L^1(T)$ is bounded. For otherwise there exists a sequence $x_n \in L^{\infty}(T)$ such that

$$(2) \qquad \| \varphi \circ x_n \|_1 = c_n \to \infty .$$

It then follows that there exists a subset $A_n \subset T$ such that

$$(3) \qquad |F(x_n \chi_{A_n})| = \left| \int_{A_n} (\varphi \circ x_n)\, d\mu \right| \geq c_n/2 \to \infty .$$

Consider first the case $\mu(T) < \infty$. Then since T is nonatomic there exists for each sufficiently large n a subset A_n' of A_n such that

$$(4) \qquad \begin{aligned} &|F(x_n \chi_{A_n'})| \geq 1 \\ &\mu(A_n') \leq 2/c_n . \end{aligned}$$

However, we have $x_n \chi_{A_n'} \to 0$ a.e., so (4_1) contradicts (iii)'.

Now suppose $T = \bigcup_{i=1}^{\infty} T_i$, $\mu(T_i) < \infty$. The preceding argument shows that for each m we have a constant N_m such that

$$\|\varphi \circ x\|_1 \leq N_m \quad \text{for } x \text{ such that } \operatorname{supp} x \subset \bigcup_{i=1}^{m} T_i .$$

By extracting a subsequence we can assume that in (2) $c_m > 3N_m$. Consequently there exist sets $A_m \subset T - \bigcup_{i=1}^{m} T_i$ such that

$$\|\varphi \circ x_m \chi_{A_m}\|_1 > 2N_m \qquad m = 1,2,\ldots .$$

It then follows that for some subset $A_m' \subset A_m$,

$$(5) \qquad |F(x_m \chi_{A_m'})| = \left| \int_{A_m'} \varphi \circ x_m d\mu \right| > N_m .$$

Now

$$x_m \chi_{A_m'} \to 0 \quad \text{a.e.,}$$

so (5) contradicts (iii)'.

Now suppose $\varphi \in \mathrm{Car}^{\infty,1}(T)$ satisfies (a) and (b). It only needs to be shown that (iii)' holds. Suppose that $x_n, x \in L^\infty(T)$ and $x_n \to x$ a.e. Then it can be shown just as in the proof of the theorem that

$$\varphi \circ x_n \to \varphi \circ x \quad \text{weakly}$$

and therefore

$$F(x_n) = \int (\varphi \circ x_n) d\mu \to \int (\varphi \circ x) d\mu = F(x) . \qquad \text{Q.E.D.}$$

Remark: It is easy to show by examples that on atomic measure spaces (i), (ii), (iii)' do not imply (b). On the other hand, the above proof shows that for all T, if $\varphi \in Car^{\infty,1}(T)$ and satisfies (a) and (b) then F satisfies (i), (ii), (iii)'.

Theorem 2. With T as in Theorem 1 let F be a real valued functional on $L^p(T)$, $1 \leq p < \infty$, which satisfies the conditions

(i) $F(x+y) = F(x) + F(y)$ when $xy = 0$ a.e.,

(ii_p) F is continuous on $L^p(T)$,

(iii_p) F is uniformly continuous relative to L^∞ norm on each bounded subset of $L^\infty(T)$ which is supported by a set of finite measure.

Then there exists a function $\varphi \in Car^{p,1}(T)$ such that

$$(*) \qquad F(x) = \int_T (\varphi \circ x)\, d\mu \qquad \text{for } x \in L^p(T).$$

Moreover φ can be taken to satisfy

(a) $\varphi(0,\cdot) = 0$ a.e.

and is then unique up to sets of the form $R \times N$ with N a null set in T.

Conversely, for every $\varphi \in Car^{p,1}(T)$ satisfying (a) the formula (*) defines a functional satisfying (i), (ii_p) and (iii_p).

Remarks. 1. Observe that when F is a linear functional, (ii_p) signifies uniform continuity on bounded subsets of $L^p(T)$ and hence implies $(iii)_p$. In addition, for such cases the function φ

necessarily has the form

$$\varphi(x,t) = xu(t)$$

for some locally summable function u. Thus the present result includes the Riesz representation theorem modulo a proof that $u \in L^q(T)$ is necessary and sufficient in order that the above φ be in $\mathrm{Car}^{p,1}(T)$.

2. For the case $u(T) < \infty$ this theorem is related to a result state in [I], though our hypotheses are weaker than the ones utilized there. See also the corollary and remark following the proof.

Proof: By hypothesis $T = \bigcup_{i=1}^{\infty} T_i$ where the T_i are disjoint subsets of finite measure. Define

$$F_i = F \,|\, L^{\infty}(T_i) \qquad\qquad i = 1,2,\ldots .$$

Then (i), (ii_p) and (iii_p) imply that each of the functionals F_i satisfies the hypotheses of Theorem 1, the validity of (iii) being a consequence of (ii_p) and the dominated convergence theorem. Hence there exist functions $\varphi_i \in Car(T_i)$, unique up to null sets, which satisfy (a) $\varphi_i(0,\cdot) = 0$ a.e. on T_i and

$$(1) \qquad F_i(x) = \int_{T_i} (\varphi_i \circ x) d\mu \qquad\qquad \text{for } x \in L^{\infty}(T_i),\ i = 1,2,\ldots .$$

Now define $\varphi : R \times T \rightarrow R$ by means of

$$(2) \qquad \varphi(h,\cdot) \,|\, T_i = \varphi_i(h,\cdot) \qquad\qquad h \in R,\ i = 1,2,\ldots .$$

It is clear that $\varphi \in Car(T)$ and that φ satisfies (a). It remains to be shown that (*) holds for $x \in L^p(T)$. Now for each simple function x we have

$$x \chi_{T_i} \in L^{\infty}(T_i) \qquad\qquad i = 1,2,\ldots, \text{ and}$$

$$x_n = x \chi_{\bigcup_{i=1}^{n} T_i} \longrightarrow x \qquad \text{in } L^p(T).$$

Hence by (i) and (ii_p)

$$F(x) = \lim_{n\to\infty} F(x_n) = \sum_{i=1}^{\infty} F(x\chi_{T_i})$$

$$= \sum_{i=1}^{\infty} \int_{T_i} (\varphi_i \circ x)\,d\mu = \int_T (\varphi \circ x)\,d\mu .$$

Therefore (*) has been established for simple functions.

To show that (*) holds for all $x \in L^p(T)$, notice that each such x is the limit a.e. as well as in norm of a sequence x_n of simple functions,

(3) $\qquad x_n \to x \quad$ a.e. and in $L^p(T)$.

Since $\varphi \in \mathrm{Car}(T)$ it follows that

(4) $\qquad \varphi \circ x_n \longrightarrow \varphi \circ x \quad$ a.e.

In addition, the indefinite integrals of the sequence $\varphi \circ x_n \in L^1(T)$ are uniformly absolutely continuous, i.e.

(5) $\qquad \int_U |\varphi \circ x_n| d\mu \to 0 \quad$ as $\mu(U) \to 0$, uniformly in n.

Otherwise there would exist for some $a > 0$ a sequence of sets $U_m \subset T$ with $\mu(U_m) < 3^{-m}$ and a corresponding sequence $\varphi \circ x_{n_m}$ such that

$$\int_{U_m} |\varphi \circ x_{n_m}| d\mu > a.$$

It follows that each U_m (even if U_m is an atom) would possess a subset U'_m satisfying

$$\left| \int_{U'_m} (\varphi \circ x_{n_m}) d\mu \right| > a/2.$$

By (3) and the Vitalli convergence theorem ([J], p.150) the functions x_n form a bounded set in $L^p(T)$ and $\lim_{\mu(U) \to 0} \int_U |x_n|^p d\mu = 0$ uniformly in n. Hence the functions $y_m = x_{n_m} \chi_{U'_m}$ lie in a bounded subset of $L^p(T)$ and satisfy

$$y_m \longrightarrow 0 \qquad \text{in } L^p(T).$$

Moreover because y_m is a simple function,

$$F(y_m) = \int_T (\varphi \circ y_m) d\mu = \int_{U'_m} (\varphi \circ x_{n_m}) d\mu .$$

However by the construction of U'_m, this formula implies that the $F(y_m)$ do not converge to zero, contradicting (ii_p).

Furthermore the sequence $\varphi \circ x_n$ has the property that for each $\varepsilon > 0$ there exists a set U_ε such that $\mu(U_\varepsilon) < \infty$ and

(6) $$\int_{T-U_\varepsilon} |\varphi \circ x_n| d\mu < \varepsilon , \qquad \text{for all } n.$$

Otherwise for some $\varepsilon > 0$ there exists an expanding sequence of sets U_m with $\mu(U_m) < \infty$, $\bigcup_1^\infty U_m = T$, and a corresponding sequence $\varphi \circ x_{n_m}$ such that

$$\int_{T-U_m} |\varphi \circ x_{n_m}| d\mu > \varepsilon .$$

Thus (even if $T-U_m$ is an atom) for some $U''_m \subset T-U_m$

$$\left| \int_{U''_m} \varphi \circ x_{n_m} d\mu \right| > \varepsilon/2.$$

By (3) and the Vitalli convergence theorem the indefinite integrals of the functions x_n are equicontinuous with respect to μ, so that the functions $y_m = x_{n_m} \chi_{U''_m}$ satisfy

$$y_m \to 0 \quad \text{in} \quad L^p(T).$$

However, the formula

$$F(y_m) = \int_T (\varphi \circ y_m) d\mu = \int_{U''_m} (\varphi \circ x_{n_m}) d\mu$$

implies that the $F(y_m)$ do not converge to zero, contradicting (ii_p).

Since the sequence $\varphi \circ x_n$ satisfies (4), (5) and (6) it follows by Vitalli's convergence theorem that $\varphi \circ x$ is in $L^1(T)$ and that $\varphi \circ x_n \to \varphi \circ x$ in $L^1(T)$, whereby

$$F(x) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \int_T (\varphi \circ x_n) d\mu$$

$$= \int_T (\varphi \circ x) d\mu.$$

Thus (*) holds for all $x \in L^p(T)$. The uniqueness of φ assuming that (a) holds follows since Theorem 1 then asserts the uniqueness of $\varphi | T_i$, $i = 1,2,\ldots$.

For the converse we proceed as follows. Suppose φ is a function in $\mathrm{Car}^{p,1}(T)$ which satisfies (a). Then (i) obviously holds. Moreover for any S such that $\mu(S) < \infty$, the restriction $\varphi | S$ is in $Car^{p,1}(S)$. This implies in particular that $\varphi | S$ is in $Car^{\infty,1}(S)$ and satisfies (a). Thus the validity of (iii_p) follows from Theorem 1. On the other hand (ii_p) is a consequence of a theorem of Nemitskii [G] which asserts that every $\varphi \in Car^{p,1}(T)$ yields a continuous transformation from $L^p(T)$ to $L^1(T)$ by

$$x \longrightarrow \varphi \circ x.$$

For, the continuity of the above transformation yields as a by-product the continuity of the functional

$$x \to \int_T (\varphi \circ x) d\mu.$$

Q.E.D.

Corollary. With T as above, there exists for every real valued functional F on $L^p(T)$, $1 \leq p < \infty$, which satisfies the conditions

(i) $F(x+y) = F(x) + F(y)$ when $xy = 0$ a.e.,

(ii') F is uniformly continuous on each bounded subset of $L^p(T)$,

a function $\varphi \in Car^{p,1}(T)$ such that

$$(*) \quad F(x) = \int_T (\varphi \circ x) d\mu \qquad \text{for } x \in L^p(T).$$

Moreover φ can be taken to satisfy (a), and is then unique up to sets of the form $R \times N$ with $N \subset T$ a null set.

<u>Remark</u>: The converse to the corollary is false except for a space T consisting of a finite number of atoms. That is, φ being in $Car^{p,1}(T)$ and satisfying (a) does not in other cases ensure that (ii') holds. To see this let $T = \bigcup_{i=1}^{\infty} T_i$ where $0 < \mu(T_i) < \infty$ and T_i are disjoint. Then the function

$$\varphi(h,t) = \sum_{i=1}^{\infty} f_i(h) h^p \chi_{T_i}(t)$$

is in $Car^{p,1}(T)$ provided that each $f_i\colon R \to R$ is continuous and satisfies $|f_i| \le 1$. However it is easy to prevent <u>uniform</u> continuity on certain bounded sets in $L^p(T)$ by selecting the f_i to have appropriate zeros.

<u>Theorem</u> 3. <u>With</u> T <u>as in Theorem</u> 1 <u>let</u> A <u>be a transformation on</u> $L^{\infty}(T)$ <u>with values in</u> $C(S)$ <u>where</u> S <u>is a compact Hausdorff space</u>. <u>Suppose</u> A <u>satisfies the conditions</u>

(i_A) $A(x+y) = A(x) + A(y)$ when $xy = 0$ a.e.,

(ii_A) A is uniformly continuous on each bounded subset of $L^{\infty}(T)$,

(iii_A) $A(x_n) \to A(x)$ whenever $\{x_n\}_{n\ge 1}$ converges boundedly a.e. to $x \in L^{\infty}(T)$.

<u>Then there exists a</u> <u>transformation</u> $\Phi\colon S \to Car^{\infty,1}(T)$ <u>such that</u>

$$(*) \qquad A(x)(s) = \int_T (\Phi(s) \circ x)\, d\mu = \int_T \Phi(s; x(t), t)\, d\mu(t).$$

<u>The transformation</u> Φ <u>can be taken to satisfy</u>

(a) $\Phi(s) \circ 0 = 0$ a.e., for all $s \in S$, in which case $\Phi(s)$ is unique, for each s, up to sets of the form $R \times N$ with N a null set in T. Moreover Φ has the following additional properties,

(b) the mapping $s \mapsto \Phi(s) \circ x$ is weakly continuous for each $x \in L^{\infty}(T)$,

(c) the mapping $x \mapsto \Phi(s) \circ x \in L^{1}(T)$ is uniformly continuous on each bounded subset of $L^{\infty}(T)$, uniformly in s,

(d) if $x_n \rightarrow x$ boundedly a.e. then

(1) $\lim_{\mu(E) \rightarrow 0} \int_E (\Phi(s) \circ x_n) d\mu = 0$ uniformly in s and n,

(2) for any expanding sequence E_j such that $\cup E_j = T$

$$\lim_{E_j \uparrow T} \int_{T-E_j} (\Phi(s) \circ x_n) d\mu = 0 \quad \text{uniformly in } s \text{ and } n.$$

Conversely, every transformation $\Phi: S \rightarrow Car^{\infty,1}(T)$ satisfying (a), (b), (c) and (d) determines by means of (*) a transformation $A: L^{\infty}(T) \rightarrow C(S)$ satisfying (i_A), (ii_A), (iii_A).

Proof: If A satisfies (i_A), (ii_A), (iii_A) then for each fixed $s \in S$ the functional defined by

$$F_s(x) = A(x)(s)$$

satisfies (i), (ii), (iii). Hence by Theorem 1 there exists a unique element $\Phi(s) \in Car^{\infty,1}(T)$ satisfying (a) for which the representation

$$F_s(x) = A(x)(s) = \int_T (\Phi(s) \circ x) d\mu$$

holds. To show that (b), (c) and (d) are satisfied we proceed as follows. According to (i_A) and (iii_A) F_s determines for each $x \in L^\infty(T)$ a μ-continuous measure ν_x by means of

$$(1) \qquad \nu_x(G) = F_s(x \chi_G) = \int_T (\Phi(s) \circ x \chi_G) \, d\mu .$$

Using (a) we can rewrite this as follows:

$$(2) \qquad \nu_x(G) = \int_G (\Phi(s) \circ x) \, d\mu = A(x \chi_G)(s) .$$

Thus for any $x, y \in L^\infty(T)$ we have

$$(3) \qquad \int_G [\Phi(s) \circ x - \Phi(s) \circ y] \, d\mu = \nu_x(G) - \nu_y(G) = A(x\chi_G)(s) - A(y\chi_G)(s) .$$

Now the total variation of the signed measure $\nu_x - \nu_y$ is given by

$$(4) \qquad \mathrm{Var}(\nu_x - \nu_y) = \int_T |\Phi(s) \circ x - \Phi(s) \circ y| \, d\mu$$

$$= \sup_{G \in \Sigma} [A(x \chi_G)(s) - A(y \chi_G)(s)]$$

$$- \inf_{G' \in \Sigma} [A(x\chi_{G'})(s) - A(y \chi_{G'})(s)] .$$

However by (ii_A) we see that on each bounded subset B of $L^\infty(T)$ there exists for each $\varepsilon > 0$ a δ, independent of s, such that for $x, y \in B$, $\|x-y\|_\infty < \delta$ the right side of equation (4) is less than ε. This yields (c).

To show that (b) holds we observe first that, as a consequence of (c), for each $x \in L^\infty(T)$ the family

$$\mathcal{R}_x = \{\Phi(s) \circ x \mid s \in S\}$$

is a bounded subset of $L^1(T)$ (here $B = B_x = \{y \in L^\infty(T) \mid \|y\|_\infty \leq \|x\|_\infty\}$). Moreover, since A maps $L^\infty(T)$ into $C(S)$ we have for each

$E \in \Sigma$ that

$$(5) \quad \int_T (\Phi(s) \circ x \chi_E)\, d\mu = \int_T \chi_E (\Phi(s) \circ x)\, d\mu = A(x\chi_E)(s)$$

is continuous with respect to s. It then follows by (i_A) and (a) that

$$(6) \quad \int_T z(\Phi(s) \circ x)\, d\mu \in C(S)$$

for every simple function z. Since the simple functions are dense in $L^{\infty}(T) = L^1(T)'$ and R_x is a bounded subset of $L^1(T)$, it follows that (6) holds for all $z \in L^{\infty}(T)$, which yields (b).

To prove (d) we argue by contradiction. If (d_1) were false then there would exist a sequence x_n converging to x boundedly a.e. and a sequence of triples (E_m, s_m, x_{n_m}) with $\mu(E_m) < \frac{1}{m}$ such that for some fixed $\alpha > 0$,

$$(7) \quad \int_{E_m} (\Phi(s_m) \circ x_{n_m})\, d\mu > \alpha, \qquad m=1,2,\ldots .$$

By compactness of s we may assume without loss of generality that $s_m \longrightarrow s_o$. Moreover, by (iii_A) we have for each $G \in \Sigma$

$$(8) \quad \nu_{x_n, s_m}(G) = \int_G (\Phi(s_m) \circ x_n)\, d\mu = A(x_n \chi_G)(s_m) \rightarrow A(x\chi_G)(s_m) = \int_G (\Phi(s_m) \circ x)\, d\mu$$

as $n \to \infty$,

uniformly in m. The continuity of $A(x\chi_G)$ now implies that

$$(9) \quad \lim_{m,n\to\infty} \nu_{x_n, s_m}(G) = A(x\chi_G)(s_o) = \nu_{x, s_o}(G).$$

Therefore it follows by the Vitalli-Hahn-Saks Theorem ([J],p.158) that

$$(10)\qquad \lim_{\mu(G)\to 0} \nu_{x_n, s_m}(E) = \lim_{\mu(E)\to 0} \int_E (\Phi(s_m)\circ x_n)\, d\mu = 0$$

uniformly in m and n,

which contradicts (7). If (d_2) were false then there would exist a sequence x_n converging boundedly to x a.e. and a sequence of triples (E'_m, s_m, x_{n_m}), with $\{E'_m\}$ an expanding family in Σ whose union is T, such that for some fixed $\alpha > 0$

$$(11)\qquad \int_{T-E'_m} (\Phi(s_m)\circ x_{n_m})\, d\mu > \alpha, \qquad m=1,2,\ldots .$$

Again we may assume $s_m \to s_o$, so that (9) holds. Therefore it follows by Nikodym's corollary to the Vitalli-Hahn-Saks Theorem ([J],p.160) that

$$(12)\qquad \lim_{m\to\infty} \int_{T-E'_m} (\Phi(s_m)\circ x_{n_m})\, d\mu = 0 \quad \text{uniformly in } m \text{ and } n,$$

which contradicts (11).

For the converse we observe by Theorem 1 that (i_A), (ii_A) and $\mathcal{R}_A \subset C(s)$ all follow directly from (a), (b), and (c). To prove (iii_A) we observe that x_n converging to x boundedly a.e. implies by (d_2) that for each $\varepsilon > 0$ there exists a set E_ε, with $\mu(E_\varepsilon) < \infty$, such that

$$\int_{T-E} (\Phi(s)\circ x_n)\, d\mu < \varepsilon \qquad \text{uniformly in } s \text{ and } n.$$

Now bounded a.e. convergence of x_n to x implies that on the set E_ε this convergence is almost uniform. Hence by (d_1) there exists a subset $F_\varepsilon \subset E_\varepsilon$ such that

$$\left| \int_{F_\varepsilon} (\Phi(s)\circ x_n)\, d\mu \right| < \varepsilon \quad \text{uniformly in } n \text{ and } s$$

while the convergence of x_n to x on $E_\varepsilon - F_\varepsilon$ is uniform. Thus by (i_A)

$$|A(x_n)(s) - A(x)(s)| = \Big| \int_{T-E_\varepsilon} (\Phi(s) \circ x_n)\, d\mu - \int_{T-E_\varepsilon} (\Phi(s) \circ x)\, d\mu + \int_{F_\varepsilon} (\Phi(s) \circ x_n)\, d\mu$$

$$(13) \qquad - \int_{F_\varepsilon} (\Phi(s) \circ x)\, d\mu + \int_{E_\varepsilon - F_\varepsilon} (\Phi(s) \circ x_n - \Phi(s) \circ x) d\mu \Big|$$

$$\leq 4\varepsilon + \int_{E_\varepsilon - F_\varepsilon} |\Phi(s) \circ x_n - \Phi(s) \circ x|\, d\mu.$$

Then by (ii_A) we have for sufficiently large n that

$$(14) \qquad |A(x_n)(s) - A(x)(s)| \leq 5\varepsilon, \quad \text{uniformly in } s.$$

Since $\varepsilon > 0$ was arbitrary this yields (iii_A). Q.E.D.

We now give an analogue for $L^p(T)$, $1 \leq p < \infty$.

__Theorem__ 4. __With__ T __as in Theorem__ 1 __let__ A __be a transformation on__ $L^p(T)$ __with values in__ $C(S)$ __where__ S __is a compact Hausdorff space__. __Suppose__ A __satisfies the conditions__

(i_A) $A(x+y) = A(x) + A(y)$ when $xy = 0$ a.e.,

(ii_{Ap}) A is continuous on $L^p(T)$,

(iii_{Ap}) A is uniformly continuous relative to L^∞ norm on each bounded subset of $L^\infty(T)$ which is supported by a set of finite measure.

__Then there exists a transformation__ $\Phi: S \to Car^{p,1}(T)$ __such that__

$$(*) \qquad A(x)(s) = \int_T (\Phi(s) \circ x)\, d\mu$$

__The transformation__ Φ __can be taken to satisfy__

(a) $\Phi(s) \circ 0 = 0$ a.e. for all $s \in S$, in which case $\Phi(s)$ is unique, for each s, up to sets of the form $R \times N$ with N a null set in T. Moreover Φ has the following additional properties,

(b_p) the mapping $s \mapsto \Phi(s) \circ x$ is weakly continuous for each $x \in L^p(T)$

(c_p) the mapping $x \mapsto \Phi(s) \circ x$ is weakly continuous on $L^p(T)$, uniformly in s

(d_p) the mapping $x \mapsto \Phi(s) \circ x$ is uniformly continuous (relative to L^∞ norm), uniformly in s, on each bounded subset of $L^\infty(T)$ which is supported by a set of finite measure.

Conversely, every transformation $\Phi: S \to Car^{p,1}(T)$ satisfying (a), (b_p), (c_p) and (d_p) determines by means of (*) a transformation $A: L^p(T) \to C(S)$ satisfying (i_A), (ii_{Ap}), (iii_{Ap}).

Proof: If A satisfies (i_A), (ii_{Ap}), (iii_{AP}) then for each fixed $s \in S$ the functional defined by

$$F_s(x) = A(x)(s)$$

satisfies (i), (ii_p), (iii_p). Hence by Theorem 2 there exists a unique element $\Phi(s) \in Car^{p,1}(T)$ satisfying (a) for which the representation

$$F_s(x) = A(x)(s) = \int_T (\Phi(s) \circ x) \, d\mu$$

holds. To show that (b_p), (c_p) and (d_p) hold we proceed as follows. According to (i_A) and (ii_{Ap}) F_s determines for each $x \in L^p(T)$ a μ-continuous measure ν_x by means of

$$(1) \qquad \nu_x(G) = F_s(x\chi_G) = \int_T (\Phi(s) \circ x\chi_G) \, d\mu.$$

Using (a) we can rewrite this as follows

(2) $$\nu_x(G) = \int_G (\Phi(s)\circ x)\,d\mu = A(x\chi_G)(s).$$

Thus the variation of the signed measure ν_x is given by

(3) $$\mathrm{Var}(\nu_x) = \int_T |\Phi(s)\circ x|\,d\mu$$
$$= \sup_{G\in\Sigma} A(x\chi_G)(s) - \inf_{G'\in\Sigma} A(x\chi_{G'})(s).$$

We now show that for each x the right side of equation (3) is bounded. Since x is in $L^p(T)$ we deduce by equicontinuity of its indefinite integral that corresponding to each ε there is a set $E_\varepsilon \in\Sigma$, $\mu(E_\varepsilon) < \infty$, such that $\|x\chi_{T-E_\varepsilon}\| < \varepsilon$. We can require without loss of generality that E_ε contain at most finitely many atoms, $E_1,\ldots,E_{n_\varepsilon}$. Moreover by absolute continuity of the indefinite integral of x there exists a δ such that

(4) $$\|x\chi_E\|_p < \varepsilon \quad \text{whenever} \quad \mu(E) < \delta.$$

Now by (ii_{Ap}) A is continuous at $0\in L^p(T)$. Hence on taking ε sufficiently small we deduce that

(5)
$$|A(x\chi_F)(s)| \le 1, \text{ uniformly in } s, \text{ whenever } \mu(F) \le \delta,$$
$$|A(x\chi_F)(s)| \le 1, \text{ uniformly in } s, \text{ whenever } F \subset T-E_\varepsilon.$$

Now for any $G\in\Sigma$ we have by (i_A)

$$|A(x\chi_G)(s)| = |A(x\chi_{G\cap(T-E_\varepsilon)})(s) + \sum_{i=1}^{n_\varepsilon} A(x\chi_{G\cap E_i})(s) + A(x\chi_{G\cap(E - \overset{n_\varepsilon}{\underset{1}{\cup}} E_i)})(s)|$$

$$(6) \qquad \leq 1 + \sum_{i=1}^{n_\varepsilon} |A(x\chi_{E_i})(s)| + |A(x\chi_{G\cap(E_\varepsilon - \overset{n_\varepsilon}{\underset{1}{\cup}} E_i)})(s)|.$$

Moreover by splitting the nonatomic subset $E_\varepsilon - \overset{n_\varepsilon}{\underset{1}{\cup}} E_i$ into parts of measure less than δ and applying (5_1) we obtain the estimate

$$(7) \qquad |A(x\chi_{G\cap(E_\varepsilon - \overset{n_\varepsilon}{\underset{1}{\cup}} E_i)})(s)| \leq \frac{\mu(E - \overset{n_\varepsilon}{\underset{1}{\cup}} E_i)}{\delta} + 1 \leq \frac{\mu(E_\varepsilon)}{\delta} + 1.$$

Combining (6) and (7) we deduce that

$$(8) \qquad |A(x\chi_G)(s)| \leq 2 + \frac{\mu(E_\varepsilon)}{\delta} + \sum_{i=1}^{n_\varepsilon} \|A(x\chi_{E_i})\|_\infty \equiv M_x, \text{ uniformly in s and G.}$$

Therefore by equation (3) it follows that the set

$$(9) \qquad B_x = \{\Phi(s)\circ x\chi_E \,|\, E\in\Sigma,\ s\in S\}$$

is a bounded subset of $L^1(T)$.

Now since A takes $L^p(T)$ into $C(S)$ we have for each $E\in\Sigma$ that

$$(10) \qquad \int_T (\Phi(s)\circ x\chi_E)\,d\mu = \int_T \chi_E(\Phi(s)\circ x)\,d\mu = A(x\chi_E)(s)$$

is continuous with respect to s. It then follows by (i_A) that

$$(11) \qquad \int_T z(\Phi(s)\circ x)\,d\mu \text{ is in } C(S)$$

for every simple function z. Since the simple functions are dense in $L^\infty(T) = L^1(T)'$ and B_x is a bounded subset of $L^1(T)$, it follows that (11) holds for all $x \in L^{\infty}(T)$, which yields (b_p).

To show that (c_p) holds let $\{x_n\}_{n \geq 1}$ denote a sequence converging to $x = x_o$ in $L^p(T)$. Then the indefinite integrals of the $\{x_n\}_{n \geq 0}$ are uniformly absolutely continuous and equi-continuous with respect to μ. Hence it follows by the technique used in deriving equation (8) that

$$B_{\{x_n\}} = \{\Phi(s) \circ x_n \chi_E \mid E \in \Sigma,\ s \in S,\ n \geq 0\}$$

is a bounded subset of $L^1(T)$.

Now for each $E \in \Sigma$, $x_n \chi_E$ converges to $x \in \chi_E$ in $L^p(T)$ and hence by (ii_{Ap}) we have

$$(12) \qquad \int_T (\Phi(s) \circ x_n \chi_E)\, d\mu = \int_T \chi_E (\Phi(s) \circ x_n)\, d\mu \rightarrow \int_T \chi_E (\Phi(s) \circ x)\, d\mu,$$

uniformly in s.

It then follows by (i_A) that

$$(13) \qquad \int_T z(\Phi(s) \circ x_n)\, d\mu \rightarrow \int_T z(\Phi(s) \circ x)\, d\mu \quad \text{uniformly in } s,$$

for every simple function z. Since the simple functions are dense in $L^{\infty}(T) = L^1(T)'$ and $B_{\{x_n\}}$ is a bounded subset of $L^1(T)$, it follows that (13) holds for all $z \in L^{\infty}(T)$, which yields (c_p). Finally, the transformation $\wedge A_1 = A|L^{\infty}(E)$, for any E such that $\mu(E) < \infty$, satisfies (i_A), (ii_A) and (iii_A) of Theorem 3, the last following from (ii_{Ap}) by virtue of the Lebesgue dominated convergence theorem. Therefore (d_p) is a consequence of Theorem 3.

The converse is immediate. Q.E.D.

Remark: Theorems 3 and 4 are well known in the linear case ([J], p.490.)

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