COMPLETE SETS OF EXPECTATIONS ON VON NEUMANN ALGEBRAS

by

Andre de Korvin

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Complete sets of Expectations on von Neumann Algebras

1) Introduction.

The first theorem shown in this paper states:

If N is a von Neumann subalgebra of M and if M has enough states which diagonalize N and if the restrictions of these states to the center of N are normal, then there will exist sufficiently many expectations from M to N. If the states are normal, the expectations will be normal. This result is an improvement of a result contained in [5]. In [5] the following is shown: if M is a finite countably decomposable von Neumann Algebra and if cr is a normal state which diagonalizes the subalgebra N of M then there exists an expectation $<p_\sigma$ invariant under cr. A corollary of this theorem is the equivalence of the following two statements:

(1) N is finite and there exists an expectation <p of £(h) on N such that $<p(UXU^{-1}) = <p(X)$ for all X in £(h) and all unitaries U in the commutant N* of N.

(2) The commutant of N is finite and there exists an expectation <p' of £(h) on N' such that $\wedge(VXV^{*1}) = <p'(X)$ for all X in £(h) and all unitaries V of N.

The second theorem states: let M be a von Neumann subalgebra of £(h), then if there exists enough expectations of X(h) on M, then M is atomic. Conversely if M is atomic in £(h) then there exists enough normal expectations of £(h) on M. A corollary of this result is that if G is a maximal abelian self-adjoint algebra of £(h)9 then there exists a faithful expectation of £(h) onto G if and only if G is totally atomic. A result by Sakai [3] and by Tomiyama [4] are necessary to carry out the proof of the above result. Another corollary to the result is that if M is a von Neumann subalgebra of £(h) onto M then there exist enough expectations of £(h) onto the commutant of M.

2) Definitions.

In this paragraph we shall give some of the basic definitions. For these definitions see [1] and [4]. Let G be a von Neumann subalgebra of the von Neumann algebra ft. An expectation $ of ft on G will be a linear positive map from ft on G such that $ preserves the identity and such that $(AB) = A$(ft) for all A in G and B in ft. $ will be faithful if Φ(T) = 0 and T positive implies T = 0. $ will be called normal if $(sup$_\alpha$ A_α) = sup$_\alpha$ <I>(A_α) where A_α are self-adjoint operators uniformly bounded in norm. Let $\$_\alpha$ be a set of expectations from ft on G such that if T is positive and if a_α(T) = 0 for all *a* then T = 0_9 the set $\$_\alpha$ will then be called complete. A state will be a linear functional of norm one. The above definitions of faithfullness, normality and completeness carry over for states.

Finally if N is a von Neumann subalgebra of the von Neumann algebra M and if p is a state of M, p will diagonalize N if p(nm) = p(mn) for all m in M and all n in N. (For example if an algebra of nxn matrices G_n has a diagonal representation respectively to some orthonormal basis XpX$_2$,...,x_n.

Then the states $\omega_{x_i}(A) = (Ax_i, x_i)$, diagonalize G_n in the above sense.)

3) The Diagonalization Theorem.

N will denote a von Neumann subalgebra of the von Neumann algebra M. Z will denote the center of N. If G is any von Neumann algebra and P is a projection in G or G' (the commutant of G) then G_P will denote the von Neumann algebra containing all operators of the form PAP, $A \in G$.

Lemma 1. Let C_α be a family of projections in Z. Assume that C_α are orthogonal and that $\Sigma C_\alpha = I$. Let Φ_α be a family of expectations of M on N_{C_α}. For each X in M define $\Phi(X) = \Sigma\Phi_\alpha(C_\alpha X C_\alpha)$. Then Φ is an expectation of M on N. If all Φ_α are normal then so is Φ.

Proof. Clearly Φ is a map of M on N. If X is positive then so is $C_\alpha X C_\alpha$ so $\Phi_\alpha(C_\alpha X C_\alpha)$ is positive and so $\Phi(X)$ is positive

$$\Phi(I) = \Sigma\Phi_\alpha(C_\alpha) = \Sigma C_\alpha = I$$

Let T be in N then $\Phi(TX) = \Sigma\Phi_\alpha(C_\alpha T X C_\alpha)$

$$= \Sigma C_\alpha T\Phi_\alpha(C_\alpha X C_\alpha) = (\Sigma C_\alpha T)(\Sigma\Phi_\alpha(C_\alpha X C_\alpha))$$

$$= T\Phi(X).$$

Hence Φ is an expectation.

Lemma 2. Suppose ρ_α is a collection of states which satisfy the following conditions:

1) Each ρ_α diagonalizes N

2) Each p_α restricted to Z is normal

3) The set of p_α is complete

then N is finite and $p_\alpha(A) = p_\alpha(A^\natural)$ for all A in N ($A^\natural$ denotes the unique point in the intersection of C(A) with Z where C(A) denotes the norm closure of the convex hull generated by UAU* as U ranges over the unitaries of N. This map is defined in [2].)

Proof. Let U be any unitary in N, let X be any element of N. Then $p_\alpha(UXU^{-1}) = p_\alpha(XUU^{-1}) = p_\alpha(X)$ for all α. Let C(X) be the norm closure of the convex hull of UXU* as U ranges over unitaries of N. By continuity p_α is constant on C(X). By [2] C(X) ∩ Z is never void, and it is sufficient to show that it reduces to only one point.

Let T be in C(X) ∩ Z. Let A be in Z. T is limit in the norm sense of elements of type $\sum_{i=1}^{n} a_i U_i X U_i^*$ where $a_i > 0$, $\sum_{i=1}^{n} a_i = 1$ and U_i are unitaries of N. So AT is limit in the norm sense of elements of the type

$$\sum_{i=1}^{n} a_i A U_i X U_i^* = \sum_{i=1}^{n} a_i U_i A X U_i^*,$$

so AT is in C(AX) ∩ Z. p_α is constant on C(AX) so

$$p_\alpha(AT) = p_\alpha(AX) \quad \text{for all } \alpha \text{ and all } A \text{ in } Z.$$

Let T_1 be another element of C(X) ∩ Z then

$$p_\alpha(AT) = p_\alpha(AX) = p_\alpha(AT_1)$$

so $p_\alpha[A(T-T_1)] = 0$ for all A in Z in particular if $A = (T-T_1)^*$

then by 3) $T = T_{\pm}$.

Also Pyy(A) =• P/y(A) • Note that since ^ is normal, it follows that p_α are normal on N.

Lemma 3• Under the above hypothesis for each *a* there exists a map $\$_a$ such that $P_a(AX) = P_a(A\hat{}(X))$ for all A in N and all X in M.

Proof, Each p_α restricted to N is a normal, finite trace. Let E_α be the carrier of p_α restricted to N. Then E_α is in Z and $p\hat{}_a$ is a finite, normal, faithful trace on N_{E_a}. N_{E_a}, $p\hat{}_a$ forms a Hilbert algebra with $(A,B) = p\hat{}_a(AB\hat{})$ • Let X be a positive operator in M and define $a_a(b) = P_\alpha(AX)$, $[A,B] = a_a(AB^*)$ for all A and B in N_{E_a}. Then [,] is a bounded hermitian form on N_{E_a} $>P_a{}^*$ In fact a_a is a positive linear functional dominated by $||x||p_\alpha$. By Riez lemma there exists a bounded operator $\$_a(X)$ defined on the completion of N_{E_a} respectively to (,) such that

$$[A,B] = (\$_a(X)(A),B) .$$

For any D in $N\hat{}_{E_a}$, let R_v be an operator defined on N_{E_a} by $R_D(T) = TD$ then it is easy to check that ^^(X) = ^(X)!^. By the commutation theorem [ ], $\$_\alpha(X)$ acts on N_{E_a}. by premultiplication (on the left) by some element which we designate by the same symbol ^y(X) . Then

$$\Phi_\alpha(x) \cdot A = *_a rx)(A) .$$

$P_a(AB^*X) = [A,B] = (p_a(X)\ A,B) = \bullet P_a(*_i(X)\ AB^*)$ as $N\hat{}_\alpha = N_{E_\alpha}$

$\rho_\alpha(AX) = \rho_\alpha(A\,\Phi_\alpha(X))$ for all A in N_{E_α}.

So $\rho_\alpha(AX) = \rho_\alpha(A\Phi_\alpha(X))$ for all A in N and all X in M and X positive. Now extend Φ_α to M, the above relation still holds.

Theorem 1. Under the above hypothesis there exists a complete set of expectations from M to N. If all ρ_α are normal in M, these expectations are normal.

Conversely if N is finite and if there exists a complete set of expectations of M on N, then there exists a set of states of M satisfying (1), (2), (3) of Lemma 2.

Proof. Consider the maps Φ_α of M on N_{E_α} of the previous lemma. When restricted to E_α, Φ_α is uniquely defined by $\rho_\alpha(AX) = \rho_\alpha(A\Phi_\alpha(X))$. The Φ_α are expectations of M on N_{E_α}. For example let us check that $\Phi_\alpha(TX) = T\Phi_\alpha(X)$ for all T in N_{E_α} and all X in M. Let A be any operator in N_{E_α} then

$$\Phi_\alpha(AT\Phi_\alpha(X)) = \rho_\alpha(A\Phi_\alpha(TX)).$$

Then $\rho_\alpha[A(T\Phi_\alpha(X) - \Phi_\alpha(TX)] = 0$ for all A in N_{E_α}. Let $A = [T\Phi_\alpha(X) - \Phi_\alpha(TX)]^*$ by faithfulness of ρ_α on N_{E_α}

$$T\Phi_\alpha(X) = \Phi_\alpha(TX).$$

The other axioms are checked out the same way.

By completeness of the ρ_α, $\sup E_\alpha = I$. One can pick (in many ways) orthogonal projections C_α in Z where $C_\alpha \le E_\alpha$ and $\Sigma C_\alpha = I$. For any such choice of C_α consider $\Phi(X) = \Sigma_\alpha \Phi_\alpha(C_\alpha X C_\alpha)$. By Lemma 1 Φ is an expectation of M on N. To each choice of C_α

we have associated an expectation $\$$. Now the claim is that one obtains a complete set of $\$$ (as one ranges over possible choices of C^) . Let X > 0, X ^ 0, X€M. If 0^(X) = 0 for all *a,* then $p_\alpha(X) = 0$ so by completeness of the p_α X = 0, a contradiction. Assume then that for a_{o} $\$_{\alpha_o}(X)$ ^ 0. Choose $C_{\alpha_o} = E_{\alpha_o}$ > then construct the C_α for a ^ a_o. Clearly then the corresponding $\$$ will verify $\$(X)$ / 0.

If all p_α are normal, then all $\$_\alpha$ are normal so all $\$$ are noraml. Conversely if N is finite then there exists a complete set of normal finite traces, all this set Tr^* Let *$g* be the complete set of expectations of M on N. Define p^$_{,\beta}$(x) = Tr$_a$($\$_\beta$(X)). Normalize the p^$_{,\beta}$. It follows trivially that the normalized p$_R$ satisfy (1), (2), (3) of Lemma 2.

Corollary 2.

If p_α is faithful for some *a,* then $\$_\alpha$ for that *a* is a faithful expectation.

Proof. If *$\$_a(X^*X)$* =0 then p^(X*X) = *p^Q^iX+X))* =0 so X = 0.

Remark. Let p be a state of M such that

(1) p restricted to Z is faithful and normal

(2) p diagonalizes N,

then there exists an expectation * of M on N such that p(AX) = p(A$(X)) for all X in M and A in N. Lemma 3 would go through replacing p_α by p and $\$_\alpha$ by <E>. Of course E_α would be replaced by I.

Corollarv 3.

Let N be a von Neumann subalgebra of £(h) . Assume that Z is countably decomposable, the following conditions are equiv-alent:

(1) N is finite and there exists an expectation $ of **£(h)** on N such that ^UXU""^{1}) = $(X) for all X in £(h) and all unitaries U of N^{T} (N^{1} = commutant of N) .

(2) N$^{!}$ is finite and there exists an expectation $» of £(h) on N' such that $' (VXV"^{1}) = $(X) for all X in £(h) all unitaries V of N.

Proof. To show (1) implies (2). (The same argument will show that (2) implies (1) .)

By [2] the following statements are true: N is countably decomposable, there exists on N a finite, faithful, normal trace, call it Tr. Define

p(X) =Tr($(X)) for all X in £(h) .

As $(UXU~^{1}) = $(X) for all unitaries U in N$^{!}$, hence $(A'X) = $(XAt) for all A* in N^{1} . Now p(A*X) = Tr$(XA») **= Φ(XA') .**

Also p(AX) = Tr$(AX) =Tr(A$(X)) =Tr($(X)A) =Tr($(XA)) = p(XA) for all A in N. Moreover p restricted to Z is nor-mal and faithful as Tr is. By the above remark, there exists an expectation from £(h) onto N$^{!}$ such that p(A$^{!}$X) = p(A«$t(X)). Let V be a unitary in N

p(A'$» (VXV""^{1})) = pfA'VXV"^{1}) = p(A'X) =. p(A'$t(X)).

It was shown in Lemma 3 that p is normal and faithful over all of N' . If one lets A» = ($» (VXV $^{-1}$) - $< (X))* then $' (VXV"^{1}) = <S' (X) .

Application.

Let G be a non discrete locally compact group. Let m be its Haar measure. Let h = L^2 (G,m). Then there does not exist any normal expectation of £(h) onto the multiplication algebra M.

First to show that no state of £(h) exists which is normal and diagonalizes M. Let x be in G, let t_x be defined on h by I f(-) = f(x" .) for each f in h. Let p be a state of £(h) which diagonalizes M. Pick an orthogonal family E of subsets of G_α such that U E = G, m(E)< oo and E_α xE^ is disjoint from E . (x ^ identity of G) . Then if ij) is the characteristic function of E_α ρ( ) = Σ ρ(ψ ℓ ψ

by normality of p.

So pU) = L p(il) IJ)) = E p(0 if) I) = 0.

Since G is non discrete there exists a net x_α which converges to e. So I converges strongly to I. So p(I) = 0 a contradiction.

Now no normal expectation $ could exist or else if Tr is a finite normal trace on M, p(X) = Tr$(X) would be a normal state diagonalizing M.

4) Complete sets of expectations from £(h) to a von Neumann subalgebra.

Theorem £. Let M be a von Neumann subalgebra of £(h), then if there exists a complete set of expectations of £(h) on M, M is atomic. Conversely if M is atomic in *Z(h),* there exists a complete set of normal expectations of £(h) on M.

Proof. Assume there exists a complete set of expectations of £(h) on M. Call this set $\$_\alpha$. Let A be a compact positive operator in £(h) with A ^ 0. Then for some *a* $\$_{\alpha_O}$(A) ^ 0. Pick a projection P in M such that P<$\$_{\alpha_O}$(A) = $\$_{\alpha_O}$(A) P and $\$_{®o}$(A) > 6P for 5 > 0. Now $\$\&_{o}$(M_A) = M-P&v (A) P. Let T be in Mp, since $M_p^2 = M_p$ the element T (P$ao (A) P)$^{-1}$ in in M_p. Hence T Is in Φ_{α_O} (M_pA) i.e. M_p c Φ_{a_O} (M_FA) . M_FA as a subset of compact operators is norm separable. So $\$_{\alpha_O}$ (M_pA) is norm separable as $\$^\alpha$ is norm continuous. By [3] this implies that M_p is finite dimensional as an algebra. Hence there exists a minimal projection Q in M, Q <^ P.

Now let C be a central projection of M such that M^C is non atomic. Let X be any compact positive operator in £(h). Then *(X(CXC) > 6Q for some 6 > 0, where Q is a minimal projection in M_C. The only minimal projection in M_C is 0 so $\$_\alpha$(CXC) = 0 for all *a.* So CXC = 0 for all X compact positive in £(h). So C = 0 and M is atomic.

Now to show the converse. Let M be atomic i.e., M is a direct sum of factors M_a of type I in £(C_a h) (C_a eZ) . Assume first that M is a factor of type I in £(h) . A result by J. Tomiyama [4] states:

Let $N_i \subset M_i$, $i=1,2$ be von Neumann algebras. Let π_i be a normal expectation of M_i on N_i. Then $\pi = \pi_1 \otimes \pi_2$ is a normal expectation of $M_1 \otimes M_2$ on $N_1 \otimes N_2$. From the proof of this result it is clear that if $\pi_1^{(\alpha)}$, $\pi_2^{(\alpha)}$ form complete sets of expectations then $\pi^{\alpha} = \pi_1^{(\alpha)} \otimes \pi_2^{(\alpha)}$ form a complete set of expectations from $M_1 \otimes M_2$ to $N_1 \otimes N_2$. Now M as a factor of type I is isomorphic to $\mathcal{L}(h_1)$ for some h_1.

Let P be a minimal projection of M'. Let $h_2 = P(h)$. Then it has been shown [2] that $\mathcal{L}(h)$ is isomorphic to $\mathcal{L}(h_1) \otimes \mathcal{L}(h_2)$. Take $M_i = \mathcal{L}(h_i)$ $i=1,2$. $N_1 = \mathcal{L}(h_1)$, $N_2 = CI_{h_2}$ (scalar operators on h_2). Let π_1 be the identity. Let ρ be a normal state of $\mathcal{L}(h_2)$ (there exists a complete set of these) and define $\pi_2(B) = \rho(B) I_{h_2}$ for all B in $\mathcal{L}(h_2)$. So

$$\pi(A \otimes B) = A \otimes \rho(B) I_{h_2} = \rho(B)(A \otimes I_{h_2}).$$

As ρ ranges over all normal states of $\mathcal{L}(h_2)$, π ranges over a complete set of expectations of $M_1 \otimes M_2$ on $N_1 \otimes N_2$. Of course $N_1 \otimes N_2$ is isomorphic to M. The following has been shown:
When M is a factor of type I in $\mathcal{L}(h)$ there exists a complete set of expectations of $\mathcal{L}(h)$ on M.

Now in general M is the direct sum of M_{C_α} where M_{C_α} is a factor of type I in $\mathcal{L}(C_\alpha h)$. For a fixed α let $\phi_\alpha^{(\beta)}$ be a complete set of expectations of $\mathcal{L}(C_\alpha h)$ on M_{C_α} then

$$\Phi^{(\beta)}(X) = \sum_\alpha \phi_\alpha^{(\beta)}(C_\alpha X C_\alpha)$$

is a set of expectations from $\mathcal{L}(h)$ on M by Lemma 1. Now let G be a positive operator in $\mathcal{L}(h)$. Say $\Phi^{(\beta)}(G) = 0$ for all β, then $\Phi_\alpha^{*P}(G) = 0$. As Φ_α form a complete set of expectations onto $M_{v*\alpha}$, then $C_\alpha G C_\alpha = (C_\alpha G^{\frac{1}{2}})(C_\alpha G^{\frac{1}{2}})^*$ as $\Sigma C_\alpha = I$ this implies $G = 0$, so $\{\Phi^{*p}\}$ forms a complete set.

Corollary 4.

Let M be a von Neumann subalgebra of £(h) and assume that there exists a complete set of expectations of £(h) on M. Then there exists a complete set of expectations of £(h) on M* .

Proof. By the above theorem it is sufficient to show that M' is atomic. Since M is atomic, M is of type I and hence [2] M* is of type I. Hence any non zero projection in M' dominates an abelian non zero projection of M. Let P be an abelian projection of M'. Let Q be a projection in M*, Q ≤ P, then Q = PC where C is a central projection. But the center Z of M is finite and atomic, so C dominates a non zero minimal projection of Z. So P dominates an atomic projection, i.e., M' is atomic.

Corollary 5.

Let M be a maximal abelian self adjoint von Neumann subalgebra of £(h) . There exists a faithful expectation of £(h) on M if and only if M is atomic.

Proof. If there exists a faithful expectation of £(h) on M, that expectation forms a complete set hence M is atomic by Theorem 2. Conversely suppose M is atomic. Let P_α be minimal orthogonal projections in M such that $\Sigma P_\alpha = I$. Put $\Phi(X) = \Sigma P_\alpha X P_\alpha$ in the strong topology ($X \in$ £(h)). Clearly $P_\alpha \Phi(X) = \Phi(X) P_\alpha$ for all a. But the P_α generate M so

$(X)eM'A£(h) = M. Trivially now it can be shown that $ is an expectation of £(h) on M. $ is faithful: indeed if <|>(T*T) = 0 then $P_a T^* T P_a = 0$ for all a i.e., (TP^) * $(TP_a) = 0$ so $TP_\alpha = 0$ so T = O.

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