EXISTENCE, UNIQUENESS AND STABILITY OF SOLUTIONS OF THE EQUATION

$$u_{tt} = \frac{\partial}{\partial x} (\sigma(u_x) + \lambda(u_x)u_{xt}) \ .$$

by

R. C. MacCamy

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1. Introduction and Statement of Results.

This paper contains an investigation of the partial differential equation

$$\text{(E)} \qquad u_{tt} = \frac{\partial}{\partial x}(\sigma(u_x) + \lambda(u_x)u_{xt}) \ .$$

The study was begun in [1] where it was assumed that λ was a positive constant. Here we allow λ to depend on u_x. This seemingly simple change leads to considerable difficulty since the equation then becomes quasi-linear rather than semi-linear.

Equations of the form (E) arise in the theory of elasticity, more precisely in what is called linearly-viscous elasticity ([2] and [3]). They result from applying what is called the slow-flow approximation to more general theories in which stress is a functional of the past history of the strain. We remark that (E) includes the equation of one-dimensional, compressible, viscous flow of a gas (formulated in so-called Lagrange co-ordinates). In the physical context it is not consistent to assume λ is a constant while allowing σ to depend nonlinearly on u_x. Thus, physically, it is much more realistic to allow λ in (E) to depend on u_x.

We study (E) with the additional conditions,

(A) $u(0,t) = u(1,t) = 0$,

(B) $u(x,0) = f(x)$,

(C) $u_t(x,0) = g(x)$.

The results here parallel those of [1] in that we establish the uniqueness, existence and stability of solutions of the problem defined by (E), (A), (B), (C) .

We assume throughout that f and g belong to $C^4[0,1]$ and $C^2[0,1]$ respectively. We impose on the functions σ and λ the following conditions:

(*) $\sigma(0) = 0$* $\quad 0 < \sigma_0 \leq \sigma'(\xi) \leq \sigma_1 \quad |\sigma''(\xi)| \leq \sigma_2$ for all ξ,

(**) $0 < \lambda_0 \leq \lambda(\xi) \leq \lambda_1; \quad |\lambda'(\xi)| \leq \lambda_2$ for all ξ,

where $\sigma_0, \sigma_1, \sigma_2, \lambda_0, \lambda_1,$ and λ_2 are constants.

We follow the notation and terminology of [1]. For functions Λ and Ω which are in $C^2[0,1]$ we set

(1.1) $$J(\Lambda,\Omega) = \sum_{1=0}^{2} \max_{x\in[0,1]} |\Lambda^{(i)}(x)| + \max_{x\in[0,1]} |\Omega^{(i)}(x)| .$$

For any positive number T we denote by S_T the strip.

$$S_T = \{(x,t)|0 \leq x \leq 1, \quad 0 \leq t \leq T\} .$$

Then if $U \in C^2(S_T)$ we write

(1.2) $$|||U|||(t) = \sum_{i=0}^{2} \sum_{\substack{k=0 \\ (i,k)\neq(2,2)}}^{i} \max_{x[0,1]} \left|\frac{\partial^i U}{\partial x^{i-k}\partial t^k}\right| .$$

An (f,g) <u>displacement</u> in S_T will be any function u such that:

(i) all derivatives appearing in (E) are continuous in S_T.

(ii) u satisfies (E) in $(0,1)\times(0,T)$ and conditions (A), (B), (C). The two main theorems are then as follows:

<u>Theorem</u> (1). <u>There exists</u> M <u>such that for any</u> (f,g) <u>displacement</u> u,

(1.3) $$|||u|||(t) \leq M$$

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*Note that this condition can be assumed without loss of generality.

on any S_T. *The constant* M *depends only on* $J(f,g),\sigma_0,\sigma_1,\sigma_2,$ $\lambda_0,\lambda_1,\lambda_2$ *and for fixed* $(\sigma_0,\sigma_1,\sigma_2,\lambda_0,\lambda_1,\lambda_2)$ M *tends to zero as* J *tends to zero*. *Moreover any* (f,g) *displacement satisfies the conditions*

$$(1.4) \qquad \lim_{t \to \infty} |||u||| = 0 .$$

Theorem (2). *There exists a unique* (f,g) *displacement in* S_T *for any* T *provided that* $J(f,g)$ *is sufficiently small*.

Remarks: 1. Theorems (1) and (2) are weaker than the corresponding ones in [1] in two respects. In [1] the expression $|||U|||(t)$ contained also U_{tt}. It would be possible to obtain the results (1.3) and (1.4) for u_{tt} in this paper also but this requires some additional computations which we do not include.

The second difference is more serious. In [1] we obtained the results under the sole assumption that σ' was positive. Thus it sufficed simply to assume that (*) holds on any compact subset and our results were independent of the size, $J(f,g)$, of the initial data. Suppose we replace (*) and (**) by the conditions:

$(*)'$ $\sigma(0) = 0 \qquad \sigma'(\xi) > 0 \qquad$ for all ξ,

$(**)'$ $\lambda(\xi) > 0 \qquad\qquad\qquad$ for all ξ .

Then conditions (*) and (**) will hold on any set $|\xi| \leq p$, with some fixed constants $\sigma_0,\sigma_1,\sigma_2,\lambda_0,\lambda_1,\lambda_2$. Theorem (1) states that we can choose $J(f,g)$ so small that u_x will remain in the interval $|u_x| \leq p$ so that (*) and (**) will in fact be satisfied.

2. Physical considerations require that in the applications of our equation in elasticity we must have $u_x > -1$ (positivity of

the density). As in Remark 1 we point out that we can always insure this condition by making $J(f,g)$ sufficiently small.

The proof of Theorem (1) is similar to that in [1]. It proceeds by a combination of energy estimates, obtained in Section three, and the use of results for linear parabolic equations. The latter presents considerable technical difficulty due to the presence of u_x in λ. The computations are presented in Section four. In sections five and six we give the existence proof which is somewhat more complicated than that of [1]. Section two contains the necessary results for linear parabolic equations. These will be used in the proofs of both theorems.

<u>Note</u>: After completion of this work the author's attention was called to a paper by Constantine M. Dafermos. This paper has been submitted to the Journal of Differential Equations. Dafermos considers a more general equation of the form,

$$u_{tt} = \frac{\partial}{\partial x} \sigma(u_x, u_{xt}) \quad .$$

The function σ satisfies the conditions,

$$\sigma_q(p,q) \geq K \ , \qquad |\sigma_p(p,q)| \leq N\sqrt{\sigma_q(p,q)} \ .$$

The methods are related to ours but somewhat different since Dafermos studies the equation with boundary conditions $\sigma = 0$ at $x = 0$ and 1. His stability results are somewhat weaker than the ones presented here; this reflects the stronger hypotheses we have made on σ.

2. Estimates for Linear Parabolic Equations.

In this section we collect some results which will be necessary for further work. These consist of a collection of estimates for linear parabolic equations. We need a number of such estimates with increasing restrictions on the coefficients in order that we can follow a step by step procedure to the desired 'a priori' bounds of Theorem 1.

The first result is essentially obtained in [4]. It concerns the problem,

$$\text{(P.1)}\qquad \begin{aligned} &v_t = \frac{\partial}{\partial x}(av_x + b), \qquad 0 < x < 1 \quad t > 0, \\ &v(0,t) = v(1,t) = 0, \\ &v(x,0) = \psi(x). \end{aligned}$$

It is assumed here that a is differentiable, with

$$\text{(2.1)}\qquad a(x,t) \geq a_o > 0 \quad \text{for all} \quad (x,t) \ ,$$

and that b is continuous. Let $k_o = \max_{0\leq x\leq 1} |\psi(x)|$. Then the theorem is as follows:

Theorem 3. *Suppose that*

$$a_o^{-1} \|b\|_{L_4[0,1]} \leq C \qquad \text{for all} \quad t \geq 0 \ .$$

Let v *be a solution of* (P.1) *such that*

$$\|v\|(t) \leq M_1 \qquad \text{for all} \quad t \geq 0 \ .$$

Define the constant M *by*

$$\text{(3.2)}\qquad M^{4/3} = M_1^2 2^{4/9}(2C/a_o)^{2/3},$$

<u>Then</u>

$$|v(x,t)| \leq k_o + 2M \quad \text{for} \quad t \geq 0 , \quad 0 \leq x \leq 1.$$

The next theorems are analogs of some results of [1]. They concern the following problem:

$$\text{(P.2)} \qquad \begin{aligned} &v_t = A(x,t)v_{xx} + B(x,t)v_x + k , \quad 0 < x < 1 \quad \tau < t < \tau+\alpha, \\ &v(0,t) = v(1,t) = 0, \\ &v(x,\tau) = \varphi(x). \end{aligned}$$

Let us denote the solution of this problem (if it exists) by $V(x,t;\tau;k;\varphi)$. In the three theorems to follow the same set of hypotheses hold namely the following:

(i) $A \in C^1([0,1]\times[0,\infty))$

(ii) $|A(x,t)-A(x',t)| \leq \bar{A}|x-x'|^\beta$ for $t \geq 0$ and for some $\beta > 0$.

(iii) $B \in C([0,1]\times[0,\infty))$

(iv) $|B(x,t)| \leq \bar{B}$ for $t \geq 0$.

(v) $\varphi \in C[0,1]$, k Hölder continuous in x and t in

$$0 \leq x \leq 1, \ \tau \leq t \leq \tau+\alpha .$$

<u>Theorem</u> 4. <u>Under hypothesis</u> (i)-(v) $V(x,t;\tau;k;\varphi)$ <u>exists and is unique</u> (This is a standard result).

In order to state our next two theorems we need some further notation, again that of [1]. For functions h defined on $[0,1]\times[\tau,\tau+\alpha]$ we let

$$\text{(2.3)} \qquad \begin{aligned} &|h|(t) = \max_{x\in[0,1]} |h(x,t)|, \quad |h|_{\tau,\alpha} = \max_{t\in[\tau,\tau+\alpha]} |h|(t), \\ &\|h\|(t) = \left(\int_0^1 h^2(x,t)\,dx\right)^{1/2}, \quad \|h\|_{\tau,\alpha} = \max_{t\in[\tau,\tau+\alpha]} \|h\|(t) . \end{aligned}$$

For functions Ω defined on $0 \leq x \leq 1$ we let

$$(2.4) \qquad |\Omega| = \max_{x\in[0,1]} |\Omega(x)|, \quad \|\Omega\| = \left(\int_0^1 \Omega(x)^2 dx\right)^{1/2} .$$

<u>Theorem</u> 5. <u>Let</u> $\alpha > 0$ <u>be</u> <u>fixed</u>. <u>There</u> <u>exists</u> <u>a</u> <u>function</u> $c_1(\bar{A},\bar{B})$, <u>such</u> <u>that</u> <u>if</u>,

$$v(x,t) = V(x,t;\tau;k;0),$$

<u>then</u> <u>we</u> <u>have</u>

$$(2.3) \qquad \begin{aligned} c_1^{-1}|v|_{\tau,\alpha} &\leq \begin{matrix} |k|_{\tau,\alpha} \\ \|k\|_{\tau,\alpha} \end{matrix}, \\ c_1^{-1}|v_x|_{\tau,\alpha} &\leq \begin{matrix} |k|_{\tau,\alpha} \\ \|k\|_{\tau,\alpha} \end{matrix}. \end{aligned}$$

[In this and subsequent theorems the functions $c_k(\bar{A},\bar{B})$ are bounded on compact $(\bar{A},\bar{B})$ sets] .

<u>Theorem</u> 6. <u>Let</u> $\alpha > 0$ <u>be</u> <u>fixed</u>. <u>There</u> <u>exists</u> $c_2(\bar{A},\bar{B})$ <u>such</u> <u>that</u> <u>if</u>,

$$\tilde{v}(x) = V(x,\tau;\tau;0;\varphi), \quad \tilde{w}(x) = V_t(x,\tau;\tau;0;\varphi),$$

<u>then</u>

$$(2.4) \qquad |\tilde{v}|, |\tilde{v}_x|, |\tilde{v}_{xx}|, |\tilde{w}| \leq \begin{matrix} c_2 \ |\varphi| \\ c_2 \ \|\varphi\| \end{matrix}.$$

<u>Remarks</u> (1) The constant c_2 in Theorem 6 tends to infinity as τ tends to zero. The content of the theorem is that the various derivatives can be bounded at values of t away from τ.

(2) Theorem 5 is very close to a standard result (Theorem 7 below). It differs in that we do not require that A be Hölder continuous in t. Its role is to give us a preliminary estimate

which enables us to get into the hypotheses of Theorem 7.

We shall not give proofs of theorems (5) and (6) since they are tedious and very close to ones in [1] and [5]. The essential ideas can be easily stated though. Following a remark in [5] we can construct a t-dependent parametrix for the differential equation in (P.2). By a method of images we can modify this parametrix so that it satisfies the boundary conditions. We can then use this modified parametrix in the usual way to construct a fundamental solution of the equation. This fundamental solution also satisfies the boundary conditions, hence is a Green's function. The role of taking the t-dependent paramatrix is that it eliminates the necessity of assuming Hölder continuity in t for the function A.

The Green's function $G(s,t,\xi,\tau)$ which we construct has the property that $V(x,t;\tau,k,\varphi)$ can be written in the form,

$$V(x,t;\tau;k;\varphi) = \int_{\tau}^{t}\int_{0}^{1} k(\xi,\tau)G(x,t,\xi,\varkappa)\,d\xi\,d\varkappa + \int_{0}^{1}\varphi(\xi)G(x,t,\xi,\tau)\,d\xi.$$

Theorem (6) follows from the fact that G, G_x, G_t and G_{xx} are all bounded for $t > \tau$. Theorem (5) follows from the fact that G and G_x satisfy estimates similar to those of [1] for the Green's function for the heat equation. Thus the proof is reduced to that in [1].

The remaining theorems are standard ones which one finds in [5]. In order to state them we need the notion of Hölder norm.

Let $s_{\tau,\tau+\alpha}$ denote the strip

$$s_{\tau,\tau+\alpha} = \{(x,t)\,|\,0\leq x\leq 1, \quad \tau\leq t\leq \tau+\alpha\} .$$

A function h will be called β-<u>Hölder continuous</u> in $s_{\tau,\tau+\alpha}$ if

$$(2.5) \qquad \sup_{\substack{(x,t)\in s_{\tau,\tau+\alpha} \\ (x',t')\in s_{\tau,\tau+\alpha}}} \frac{|h(x',t')-h(x,t)|}{(|x-x'|^2 + (t-t'))^{\beta/2}} = H_\beta(h) < \infty .$$

If h is β-Hölder continuous in $s_{\tau,\tau+\alpha}$ then we define

$$(2.6) \qquad |h|_\beta^{\tau,\alpha} = |h|_{\tau,\alpha} + H_\beta(h) .$$

The following two results are well known and easily verified.

<u>Lemma</u> 3.1. <u>The set</u> $C_\beta^{\tau,\alpha}$ <u>of functions which are</u> β-<u>Hölder continuous in</u> $s_{\tau,\tau+\alpha}$ <u>forms a Banach space under the norm</u> (2.6)

<u>Lemma</u> 3.2. <u>Closed, bounded sets in</u> $C_{\beta'}^{\tau,\alpha}$ <u>are compact in</u> $C_\beta^{\tau,\alpha}$ <u>if</u> $\beta' > \beta$.

We return now to problem (P.2); more specifically to the case $\varphi = 0$. We replace hypothesis (ii) with the following:

$$(ii)' \qquad A\in C_\beta^{\tau,\alpha} , \qquad |A|_\beta^{\tau,\alpha} \leq \bar{A} ,$$

with hypotheses (i),(ii),(iv) and (v) remaining the same. Then the following theorem is proved in [4].

<u>Theorem</u> (7). <u>Let</u> $\alpha > 0$ <u>and</u> $\beta',\beta < \beta' < 1$ <u>be fixed</u>. <u>There exists a function</u> $C_3(\bar{A},\bar{B})$ <u>such that if</u>,

$$v(x,t) = V(x,t;\tau;k;0),$$

<u>then</u>

$$(2.7) \qquad |v|_{\beta'}^{\tau,\alpha}, \quad |v_x|_{\beta'}^{\tau,\alpha} \leq C_3|k|_{\tau,\alpha} .$$

Next we strengthen the hypotheses (v) to

$$(v') \qquad k \in C_{\beta}^{\tau,\alpha} \quad |k|_{\beta}^{\tau,\alpha} \leq \overline{K} .$$

Then again the following is a theorem from [5].

Theorem (8). _Under hypotheses_ (i), (ii)', (iii), (iv) _and_ (v)' _there exists a constant_ $C_4(\overline{A},\overline{B})$ _such that if_,

$$v(x,t) = V(x,t;\tau;k;0),$$

then

$$(2.8) \qquad |v|_{\beta}^{\tau,\alpha}, \quad |v_x|_{\beta}^{\tau,\alpha} \leq \overline{K} .$$

3. Energy Inequalities.

In this section we derive a number of bounds for solutions of the problem (E), (A), (B), (C) of Section 1. These are almost exactly like those of [1] and we only outline the proofs. We use the notations

$$(3.1)\qquad |\Phi|(t) = \max_{x\in[0,1]} |\Phi(x,t)|, \quad \|\Phi\|(t) = \left[\int_0^1 \Phi^2(x,t)\,dx\right]^{1/2},$$

of the last section. Then the following result holds for any function $\Phi \in C^2[0,1]\times[0,\infty)$ which vanishes at $x=0$ and $x=1$:

$$(3.2)\qquad \|\Phi\|(t) \le |\Phi|(t) \le \|\Phi_x\|(t) \le |\Phi_x|(t) \le \|\Phi_{xx}\|(t) \le |\Phi_{xx}|(t) \quad t \ge 0.$$

We observe also the following elementary result:

Lemma (3.1). *If* $\psi(t)$ *is uniformly continuous and integrable on* $[0,\infty)$ *then*

$$\lim_{t\to\infty} \psi(t) = 0.$$

In the remainder of this section it is assumed that u is an (f,g) displacement on S_∞.

We multiply (E) by u_t and integrate the resulting expression over $(0,1)\times(t_1,t_2)$. If we use conditions (A) we obtain the relation,

$$(3.3)\qquad \|u_t\|^2(t_2) + 2\int_0^1 \int_{u_x(x,t_1)}^{u_x(x,t_2)} \sigma(\xi)\,d\xi\,dx + 2\int_{t_1}^{t_2}\int_0^1 \lambda(u_x(\xi,\tau))\,u_{xt}^2(\xi,\tau)\,d\xi\,d\tau$$

$$= \|u_t\|^2(t_1) + 2\int_0^1 \int_{u_x(x,t_1)}^{u_x(x,t_1)} \sigma(\xi)\,d\xi\,dx.$$

Next we multiply (E) by u_{xx} obtaining

$$u_{tt}u_{xx} = \sigma'(u_x)u_{xx}^2 + \lambda'(u_x)u_{xx}^2\,u_{xt} + \lambda(u_x)u_{xxt}u_{xx}\,.$$

Hence

$$(3.4) \quad \lambda u_{xx}u_{xxt} + \lambda\frac{\partial}{\partial t}(\lambda(u_x))u_{xx}^2 + \sigma'(u_x)u_{xx}^2 = u_{tt}u_{xx}$$

$$= (u_t u_{xx})_t - (u_t u_{tx})_x + u_{tx}^2 .$$

Note that

$$\lambda^2 u_{xx}u_{xxt} + \lambda\frac{\partial}{\partial t} u_{xx}^2 = \frac{1}{2}\frac{\partial}{\partial t}(\lambda^2 u_{xx}^2) .$$

Hence (3.4) can be written

$$(3.5) \quad \frac{\partial}{\partial t}(\lambda^2 u_{xx}^2) + 2\lambda\sigma'(u_x)u_{xx}^2 = 2\lambda[(u_t u_{xx})_t - (u_t u_{tx})_x + u_{tx}^2]$$

$$= 2[(\lambda u_t u_{xx})_t - (\lambda u_t u_{tx})_x + \lambda u_{tx}^2].$$

Now integrate over $(0,1)\times(t_1,t_2)$ and use condition (A). The result is,

$$(3.6) \quad \int_0^1 \lambda^2 u_{xx}^2(x,t_2)dx + \int_{t_1}^{t_2}\int_0^1 \sigma'(u_x)\lambda(u_x)u_{xx}^2 (x,\tau)dxd\tau$$

$$= 2 \int_0^1 \lambda^2 u_{xx}^2(x,t_1) + 2 \int_0^1 [\lambda u_t u_{xx}(x,t_2) - \lambda u_t u_{xx}(x,t_1)] dx$$

$$+ 2 \int_0^1 \int_{t_1}^{t_2} \lambda u_{tx}^2 (x,\tau)d\tau dx^*.$$

We can now derive various estimates from the two formulas (3.3) and (3.6). <u>In the succeeding results</u> M, M_* <u>and so on will all denote constants which depend only on</u> $J(f,g), \lambda_0$ <u>and</u> λ_1 <u>of</u> (**) <u>and tend to zero as</u> $J(f,g)$ <u>tends to zero</u>. We define $E_0(a)$ and $E_1(a)$ by the formulas,

$$(3.7) \quad E_0(a) = \inf_{|\xi|\le a} \sigma'(\xi), \; E_1(a) = \sup_{|\xi|\le a} \sigma'(\xi).$$

Both E_0 and E_1 are positive for any $a > 0$ by condition (*). Then (3.3) yields, just as in [1], the following result.

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*This argument is due to Professor James Greenberg.

Lemma (3.2). There exists M such that

$$\|u_t\|^2(t) \leq M^2 \quad \text{and} \quad \int_0^t \|u_{x\tau}\|^2(\tau)d\tau \leq M^2 \qquad t \geq 0 .$$

The constant M is given by,

$$M^2 = \max(M_1^2, \lambda_0^{-1}M_1^2),$$

where

$$M_1^2 = J(f,g)^2 + E_1(J(f,g))J(f,g)^2.$$

Corollary. $\|u_t\|^2(\cdot)$ is integrable on $[0,\infty)$ and

$$\int_0^\infty \|u_t\|^2(\tau)d\tau \leq M^2 .$$

The next result comes from formula (3.6) and again is analogous to one in [1].

Lemma (3.3). There exists M_* such that,

$$\|u_{xx}\|^2(t) \leq M_*^2 \quad \text{and} \quad \int_0^t \|u_{xx}\|^2(\tau)d\tau \leq M_*^2 \qquad t \geq 0 .$$

Corollary. $|u|(t) \leq |u_x|(t) \leq M_* \qquad t \geq 0$.

The following result is an exact analog of one in [1], hence we omit the proof.

Lemma (3.4). $\lim_{t\to\infty} \|u_t\|^2(t) = 0$; $\lim_{t\to\infty} \|u_{xx}\|^2(t) = 0$.

The second inequality of Lemma (3.4) together with formula (3.2) yields immediately the following result.

Corollary. $\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} u_x(x,t) = 0$.

4. A priori Estimates (Proof of Theorem 1).

Our first step in this section is to use Theorem 3 to obtain a bound for the t derivative of an (f,g) displacement u. Let u be such a displacement and let $v = u_t$. Then by (E), (A) and (C) we have,

$$(4.1) \qquad v_t = \frac{\partial}{\partial x}(\lambda(u_x)v_x + \sigma(u_x)),$$

$$(4.2) \qquad v(0,t) = v(1.t) = 0,$$

$$(4.3) \qquad v(x,0) = g(x).$$

This is for the form considered in Theorem 3. We have by (**) and Lemma (3.2)

$$(4.4) \qquad \lambda(u_x) \geq \lambda_0 > 0,$$

$$(4.5) \qquad k_0 = \max|g(x)| \leq J(f,g),$$

$$(4.6) \qquad \|v\|_{L_2[0,1]} = \|u_t\|\,(t) \leq M\ .$$

We know that $|u_x|(t) \leq M_*$ by the Corollary of Lemma (3.3). Hence by (3.7)

$$|\sigma(u_x)|(t) \leq E_1(M_*)|u_x|(t) \leq E_1(M_*)M_*\ .$$

Thus we have

$$(4.7) \qquad \lambda_0^{-1}\|\sigma\|_{L4[0,1]} \leq \lambda_0^{-1}E_1(M_*)M_* = C.$$

Now we can apply Theorem 3 and deduce that

$$(4.8) \qquad |v|(t) = |u_t|(t) \leq k_0 + \{M^2 2^{4/9}(2C/\lambda_0)^{2/3}\}^{3/4}.$$

Note that both terms on the right side of (4.8) tend to zero as $J(f,g)$ tends to zero hence we have proved the following result:

<u>Lemma</u> (4.1). <u>There exists</u> $M_{\#}$ <u>such that</u>

$$|u_t|(t) = M_{\#}$$

We make use of the estimate for u_t to derive an estimate for u_{xx}. Here we use a device which again is analoqous to one in [1]. We write equation (E) as

$$\begin{aligned} (4.9) \qquad u_{tt} &= \sigma'(u_x)u_{xx} + \lambda'(u_x)u_{xx}u_{xt} + \lambda(u_x)u_{xxt} \\ &= \frac{\sigma'(u_x)}{\lambda(u_x)} (\lambda(u_x)u_{xx}) + \frac{\partial}{\partial t} (\lambda(u_x)u_{xx}) \ . \end{aligned}$$

Now we consider this as an ordinary differential equation for $\lambda(u_x)u_{xx}$. Let

$$\gamma(x,t) = \frac{\sigma'(u_x(x,t))}{\lambda(u_x(x,t))} \ ; \qquad \epsilon(x,t,\tau) = e^{\int_\tau^t \gamma(x,\varkappa)\,d\varkappa}.$$

Then (4.9) yields,

$$\lambda(u_x(x,t)u_{xx}(x,t) = \epsilon(x,t,\tau)\{\lambda(u_x(x,\tau) + \int_\tau^t u_{tt}(x,s)\epsilon^{-1}(x,s,\tau)ds.$$

If we integrate by parts we can write this in the form,

$$\begin{aligned} (4.10) \qquad \lambda(u_x(x,t))u_{xx}(x,t) &= u_t(x,t) + \epsilon(x,t,\tau)\{\lambda(u_x(x,\tau))u_{xx}(x,\tau) \\ &\quad - u_t(x,\tau) - \int_\tau^t \gamma(x,s)u_t(x,s)\epsilon^{-1}(x,s,\tau)ds\} \ . \end{aligned}$$

We deduce two kinds of information from (4.10). First one derives immediately, from (4.10) with $\tau = 0$, Lemma (4.1) and our assumptions about σ and λ, the following result:

<u>Lemma</u> (4.2). <u>There exists</u> M_{**} <u>such that</u>

$$|u_{xx}|(t) \le M_{**}.$$

We can also deduce some smoothness results from (4.10). Let $\tau=0$ in (4.10). Then $\lambda(u_x(x,\tau)u_{xx}(x,\tau)-u_t(x,\tau)=\lambda(f'(x))f''(x)-g(x)=\psi(x)$. The smoothness assumptions on f and g imply that the function ψ has two continuous derivatives. On the other hand σ and λ were assumed to be C^3 and C^2 respectively while for an (f,g) displacement $u_x, u_t, u_{xx}, u_{xt}, u_{xxt}$ are all continuous. One then derives the following result from (4.10).

Lemma (4.3). For an (f,g) displacement u_{xxx} and u_{xxxx} are continuous.

Now that we have a bound on u_{xx} we can obtain a bound for the x-Hölder norm of $\lambda(u_x)$. Indeed we have by (**)

(4.11) $$|\lambda(u_x(x,t)) - \lambda(u_x(x',t))| \leq \lambda_2 |u_x(x,t) - u_x(x',t)|$$

$$\leq \lambda_2 |u_{xx}|\ |x-x'| \leq \lambda_2 M_{**} 2^{1-\beta} |x-x'|^{\beta} \equiv \bar{A}|x-x'|^{\beta}$$

for any β, $0 < \beta < 1$. We also have the bound,

(4.12) $$|\lambda'(u_x)u_{xx}| \leq \lambda_2 M_{**} = \bar{B}.$$

We can now use Theorems 5 and 6. Let $v = u_t$. Then v is a solution of the problem (P.2) with,

$$A = \lambda(u_x),\ B = \lambda'(u_x)u_{xx},\ k = \sigma'(u_x)u_{xx},\ \varphi(x) = u_t(x,\tau) .$$

A is differentiable with respect to x while B and φ are continuous. k is Hölder continuous (in fact differentiable). Thus hypotheses (i)-(v) of Theorems (4), (5), and (6) are satisfied with $\bar{A},\bar{B}$ given by (4.11) and (4.12).

We observe that $u_t = v = v^1 + v^2$ where,

(4.13) $$v^1(x,t) = V(x,t;\tau;k;0),\quad v^2(x,t) = V(x,t;\tau;0,\varphi).$$

From Theorem 5, (3.7), Lemma 4.2 and the Corollary to Lemma 3.3 we deduce that,

$$(4.14) \quad |v_x^1| \leq C_1 |k|_{\tau,\alpha} = C_1 |\sigma'(u_x) u_{xx}|_{\tau,\alpha} \leq C_1 E_1(M_*) M_{**} .$$

From Theorem 6 and Lemma (4.1) we find that

$$(4.15) \quad |v_x^2| \leq C_2 |\varphi| = C_2 |u_t| \leq C_2 M_{\#} .$$

We combine (4.14) and (4.15) and deduce the following result.

<u>Lemma</u> (4.4). <u>There exists</u> M <u>such that</u>

$$|u_{tx}|(t) \leq M .$$

Observe that Theorem 6 and Lemma (4.1) also yield the estimates,

$$(4.16) \quad |v_{xx}^2| \leq C_2 M_{\#}, \quad |v_t^2| \leq C_2 M_{\#} .$$

We can obtain some further estimates by using those formulas from Theorems (5) and (6) which involve L_2 norms. By formulas (2.3) and (2.4) we have,

$$|v^1| \leq C_1 \|\sigma'(u_x) u_{xx}\|_{\tau,\alpha}, \quad |v^2| \leq C_1 \|\sigma'(u_x(\cdot,\tau)) u_{xx}(\cdot,\tau)\| .$$

It follows then from the second inequality of Lemma (3.4) that $|v^1|$ and $|v^2|$ tend to zero as t tends to infinity. In the same way we deduce also that $|v_x^1|$ and $|v_x^2|$ tend to zero as t tends to infinity. Since $v^1 + v^2 = u_t$ this yields the following result.

<u>Lemma</u> (4.5). $\lim\limits_{t\to\infty} u_t(x,t) = \lim\limits_{t\to\infty} u_{xt}(x,t) = 0$.

From Lemma (4.5) and Equation (4.10) one obtains, exactly as in the proof of Lemma (5.3) of [1], the following additional result.

<u>Lemma</u> (4.6). $\lim_{t\to\infty} u_{xx}(x,t) = 0.$

Lemmas (4.2) and (4.4) show that the function $A = \lambda(u_x)$ has a bounded Hölder norm. Indeed we have

$$|A(x',t') - A(x,t)| \le \lambda[|u_{xx}|\ |x-x'| + |u_{xt}|\ |t-t'|]$$
$$\le A[M_{**}z^{1-\beta}|x-x'|^{\beta} + M\tau^{1-\beta/2}|t-t'|^{\beta/2}$$
$$\le \bar{\bar{A}}(|x-x'|^2 + |t-t'|)^{\beta/2} .$$

The constant $\bar{\bar{A}}$ depends only on τ and β .

Now we apply Theorem 7 to the function v^1 of (4.13). We find

(4.17) $|v^1|_{\beta}^{\tau,\alpha} \le C_3|k|_{\tau,\alpha} \le C_3E_1(M_*)M_{**}$.

On the other hand it follows from (4.16) that the β-norm of v^2 is bounded by a constant C times M where C depends only on τ and β. Combining this fact with (4.17) we obtain the following result.

<u>Lemma</u> (4.7). <u>There exists a constant</u> K_1 <u>depending only on</u> α,β <u>and</u> $J(f,g)$ <u>such that</u>

(4.18) $|u_t|_{\beta}^{\tau,\alpha} \le K_1$.

<u>Moreover</u> K_1 <u>tends to zero as</u> $J(f,g)$ <u>tends to zero</u>.

From Lemma (4.2) and Lemma (4.4) we deduce that there is another constant K_2 with the same properties as K_1 such that

(4.19) $|u_x|_{\beta}^{\tau,\alpha} \le K_2$.

We emphasize that the estimates (4.18) and (4.19) depend only on α, the width of the strip we considered, and not on τ the starting position of the strip. Thus by fixing α we obtain estimates which are uniform in $t \geq 0$. This yields the following result which we need for the existence theorem in Section 6. Let $N > 0$ be fixed. Let $C_\beta^{0,N}$ denote the Banach space of functions which are β-Hölder continuous in $S_{0,N}$. Let K_β^N denote the space $C_\beta^{0,N} \times C_\beta^{0,N}$ with the norm,

$$(4.20) \quad |(m,n)|_\beta^N = \max(|m|_\beta^{0,N}, |n|_\beta^{0,N}),$$

(see 2.6). K_β^N is then also a Banach space and closed, bounded subsets of $K_{\beta'}^N$ are compact in K_β^N if $\beta' > \beta$.

Lemma (4.8). If u is an (f,g) displacement then $(u_t, u_x) \in K_\beta^N$ for any N. Moreover there exists a constant K, depending only on β and $J(f,g)$; such that for any N.

$$|(u_t, u_x)|_\beta^N \leq K .$$

K tends to zero as $J(f,g)$ tends to zero.

If we collect all our results we find that the proof of Theorem (1) is complete. The boundedness of $|||u|||$ follows from the Corollary to Lemma (3.3) and Lemmas (4.1), (4.2) and (4.4). The relation (1.4) follows from the Corollary to Lemma (3.4) and Lemmas (4.5) and (4.6).

5. Functional Equations.

We are going to reformulate the problem (E), (A), (B), (C) as a fixed point problem. To this end we introduce certain operators analogous to those in [1]. Throughout we assume that f and g are a fixed pair of functions suitable for an (f,g) displacement. The first operator derives from formula (4.10) of the last section. Let v and w be continuous functions and define Γ and E by the formulas

$$(5.1)\qquad \Gamma(w)(x,t) = \frac{\sigma'(w(x,t))}{\lambda(w(x,t))}\ ;\quad E(w)(x,t) = e^{-\int_0^t \Gamma(w)(x,\varkappa)\,d\varkappa} .$$

Then $h(v,w)$ is defined by,

$$(5.2)\qquad h(v,w)(x,t) = E(w)(x,t)\ \{\lambda(f'(x))f''(x) - g(x) - \int_0^t \Gamma(w)(x,s)v(x,s)\ [E(w)(x,s)]^{-1}ds\} .$$

We observe that if u is an (f,g) displacement and we set $v = u_t$, $w = u_x$ then (4.10) yields the relation,

$$(5.3)\qquad \lambda(w)w_x = v + h(v,w) .$$

Our aim is to find functional equations satisfied by $v = u_t$ and $w = u_x$ if u is an (f,g) displacement. Our next step is to write Equation (E) in a different form. We have,

$$v_t = \sigma'(w)w_x + \lambda'(w)w_x v_x + \lambda(w)v_{xx} ,$$

or, if we substitute from (5.3),

$$(5.4)\qquad v_t = \mathcal{A}(w)v_{xx} + \mathcal{B}(v,w)v_x + \mathcal{C}(v,w),$$

where

$$\mathcal{A}(w) = \lambda(w), \quad \mathcal{B}(v,w) = \lambda'(w)\lambda^{-1}(w)(h(v,w) + v)$$

$$\mathcal{C}(v,w) = \sigma'(w)\lambda^{-1}(w)(h(v,w) + v) .$$

Our second operator is suggested by (5.4). For a given (v,w) we define $T_1(v,w)$ as the solution Φ of the problem,

(5.5) $$\Phi_t = \mathcal{A}(w)\Phi_{xx} + \mathcal{B}(v,w)\Phi_x + \mathcal{C}(v,w),$$

(5.6) $$\Phi(0,t) = \Phi(1,t) = 0 ; \quad \Phi(x,0) = g(x).$$

By Theorem 4, Φ will exist and be uniquely determined provided that $\mathcal{B}(v,w)$ is continuous and $\mathcal{A}$ and $\mathcal{C}$ are β-Hölder continuous. Thus if $v = u_t$ and $w = u_x$ for an (f,g) displacement then

(5.7) $$v = T_1(v,w) .$$

The final operator we introduce is similar to, but more complicated than the corresponding one in [1]. It arises from 'solving' (5.3) for w. Define $\Lambda(w)$ by the formula,

$$\Lambda(w) = \int_0^w \lambda(\xi)\,d\xi \quad \text{or} \quad \Lambda'(w) = \lambda(w) \quad \Lambda(0) = 0 .$$

Observe that Λ is a monotone increasing function. Then if $w(x,t)$ is a solution of (5.3) it must satisfy the equation,

(5.8) $$\Lambda(w(x,t)) = \int_0^x (h+v)(\xi,t)\,d\xi + r(t),$$

for some function r. Now if w is to be u_x for an (f,g) displacement then we must have,

$$\int_0^1 w(x,t)\,dx = \int_0^1 u_x(x,t)\,dx = 0 .$$

Hence we impose on r in (5.8) the condition,

$$(5.9) \qquad \int_0^1 \Lambda^{-1}\left[\int_0^x (h+v)(\xi,t)d\xi + r(t)\right] dx = 0.$$

Since Λ is monotone it is easy to see that (5.9) uniquely determines $r(t)$ as an operator $r(t) = R(v,w)(t)$ on v and w. Then we define our final operator $T_2(v,w)$ by the formula,

$$(5.10) \qquad T_2(v,w)(x,t) = \Lambda^{-1}\left(\int_0^x (h(v,w) + T_1(v,w)(\xi,t)d\xi + R(T_1(v,w),w)\right),$$

so that if $v = u_t$, $w = u_x$ for an (f,g) displacement then

$$(5.11) \qquad w = T_2(v,w) .$$

Equations (5.7) and (5.11) suggest that if (v,w) is a fixed point of the map $\Phi:(v,w) \to (T_1(v,w),T_2(v,w))$ then v and w will be the t and x derivatives of an (f,g) displacement. The main result of this section is that such is indeed the case.

We fix $N > 0$ and $\beta, 0 < \beta < 1$. Then our basic space will be K_β^N as defined in Lemma (4.7). We shall prove the following result.

Theorem (9). Let (v,w) be a fixed point of Φ. Then there exists a function u such that

$$u_t = v , \quad u_x = w$$

and u is an (f,g) displacement.

We begin the proof by observing as above that v, v_t, v_x, v_{xx} are all continuous. One sees by (5.2) that h is continuous and $w = T_2(v,w)$ hence it follows immediately from (5.3) that w_x is continuous and,

$$(5.12) \qquad w_x = (\lambda(w))^{-1}(v+h(v,w)) .$$

It also follows from (5.2) that h is differentiable with respect to t and

$$h_t(v,w) = -\frac{\sigma'(w)}{\lambda(w)} h(v,w) - \frac{\sigma'(w)}{\lambda(w)} v \ . \tag{5.13}$$

It is not clear immediately that w is differentiable with respect to t. This is however true and is the key point of the proof. We state it separately.

Lemma (5.1). *w is differentiable with respect to t and*

$$w_t = v_x ; \quad w(x,0) = f'(x) \ . \tag{5.14}$$

Let us assume the lemma for the moment and indicate the rest of the proof of Theorem (9). In view of (5.14) we can define a function $u(x,t)$ by either of the formulas,

$$\begin{aligned} u(x,t) &= U^1(x,t) = \int_0^x w(\xi,t)\,d\xi \\ u(x,t) &= U^2(x,t) = f(x) + \int_0^t v(x,\varkappa)\,d\varkappa \end{aligned} \tag{5.15}$$

Note that $U_t^2 = v$ while,

$$U_t^1(x,t) = \int_0^x w_t(\xi,t)\,d\xi = \int_0^x v_\xi(\xi,t)\,d\xi = v(x,t) - v(0,t) = v(x,t),$$

Moreover $U^2(x,0) = f(x)$ while,

$$U^1(x,0) = \int_0^x w(\xi,0)\,d\xi = \int_0^x f'(\xi)\,d\xi = f(x) - f(0) = f(x) \ .$$

Thus $U^1 \equiv U^2$ and u is well defined. The second of formulas (5.15) shows that u possesses all the necessary derivatives for an (f,g) displacement. Also we have

$$u(x,0) = U^2(x,0) = f(x) \ , \ u_t(x,0) = U_t^2(x,0) = v(x,0) = g(x) .$$

Now v satisfies (5.4) and $v_t = u_{tt}$. Thus if we substitute (5.14) into (5.8) and observe that $w = u_x$ we see that u satisfies (E). Finally

$$u(0,t) = U^1(0,t) = 0, \; u(1,t) = U^2(1,t) = f(1) + \int_0^t v(1,\varkappa)\,d\varkappa = 0.$$

Hence u is an (f,g) displacement.

We return to the proof of Lemma (5.1). Recall that $h(v,w)$ is differentiable with respect to t and so is v hence the function

$$\psi(x,t) = h(v,w)(x,t) + v(x,t)$$

is differentiable with respect to t. But then it follows from (5.9) that the function r is differentiable. In fact we have

$$\dot{r}(t) = -\left[\int_0^1 \lambda^{-1}\left(\int_0^x (h+v)(\xi,t)\,d\xi + r(t)\right)dx\right]^{-1} \int_0^1 \left[\lambda^{-1}\left(\int_0^x (h+v)(\xi,t)\,d\xi + r(t)\right) \int_0^x (h_t+v_t)(\xi,t)\,d\xi\right] dx.$$

If follows from (5.10) that $T_2(v,w)$ is differentiable with respect to t and then so also is $w = T_2(v,w)$.

Observe that by (5.2) we have $\psi(\xi,0) = \lambda(f'(x))f''(x)$. Hence by (5.9) we have,

$$\int_0^1 \Lambda^{-1}(\Lambda(f'(x)) - \Lambda(f'(0)) + r(0))\,dx = 0 \,.$$

This equation has a unique solution and that solution is clearly $r(0) = \Lambda(f'(0))$ (recall that $f(0) = f(1) = 0$). Then by (5.10) we have,

$$w(x,0) = T_2(v,w)(x,0) = \Lambda^{-1}\; \Lambda(f'(x)) - \Lambda(f'(0) + \Lambda(f'(0)) = f'(x).$$

We have still to verify the first of formulas (5.14). From (5.4) and (5.15) we have,

$$(5.16)\qquad (h_t + v_t)/\lambda(w) = (h+v)\lambda'\lambda^{-2}v_x + v_{xx}.$$

From the construction of T_2 we see that if $w = T_2(v,w)$ then w is a solution of (5.3) with

$$\int_0^1 w(\xi)d\xi = 0.$$

Hence w must satisfy the equation

$$(5.17)\qquad w(x,t) = \int_0^x \frac{(h+v)}{\lambda}\,d\xi - \int_0^1 d\xi \int_0^\xi \frac{(h+v)}{\lambda}\,d\tau.$$

Combining (5.16) and (5.17) we have

$$\begin{aligned}(5.18)\qquad w_t(x,t) &= \int_0^x \left(\frac{(h+v)}{\lambda}\right)_t d\xi - \int_0^1 d\xi \int_0^\xi \left(\frac{(h+v)}{\lambda}\right)_t d\tau \\ &= \int_0^x \frac{(h+v)}{\lambda^2}\,\lambda'(v_x - w_t)d\xi - \int_0^1 d\xi \int_0^\xi \frac{h+v}{\lambda^2}\,\lambda'(v_x - w_t)d\tau \\ &\quad + \int_0^x v_{\xi\xi} - \int_0^1 d\xi \int_0^\xi v_{\tau\tau}d\tau .\end{aligned}$$

The last two integrals combine to yield $v_x(x,t)$. Thus on differentiation we have,

$$(5.19)\qquad (w_t(x,t) - v_x(x,t))_x = \frac{h+v}{\lambda^2}\,\lambda'(w_t(x,t) - v_x(x,t)) .$$

Equation (5.18) shows that,

$$\int_0^1 (w_t(x,t)dx = 0.$$

On the other hand v is 0 at $x = 0$ and $x = 1$ hence

$$\int_0^1 v_x(x,t)dx = 0.$$

$w_t - v_x$ is thus a solution of equation (5.19) which has integral from 0 to 1 equal to 0. Hence $w_t = v_x$.

6. Existence and Uniqueness of a Solution (Proof of Theorem 2)

According to the last section the proof of the existence of an (f,g) displacement is reduced to showing that the map Φ has a fixed point. Consider the family of problems (P_τ).

$$\text{(E)} \qquad u^\tau_{tt} = \frac{\partial}{\partial x}(\sigma(u^\tau_x) + \lambda^\tau(u^\tau_x)u^\tau_{xt}); \quad \lambda^\tau(\xi) = \tau\lambda(\xi) + (1-\tau)\lambda_0,$$

$$\text{(A)} \qquad u^\tau(0,t) = u^\tau(1,t) = 0,$$

$$\text{(B)} \qquad u^\tau(c,0) = \tau f(x),$$

$$\text{(C)} \qquad u^\tau_t(x,0) = \tau g(x).$$

for $0 \le \tau \le 1$. λ_0 is the constant in condition (**). (P_1) is the problem of this paper while (P_0) is the problem of [1] in the special case $f = g = 0$. The results of [1] thus yield the following lemma.

Lemma (6.1) Problem (P_0) has the unique solution $u = 0$.

Observe that the function $\lambda^\tau(\xi)$ satisfies condition (**) for all τ in $[0,1]$. Observe also that the quantity $J(\tau f,\tau g)$ which bounds the initial data of P_τ is simply $\tau J(f,g)$ which is less than or equal to $J(f,g)$ in $0 \le \tau \le 1$.

We assume from now on that f,g,β,N are fixed.

We now form the operators h^τ, T_1^τ and T_2^τ which derive from (P_τ) in the same way as in (P_1). We need not write these down since they differ only in that λ^τ replaces λ. As in Theorem (9), fixed points of the map $\Phi^\tau: (V,W) \to (T_1(V,W), T_2(V,W))$ on K_β^N will yield solutions of u^τ of (P_τ) with $v = u^\tau_t$ and $w = u^\tau_x$. From Lemma (4.8) we obtain the following estimate.

Lemma (6.2). There exists a constant L, independent of τ in $0 \le \tau \le 1$ such that if (v,w) is a fixed point of Φ^τ then

$$|(v,w)|_{\beta}^{N} \leq L.$$

Our next result establishes the compactness of the maps Φ^{τ}.

<u>Lemma</u> (6.3). <u>Let</u> $\beta', \beta < \beta < 1$ <u>be fixed</u>. <u>Then there exists a function</u> $X(\rho)$ <u>such that if</u>,

$$|(v,w)|_{\beta}^{N} \leq \rho,$$

<u>then</u>

$$|T_1^{\tau}(v,w),\ T_2^{\tau}(v,w)|_{\beta'}^{N} \leq X(\rho).$$

We are assuming that $|v|_{\beta}^{N} \leq \rho$ and $|w|_{\beta}^{N} \leq \rho$. In particular it follows that $|v|_{0,N}$ and $|w|_{0,N}$ are both bounded by ρ (see(2.3)). Also we have by (**)

(6.1) $$|\lambda^{\tau}(w)|_{\beta}^{N} \leq \lambda_2\rho, \quad \text{for all } \tau, 0 \leq \tau \leq 1.$$

Now $T_1^{\tau}(V,W)$ is obtained as the solution of the problem (5.4)-(5.6) with the appropriate modifications in $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$. Call the modified coefficients $\mathcal{A}^{\tau}, \mathcal{B}^{\tau}$ and $\mathcal{C}^{\tau}$. Then it is easy to check from the preceding remarks, formula (5.2) and conditions (*) and (**) that,

(6.2) $$|\mathcal{A}^{\tau}|_{\beta}^{N} \leq \lambda_2\rho, \quad |\mathcal{B}|_{0,N} \leq X_1(\rho), \quad |\mathcal{C}|_{0,N} \leq X_2(\rho),$$

where X_1 and X_2 depend only on N and the constants in conditions (*) and (**). Then Theorem (7) yields,

(6.3) $$|T_1^{\tau}|_{\beta'}^{N} \leq C_3(\lambda_2\rho, X_1(\rho))X_2(\rho) \equiv X_3(\rho).$$

Given the result (6.3) it is tedious but straightforward calculation to verify that T_2^{τ} satisfies

(6.4) $$|T_2^\tau|_{\beta'}^N \leq X_4(\rho)$$

and that completes the proof of Lemma (6.3).

It remains only to discuss the continuity of the maps Φ^τ. In order to do this we need some estimates of second derivatives. Consider again the modified Equation (5.4) with $|(v,w)|_\beta^N \leq \rho$. We observe that the third inequality of (6.2) can be sharpened as follows. From (5.2) (*) and (**) one can show that

(6.5) $$|C^\tau|_\beta^N = |\sigma'(w)\lambda^{-1}(w)(h(v,w) + v)|_\beta^N$$

$$\leq |\sigma'(w)|_{0,N}\, |\lambda^{-1}(w)(h(v,w) + v)|_\beta^N$$

$$+ |\sigma'(w)|_\beta^N |\lambda^{-1}(w)(h(v,w) + v)|_{0,N} \leq X_5(\rho) .$$

Now we can apply Theorems 6 and 8 with $\tau = 0$. If $\Phi = T_1^\tau(v,w)$ then these theorems together yield the estimates,

(6.6) $$|\Phi_x^\tau|_{0,N} ,\ |\Phi_{xx}^\tau|_{0,N} \leq C_4(\lambda_2\rho, X_1)(\rho))X_5(\rho)$$

$$+ C_2(\lambda_2\rho, X_1))|g| = X_6(\rho).$$

Lemma (6.4). For any fixed τ the map Φ^τ is continuous in (v,w). Φ^τ is continuous with respect to τ uniformly for (v,w) in bounded sets.

Suppose $|(v,w)|_\beta^N \leq \rho$. Let $\Phi^1 = T_1^{\tau_1}(v,w)$ and $\Phi^2 = T_1^{\tau_2}(v,w)$. Then we have,

$$\Phi_t^1 = \mathcal{A}^{\tau_1}(w)\Phi_{xx}^1 + \mathcal{B}^{\tau_1}(v,w)\Phi_x^1 + \mathcal{C}^{\tau_1}(v,w)$$

$$\Phi_t^2 = \mathcal{A}^{\tau_2}(w)\Phi_{xx}^2 + \mathcal{B}^{\tau_2}(v,w)\Phi_x^2 + \mathcal{C}^{\tau_2}(v,w) .$$

Thus the difference $\psi = \Phi^1 - \Phi^2$ satisfies the equation,

$$\psi_t = \mathcal{A}^{\tau_1}(w)\psi_{xx} + \mathcal{B}^{\tau_1}(v,w)\psi_x + S ,$$

where

$$S = (\mathcal{A}^{\tau_1}(w) - \mathcal{A}^{\tau_2}(w)\Phi^2_{xx} + (\mathcal{B}^{\tau_1}(v,w) - \mathcal{B}^{\tau_2}(v,w)\Phi^2_x$$
$$+ (C^{\tau_1}(v,w) - C^{\tau_2}(v,w)) .$$

It is not too difficult to check that the differences in S are uniformly small with $|\tau^1 - \tau^2|$ for $|(v,w)|^N_\beta \leq \rho$. On the other hand by (6.6) $|\Phi^2_{xx}|$ and $|\Phi^2_x|$ are bounded uniformly. Hence given any ϵ we can find ζ, independent of (v,w) in $|(v,w)|^N_\beta \leq \rho$ such that

$$|S|_{0,N} \leq \epsilon , \quad f\ |\tau^1 - \tau^2| < \zeta .$$

ψ is a solution of Problem (5.4)-(5.5) with $\mathcal{A}$ and $\mathcal{B}$ replaced by $\mathcal{A}^{\tau_1}$ and $\mathcal{B}^{\tau_1}$ and C replaced by S. It follows by Theorem 7 that $|\psi|^N_\beta$ can be made uniformly small with $|\tau_1 - \tau_2|$.

We have shown that T^τ_1 is continuous with respect to τ uniformly with respect to (v,w). Similar calculations establish its continuity with respect to (v,w). The continuity of T_2 is a straightforward calculation.

It follows from Lemmas (6.1)-(6.4) and the Leray-Schauder Theorem that the map Φ^1 has a fixed point. Hence by Theorem 9 we infer the existence of an (f,g) displacement.

In order to complete the proof of Theorem 2 it remains to verify the uniqueness. Let u be an (f,g) displacement and let η be a twice differentiable function in S_T which vanishes at $x = 0$ and $x = 1$. Multiply (E) by η_t and integrate over S_T. This yields,

$$(6.7) \qquad \int_{S_T} \eta_t u_{tt} = \int_{S_T} \eta_t \frac{\partial}{\partial x} (\sigma(u_x) + \lambda(u_x) u_{xt}) .$$

Form (6.7) for two (f,g) displacements u^1 and u^2, subtract the results and then set $\eta = u^1 - u^2$. The result is the equation,

$$(6.8) \quad \int_{S_T} (u_t^1 - u_t^2)(u_{tt}^1 - u_{tt}^2) = \frac{1}{2}\int_0^1 (u_t^1(x,T) - u_t^2(x,T))^2 dx$$

$$= \int_0^1 \int_0^T (u_t^1(x,\tau) - u_t^2(x,\tau)) [\, \sigma'(u_x^1) u_{xx}^1 - \sigma'(u_x^2) u_{xx}^2$$

$$+ \lambda'(u_x^1) u_{xx}^1 u_{xt}^1 - \lambda'(u_x^2) u_{xx}^2 \, u_{xt}^2 + \lambda(u_x^1) u_{xxt}^1$$

$$- \lambda(u_x^2) u_{xxt}^2] \, d\tau dx \; .$$

Now the quantity in square brackets on the right side of (6.8) can be bounded by the following argument. By Lemma (4.8) u_t and u_t^2 have bounded Hölder norms. From (4.10) (with $\tau = 0$) one sees that u_{xx}^1 and u_{xx}^2 have bounded Hölder norms. Then one can apply Theorems 6 and 8 to u_t^1 and u_t^2, considered as solutions of (4.9) to deduce that u_{txx}^1 and u_{txx}^2 are bounded on any S_T. Then (6.8) yields,

$$\frac{1}{2} \int_0^1 (u_t^1(x,T) - u_T^2(x,T)^2 dx \leq M \int_0^T d\tau \int_0^1 (u_t^1(x,\tau) - u_t^2(x,\tau))^2 dx.$$

Since $u_t^1(x,0) - u_t^2(x,0) = 0$ it follows that $u_t^1(x,t) \equiv u_t^2(x,t)$ and finally since $u^1(x,0) = u^2(x,0)$ u^1 and u^2 must be identical.

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