ON THE STABILITY OF SOLUTIONS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS

by

Bernard D. Coleman

and

Victor J. Mizel

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Contents

1. Introduction

The functional-differential equations studied here have the form

$$\dot{x}(t) = f(x^t), \tag{1.1}$$

where the superposed dot denotes a right-hand derivative, x^t is the function on $[0,\infty)$ defined by $x^t(s) = x(t-s)$, and f is a preassigned, continuous, locally bounded function mapping a function space V into a finite dimensional vector space E^n. By a solution up to A ($A > 0$) of (1.1) we mean a function $x(\cdot)$ on $(-\infty,A)$ which obeys (1.1) for all t in $[0,A)$. Assuming that the domain V of f is a history space of the type occurring in the theory of fading memory,# we find sufficient conditions for the stability of solutions of (1.1).

#See, in particular, [1968, 1].

In a recent essay [1968, 2], we considered evolving systems more general than those described by (1.1), but we sought sufficient conditions for only simple stability; here the emphasis is laid on asymptotic stability.

After listing in Section 2 the basic properties of a history space V, we show, in Section 3, how results obtained in [1968, 2] may be applied to the equation (1.1). In Section 4 we prove lemmas about the compactness in $\mathbf{V}$ of the closure of trajectories, and the existence, invariance, and attracting power of positive limit sets of solutions of (1.1). The utility of such lemmas for the investigation of asymptotic stability is brought out by the work of La Salle [1960, 2], [1967, 4] on ordinary differential equations and of Hale [1963, 1], [1965, 5] on functional-differential equations. Indeed, the proofs given in Section 4 are modeled after those employed by Hale [1965, 1] for the case in which the domain of $\mathbf{f}$ is a space $\mathcal{C}$ of continuous functions on $[0,\infty)$ endowed with the compact open topology. Because the Banach function spaces $\mathbf{V}$ of the theory of fading memory are likely to be less familiar to the cultivators of stability theory than the space $\mathcal{C}$ employed by Hale, we have thought it desirable to outline in detail our proofs of the basic lemmas. The theorems proven in Section 5 give sufficient conditions for the asymptotic stability of a solution of (1.1). In Section 6 we apply Theorems 3.1 and 5.1 to an easily visualized physical problem. In the Appendix we discuss extensions of stability theorems of Hale [1963, 1], [1965, 3] to our history spaces $\mathbf{V}$.

Although we assume that the generally non-linear functional $\mathbf{f}$ in (1.1) is continuous on a history space $\mathbf{V}$ and maps bounded sets in $\mathbf{V}$ into bounded sets in E^n, our main theorems do not require that $\mathbf{f}$ be Lipschitz continuous.#

#Although Hale [1963, 1], [1965, 1], in the proof of his theorems on the extent of stability, assumes that the functional $\mathbf{f}$ is locally Lipschitzian, he points out [1965, 1, p. 455] that no such condition is required for the proof of his basic lemmas.

The present investigation arose out of two observations.

(I) There are differences between the function spaces usually employed in the theory of functional-differential equations and the history spaces $\underline{\underline{V}}$ which occur in studies of the mechanics and thermodynamics of materials with gradually fading memory. An example of a material with fading memory is one obeying Boltzmann's linear theory of viscoelasticity; for such a material, in the one-dimensional case, the stress T(t) is given by the following type of functional of the history of the strain E:

$$T(t) = G(0)E(t) + \int_0^\infty G'(s)E(t-s)ds. \tag{1.2}$$

Here G is a smooth function on $[0,\infty)$ with its derivative G'(s) negative for all $s > 0$. A functional with the form (1.2) is clearly not continuous in the compact open topology, but is continuous over an appropriate history space from the theory of fading memory,# provided only that G' is dominated by a positive measurable function ℓ with the properties (I)-(IV) listed in Remark 2.4 below. We have, therefore, been interested in determining whether the theories of stability developed recently by Krasovskii, La Salle, and Hale can be extended to cover functional-differential equations compatible with the theory of fading memory used in continuum physics.

#Cf. Coleman & Noll [1960, 1], [1961, 1]; Coleman & Mizel [1966, 1].

(II) For many problems with physical applications, known results in the thermodynamics of materials with memory# can supply

#Coleman [1964, 1], Coleman & Mizel [1967, 2].

directly free-energy functionals which have properties similar to, but not identical with the Liapunov functionals occurring in the work of Krasovskii##

##See [1959, 1, Theorems 30.1 and 30.3].

and Hale###. It is natural to ask whether one may use free energy

###See [1965, 1, Corollary 2].

functionals to investigate stability. Our present Theorems 3.1 and 5.1 show that the answer is yes. The criterion for stability which results seems to justify the common practice in physics of declaring that <u>states of stable equilibrium are those which minimize the "equilibrium free energy"</u>.#### In this regard the present study continues and extends our

####Cf. Gibbs [1875, 1].

recent investigations##### of the relation of thermodynamic principles

#####Coleman & Mizel [1967, 2], [1968, 2]; see also Coleman & Greenberg [1967, 1].

to criteria for dynamical stability.

2. The Space of Histories

Let E^n be a real Euclidean space of n-vectors with norm $|x| = \sqrt{x \cdot x}$, and let V_r be a past-history space, i.e. a Banach function space of the type discussed by Coleman & Mizel# formed from functions $\psi, \phi, \ldots$ mapping $(0,\infty)$ into E^n. We list below the basic properties assumed for V_r:

A. Constants are V_r; i.e. if a is in E^n then α, defined by $\alpha(s) = a$ for $s \in (0,\infty)$, is in V_r.

B. V_r contains all right and left translates of its elements; i.e. if ψ is in V_r then $T^{(\sigma)}\psi$ and $T_{(\sigma)}\psi$, defined by

$$T^{(\sigma)}\psi(s) = \begin{cases} 0 & \text{for } s \in (0,\sigma], \\ \psi(s-\sigma) & \text{for } s \in (\sigma,\infty), \end{cases} \tag{2.1}$$

and

$$T_{(\sigma)}\psi(s) = \psi(s+\sigma) \quad \text{for} \quad s \in (0,\infty), \tag{2.2}$$

are in V_r for each $\sigma \geq 0$.

C. The norm $\|\cdot\|_r$ on V_r is compatible with the natural partial ordering of functions on $(0,\infty)$ in the sense that if ψ is in V_r and if ϕ is a measurable function mapping $(0,\infty)$ into E^n which obeys $|\phi| \overset{\circ}{\leq} |\psi|$,##

#[1967, 3], [1968, 1]. V_r can be constructed by employing an appropriate seminorm ν on measurable functions mapping $(0,\infty)$ into itself and asserting that a measurable function ψ mapping $(0,\infty)$ into E^n corresponds to an element of V_r if and only if $\nu(|\psi|) < \infty$. The norm on V_r is then $\|\psi\|_r = \nu(|\psi|)$ and two functions ϕ and ψ on $(0,\infty)$ are considered the same if $\nu(|\psi - \phi|) = 0$.

##The superposed ° indicates that the given relation holds pointwise a.e., i.e. at all points in $(0,\infty)$ with the possible exception of a set with Lebesgue measure zero.

then $\underset{\sim}{\phi}$ is in $\underline{\underline{V}}_r$ and $\|\underset{\sim}{\phi}\|_r \leq \|\underset{\sim}{\psi}\|_r$. Furthermore, if $\|\cdot\|_r$ is not identically zero, then $\|\underset{\sim}{\psi}\|_r = 0$ only if $\underset{\sim}{\psi} \overset{\circ}{=} \underset{\sim}{0}$. It is also assumed that $\underline{\underline{V}}_r$ has the following Fatou property: If $\underset{\sim}{\phi}_1,\ldots,\underset{\sim}{\phi}_n,\ldots$ are in $\underline{\underline{V}}_r$, if $\|\underset{\sim}{\phi}_n\|_r \leq K < \infty$, and if $|\underset{\sim}{\phi}_n| \uparrow |\underset{\sim}{\psi}|$, pointwise a.e., with $\underset{\sim}{\psi}$ measurable, then $\underset{\sim}{\psi}$ is in $\underline{\underline{V}}_r$ and

$$\lim_{n\to\infty} \|\underset{\sim}{\phi}_n\|_r = \|\underset{\sim}{\psi}\|_r \leq K.$$

D. $\underline{\underline{V}}_r$ is a separable Banach space.

E. $\underline{\underline{V}}_r$ has the relaxation property:#

#Cf. [1966, 1, §6], [1968, 1, §§4,5].

$$\lim_{\sigma\to\infty} \|\underset{\sim}{T}^{(\sigma)}\underset{\sim}{\psi}\|_r = 0 \quad \text{for each } \underset{\sim}{\psi} \in \underline{\underline{V}}_r. \tag{2.3}$$

Remark 2.1. It follows from known results in the theory of Banach function spaces## that Properties C and D imply (a) that continuous functions of compact support are dense in $\underline{\underline{V}}_r$, and (b) that the dominated-convergence theorem holds for $\underline{\underline{V}}_r$: For each $\underset{\sim}{\phi}$ in $\underline{\underline{V}}_r$ and each sequence $\underset{\sim}{\psi}^n$ in $\underline{\underline{V}}_r$ such that $|\underset{\sim}{\psi}^n| \overset{\circ}{\leq} |\underset{\sim}{\phi}|$ for all n and $\underset{\sim}{\psi}^n \to \underset{\sim}{\psi}^0$ pointwise a.e., we have $\|\underset{\sim}{\psi}^n - \underset{\sim}{\psi}^0\|_r \to 0$.

##Luxemburg [1965, 2, Theorem 46.2, p. 241]; Lorentz & Wertheim [1953, 1, see the proof of Theorem 1, pp. 570, 571]; Luxemburg & Zaanen [1956, 1, Theorem 4, p. 117], [1963, 2, Theorems 2.2 - 2.4, p. 157]; Dunford & Schwartz [1958, 1, Exercise 17, p. 170].

Remark 2.2.[#] Properties B-D of $\underline{\underline{V}}_r$ imply that for each ψ in $\underline{\underline{V}}_r$ and each $\sigma \geq 0$

[#] [1967, 3, Appendix 1]. The proof given there uses Property A, but clearly this property of $\underline{\underline{V}}_r$ is not needed. See also [1968, 1, Theorem 3.4].

$$\lim_{\zeta \to \sigma} \| T^{(\zeta)}\psi - T^{(\sigma)}\psi \|_r = 0, \qquad \lim_{\zeta \to \sigma} \| T_{(\zeta)}\psi - T_{(\sigma)}\psi \|_r = 0.$$

Remark 2.3. It follows from Properties B-E of $\underline{\underline{V}}_r$ that the norm

$$\| T^{(\sigma)} \|_r = \sup_{\|\psi\|_r = 1} \| T^{(\sigma)}\psi \|_r$$

of the linear operator $T^{(\sigma)}$, as a function of $\sigma \in [0,\infty)$, is not only submultiplicative, but is also bounded:[##]

[##] [1968, 1, Theorems 3.3 and 4.1]; see also [1966, 1, Remark 6.4].

$$\sup_{\sigma \geq 0} \| T^{(\sigma)} \|_r = M < \infty. \tag{2.4}$$

Given a past-history space $\underline{\underline{V}}_r$ and its norm $\|\cdot\|_r$ we may consider the set $\underline{\underline{V}}$ of measurable functions Ψ which map the half open interval $[0,\infty)$ into E^n and satisfy $\|\Psi_r\|_r < \infty$, where Ψ_r, called the <u>past history</u> of Ψ, is the restriction of Ψ to $(0,\infty)$. The function $\|\cdot\|$ given by

$$\|\Psi\| = |\Psi(0)| + \|\Psi_r\|_r \tag{2.5}$$

is a well defined semi-norm on $\underline{\underline{V}}$. If we identify, in the usual way, functions $\underset{\sim}{\Phi}$, $\underset{\sim}{\Psi}$ in $\underline{\underline{V}}$ obeying $\|\underset{\sim}{\Phi}-\underset{\sim}{\Psi}\| = 0$, then $\underline{\underline{V}}$ becomes a Banach space with $\|\cdot\|$ as its norm. A Banach space so constructed is called a history space.#

#Cf. [1967, 3, §3], [1968, 1, §3]. $\underline{\underline{V}}$ may be referred to also as the space of total histories, to emphasize its distinction from the space $\underline{\underline{V}}_r$ of past histories.

The elements $\underset{\sim}{\Psi}$ of $\underline{\underline{V}}$ are called histories; their independent variable is called the elapsed time and is denoted by s. It follows from (2.5) that, even after identification, each history $\underset{\sim}{\Psi}$ has a well defined value $\underset{\sim}{\Psi}(0)$ at $s = 0$; $\underset{\sim}{\Psi}(0)$ is called the present value of $\underset{\sim}{\Psi}$. It is clear that a continuous functional over $\underline{\underline{V}}$ must have a "special dependence" on the present values of the histories in $\underline{\underline{V}}$.##

##See also [1968, 1, Remark 3.3].

If $\underset{\sim}{x}$ is a vector in E^n, we denote by $\underset{\sim}{x}^{\dagger}$ the constant function on $[0,\infty)$ with value $\underset{\sim}{x}$:

$$\underset{\sim}{x}^{\dagger}(s) \equiv \underset{\sim}{x}, \qquad s \in [0,\infty). \tag{2.6}$$

It follows from (2.5) and Property A of $\underline{\underline{V}}_r$ that $\underset{\sim}{x}^{\dagger}$ is in $\underline{\underline{V}}$. The restriction of $\underset{\sim}{x}^{\dagger}$ to $(0,\infty)$ is denoted by $\underset{\sim}{x}^{\dagger}_r$. Let α and β be the numbers defined by

$$\alpha = \|\underset{\sim}{e}_r^\dagger\|_r \quad \text{and} \quad \beta = \|\underset{\sim}{e}^\dagger\|, \quad \text{when} \quad \underset{\sim}{e}\cdot\underset{\sim}{e} = 1. \tag{2.7}$$

Clearly, $0 \leq \alpha < \infty$ and $\beta = \alpha + 1$.

We write $\underset{\sim}{0}^\dagger$ for the constant function in $\underline{\underline{V}}$ whose value is the zero element $\underset{\sim}{0}$ in E^n, and we denote by $\underline{\underline{S}}(h)$ the open ball in $\underline{\underline{V}}$ about $\underset{\sim}{0}^\dagger$ with radius $h > 0$:

$$\underline{\underline{S}}(h) = \left\{ \underset{\sim}{\Psi} \;\middle|\; \underset{\sim}{\Psi} \in \underline{\underline{V}},\ \|\underset{\sim}{\Psi}\| < h \right\}. \tag{2.8}$$

Given any function Ξ on $\underline{\underline{S}}(h)$, we can define a function Ξ° on a neighborhood S_h of $\underset{\sim}{0}$ in E^n by the formula

$$\Xi^\circ(\underset{\sim}{x}) = \Xi(\underset{\sim}{x}^\dagger); \tag{2.9}$$

Ξ° is called the <u>equilibrium response function corresponding to</u> Ξ.

A measurable function $\underset{\sim}{x}(\cdot)$ mapping $(-\infty,a)$ into E^n is said to be <u>admissible on</u> $(-\infty,a)$ <u>with respect to</u> $\underline{\underline{V}}$ if, for each t in $(-\infty,a)$, we have $\|\underset{\sim}{x}^t\| < \infty$, where $\underset{\sim}{x}^t$, called the <u>history of</u> $\underset{\sim}{x}(\cdot)$ <u>up to</u> t, is the function on $[0,\infty)$ defined by

$$\underset{\sim}{x}^t(s) = \underset{\sim}{x}(t-s). \tag{2.10}$$

Unless we state otherwise, we regard the histories $\underset{\sim}{x}^t$ of an admissible function $\underset{\sim}{x}(\cdot)$ as elements of the Banach space $\underline{\underline{V}}$ rather than as functions.

Elsewhere# we have given arguments which we believe strongly

#[1968, 1, Theorems 2.1 and 3.1].

motivate the present method of constructing the space $\underline{\underline{V}}$ of histories. Here we shall content ourselves with the mention of an example often employed in continuum physics.##

##See Coleman & Noll [1960, 1], [1961, 1]; Coleman [1964, 1]; Coleman & Mizel [1966, 1].

Remark 2.4. For fixed $p \geq 1$, the norm $\|\cdot\|$ given by

$$\|\underset{\sim}{\Psi}\| = |\underset{\sim}{\Psi}(0)| + \|\underset{\sim}{\Psi}_r\|_r, \qquad \|\underset{\sim}{\Psi}_r\|_r^p = \int_0^\infty |\underset{\sim}{\Psi}_r(s)|^p \ell(s)ds, \tag{2.11}$$

and the corresponding set $\underline{\underline{V}}$ of functions $\underset{\sim}{\Psi}$ mapping $[0,\infty)$ in E^n with $\|\underset{\sim}{\Psi}\| < \infty$ define a history space whenever ℓ is a fixed measurable function on $(0,\infty)$ obeying the following four conditions:

I. $$\int_0^\infty \ell(s)ds < \infty;$$

II. $$\ell \overset{\circ}{>} 0\ ;$$

III. the functions $\overline{K}$ and $\underline{K}$ defined by

$$\overline{K}(\sigma) = \operatorname*{ess.sup}_{s>0} \frac{\ell(s+\sigma)}{\ell(s)}, \qquad \underline{K}(\sigma) = \operatorname*{ess.sup}_{s>0} \frac{\ell(s)}{\ell(s+\sigma)}$$

have finite values for all $\sigma \in [0,\infty)$;

IV. $$\sup_{s>0} \overline{K}(s) < \infty.$$

In fact, if $\ell(s)$ is not almost everywhere zero, I-IV supply necessary and sufficient conditions that the norm $\|\cdot\|_r$ defined by $(2.11)_2$ and the corresponding set of functions ψ mapping $(0,\infty)$ into E^n with $\|\psi\|_r < \infty$ form a past history space. Condition I is clearly equivalent to Property A; it can be shown that II and III together are equivalent to Property B;# given I, II, and III, the condition IV is equivalent to Property E;## and for $1 \leq p < \infty$ these $\mathcal{L}_p$-spaces automatically have Properties C and D. In this example, $\|T^{(\sigma)}\|_r = \overline{K}(\sigma)^{1/p}$, $\|T_{(\sigma)}\|_r = \underline{K}(\sigma)^{1/p}$, and a sufficient condition for Property E (i.e. for IV) is that ℓ be monotone decreasing on $(0,\infty)$.

#[1966, 1, Theorem 3, p. 101].

##[1966, 1, Remark 6.4, p. 111, and Eq. (4.26), p. 101].

To illustrate a method of working with a general history space V, we now prove an easy, but useful, lemma.

Lemma 2.1. Given any $\eta > 0$, there is a $\zeta = \zeta(\eta) > 0$ such that any measurable function $x(\cdot)$ mapping $(-\infty,\infty)$ into E^n that obeys $|x(t)| < \zeta$ for all $t > 0$ and has x^0 in $S(\zeta)$ must have x^t in $S(\eta)$ for each $t \geq 0$.

Proof. By (2.5), (2.10), (2.1), and the triangle inequality

$$\|x^t\| = |x^t(0)| + \|x_r^t\|_r = |x(t)| + \|\chi_{(0,t)}x_r^t + T^{(t)}x_r^0\|_r \leq |x(t)| + \|\chi_{(0,t)}x_r^t\|_r + \|T^{(t)}x_r^0\|_r, \tag{2.12}$$

where $\chi_{(0,t)}$ is the characteristic function of $(0,t)$. Suppose that, for some number $\zeta > 0$, we have $|\underset{\sim}{x}(t)| < \zeta$ for $t > 0$ and also $\|\underset{\sim}{x}^0\| < \zeta$. Then, by (2.10) and (2.5), we have

$$|\underset{\sim}{x}(t)| < \zeta \tag{2.13}$$

not only for $t > 0$ but also for $t = 0$. Furthermore, for each $t \geq 0$ we have

$$|\chi_{(0,t)}(s)\underset{\sim}{x}_r^t(s)| \leq \zeta = |\zeta \underset{\sim}{e}_r^\dagger(s)| \quad \text{for all} \quad s > 0,$$

where $\underset{\sim}{e}_r^\dagger$ is a constant function on $(0,\infty)$ whose value is a unit vector in E^n; thus, by (2.7) and Property C of $\underline{\underline{V}}_r$,

$$\|\chi_{(0,t)}\underset{\sim}{x}_r^t\|_r \leq \|\zeta \underset{\sim}{e}_r^\dagger\|_r = \zeta\alpha. \tag{2.14}$$

Of course,

$$\|\underset{\sim}{T}^{(t)}\underset{\sim}{x}_r^0\|_r \leq \|\underset{\sim}{T}^{(t)}\|_r\|\underset{\sim}{x}_r^0\|_r,$$

and, by Remark 2.3, this yields, for $t \geq 0$,

$$\|\underset{\sim}{T}^{(t)}\underset{\sim}{x}_r^0\|_r \leq M\|\underset{\sim}{x}_r^0\|_r < M\zeta, \tag{2.15}$$

with M the constant in (2.4). Substituting (2.13)-(2.15) into (2.12) we find that for $t \geq 0$

$$\|\underset{\sim}{x}^t\| < \zeta(1+\alpha+M),$$

and if we now put

$$\zeta(\eta) \stackrel{\text{def}}{=} \frac{\eta}{1+\alpha+M},$$

the lemma is proven.

As a corollary to the proof of Lemma 2.1 we have

Lemma 2.2. If $\underset{\sim}{x}(\cdot)$ is a measurable function mapping $(-\infty,\infty)$ into E^n with $\underset{\sim}{x}^0$ in $\underline{V}$ and

$$\sup_{t\in[0,\infty)} |\underset{\sim}{x}(t)| \stackrel{\text{def}}{=} \zeta < \infty,$$

then for each $t \geq 0$

$$\|\underset{\sim}{x}^t\|_r \leq \zeta\alpha + M\|\underset{\sim}{x}_r^0\| \stackrel{\text{def}}{=} H < \infty$$

$$\|\underset{\sim}{x}^t\| \leq H + \zeta < \infty,$$

with α and M defined in (2.7) and (2.4).

3. Basic Definition, Free Energy Functionals, and Stability

Let a history space $\underline{\underline{V}}$ be given, let h be a positive real constant, and let $\underset{\sim}{f}$ be a function mapping $\underline{\underline{S}}(h)$ into E^n and enjoying the following properties:

(1) $\underset{\sim}{f}$ is continuous over $\underline{\underline{S}}(h)$.

(2) $\underset{\sim}{f}$ is bounded in the sense that $\underset{\sim}{f}\big(\underline{\underline{S}}(h)\big)$ is a set in E^n with finite diameter.

(3) $\underset{\sim}{f}(\underset{\sim}{0}^\dagger) = \underset{\sim}{0}$.

If $\underset{\sim}{\Psi}$ is in $\underline{\underline{V}}$ and if A is in $(0,\infty]$, then we say that a function $\underset{\sim}{x}(\cdot)$ mapping $(-\infty,A)$ into E^n is a <u>solution up to A of the equation</u>

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{f}(\underset{\sim}{x}^t), \tag{3.1}$$

<u>with initial history</u> $\underset{\sim}{\Psi}$, provided

(a) $\underset{\sim}{x}(\cdot)$ is admissible on $(-\infty,A)$ with respect to $\underline{\underline{V}}$, and $\underset{\sim}{x}^t$ is in $\underline{\underline{S}}(h)$ for t in $(0,A)$;

(b) $\underset{\sim}{x}(t)$ is continuously differentiable in the classical sense for t in $(0,A)$ and has a right-hand derivative, $\dot{\underset{\sim}{x}}(0)$, at $t = 0$;

(c) (3.1) holds for all t in $[0,A)$ with $\underset{\sim}{x}^t$ the history of $\underset{\sim}{x}(\cdot)$ up to t;

(d) $\underset{\sim}{x}^0$, the history of $\underset{\sim}{x}(\cdot)$ up to 0, is equal to $\underset{\sim}{\Psi}$.

Remark 3.1. Properties (1) and (2) of $\underset{\sim}{f}$ insure that for each $\underset{\sim}{\Psi}$ in $\underline{\underline{S}}(h)$ there exists an $A > 0$ and a function $\underset{\sim}{x}(\cdot)$ which is a solution up to A of (3.1) with initial history $\underset{\sim}{\Psi}$. Furthermore, this solution can be extended, i.e. A can be increased, until $\underset{\sim}{x}^t$ reaches the boundary of $\underline{\underline{S}}(h)$.

Since we do not assume that $\underset{\sim}{f}$ is Lipschitz continuous on $\underline{\underline{S}}(h)$, we know nothing about the uniqueness of solutions corresponding to an arbitrary initial history $\underset{\sim}{\Psi}$ in $\underline{\underline{S}}(h)$.

Of course, it follows from Property (3) that $\underset{\sim}{x} \equiv \underset{\sim}{0}$ is a solution of (3.1) up to ∞ with initial history $\underset{\sim}{0}^{\dagger}$; we call this the <u>zero solution</u>.

The solution $\underset{\sim}{x} \equiv \underset{\sim}{0}$ of (3.1) is said to be <u>stable</u> if for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that if $\underset{\sim}{\Psi}$ is in $\underline{\underline{S}}(\delta)$ then (a) every solution $\underset{\sim}{x}(\cdot)$ of (3.1) with initial history $\underset{\sim}{\Psi}$ has $|\underset{\sim}{x}(t)| < \epsilon$ for each $t \geq 0$ in its domain of existence, and (b) a solution of (3.1) with initial history $\underset{\sim}{\Psi}$ exists for all $t < \infty$. If, in addition, there is a $\zeta > 0$ such that each solution $\underset{\sim}{x}(\cdot)$ of (3.1) with its initial history $\underset{\sim}{x}^0$ in $\underline{\underline{S}}(\zeta)$ obeys $\lim_{t\to\infty} |\underset{\sim}{x}(t)| = 0$, then the solution $\underset{\sim}{x} \equiv \underset{\sim}{0}$ is said to be <u>asymptotically stable</u>.

It follows from Lemma 2.1 and Remark 3.1 that <u>the zero solution of (3.1) is stable if and only if for each ϵ in (0,h) there is a $\delta = \delta(\epsilon) > 0$ such that $\underset{\sim}{\Psi}$ in $\underline{\underline{S}}(\delta)$ implies that every solution $\underset{\sim}{x}(\cdot)$ of (3.1) with $\underset{\sim}{x}^0 = \underset{\sim}{\Psi}$ has $\underset{\sim}{x}^t$ in $\underline{\underline{S}}(\epsilon)$ for each $t \geq 0$</u>. (We shall discuss this point in more detail in our proof of Theorem 3.1.)

A real-valued function Ξ on $\underline{\underline{S}}(h)$ is called a free energy functional for the equation (3.1) if

(i) Ξ is continuous over $\underline{\underline{S}}(h)$,

(ii) for each solution up to A of (3.1), $\Xi(\underset{\sim}{x}^t)$ is a non-increasing function of t for $t \in [0,A)$, and

(iii) for each history $\underset{\sim}{\Psi}$ in $\underline{\underline{S}}(h)$ with $\underset{\sim}{\Psi}(0)$ in the domain S_h of the equilibrium response function Ξ° corresponding to Ξ,

$$\Xi(\underset{\sim}{\Psi}) \geq \Xi^\circ\big(\underset{\sim}{\Psi}(0)\big). \tag{3.2}$$

We say that a real-valued function, such as Ξ°, defined on a neighborhood of $\underset{\sim}{0}$ in E^n has a strict local minimum at $\underset{\sim}{0}$ if there exists an $\eta > 0$ such that

$$\underset{\sim}{x} \in E^n, \quad 0 < |\underset{\sim}{x}| < \eta \quad \Longrightarrow \quad \Xi^\circ(\underset{\sim}{x}) > \Xi^\circ(\underset{\sim}{0}). \tag{3.3}$$

Theorem 3.1. *If there exists a free energy functional Ξ for (3.1) whose equilibrium response function Ξ° has a strict local minimum at $\underset{\sim}{0}$, then the zero solution of (3.1) is stable.*

Proof. A proof we have given elsewhere# shows that in the present

#[1968, 2, Theorem 1]. The metric $\rho(\Phi,\Psi) = \|\Phi - \Psi\|$, with $\|\cdot\|$ the norm on the history space V, clearly obeys the postulates laid down in that paper. The set F mentioned there should here be chosen to be a small neighborhood of 0 in E^n.

situation there does exist, for each $\epsilon > 0$, a $\delta_1 = \delta_1(\epsilon) > 0$ such that each solution $x(\cdot)$ of (3.1) with x^0 in $S(\delta_1)$ obeys $|x(t)| < \epsilon$ for all $t \geq 0$ which are in its domain of existence. If we can now produce a $\delta_2 > 0$ such that for each Ψ in $S(\delta_2)$ a solution of (3.1) with initial history Ψ exists for all t, then Theorem 1 will be proven, for $\delta = \min(\delta_1, \delta_2)$ will then have the properties (a) and (b) mentioned in the definition of stability. Clearly, by Lemma 2.1, there exists a $\zeta > 0$ such that if $x(\cdot)$, with domain $(-\infty, A)$, $A > 0$, has x^0 in $S(\zeta)$ and $|x(t)| < \zeta$ for t in $[0,A)$, then $x(\cdot)$ will have x^t in $S(h)$ for each t in $[0,A)$. Now, if we put $\delta_2 = \min(\zeta, \delta_1(\zeta))$, then each solution $x(\cdot)$ of (3.1) with x^0 in $S(\delta_2)$ will have $|x(t)| < \zeta$ for all $t \geq 0$ in its domain of existence, and hence no solution with x^0 in $S(\delta_2)$ will have x^t reaching the boundary of $S(h)$ at a time $t \geq 0$. But, by Remark 3.1, this implies that every solution of (3.1) with its initial history in $S(\delta_2)$ can be extended indefinitely; q.e.d.

4. Some Lemmas

We have observed that any solution $\underset{\sim}{x}(\cdot)$ of (3.1) with $\underset{\sim}{x}^0$ in $\underline{\underline{S}}(h)$ can be extended until a time t at which $\underset{\sim}{x}^t$ reaches the boundary $\partial\underline{\underline{S}}(h)$ of $\underline{\underline{S}}(h)$. Henceforth, whenever it is known in advance that $\underset{\sim}{x}^t$ cannot reach $\partial\underline{\underline{S}}(h)$ at a finite value of t let us suppose that the solution has been "extended to infinity", i.e. that $\underset{\sim}{x}(\cdot)$ is defined on $(-\infty,\infty)$. Given such a solution we may consider at each time t the <u>truncated history</u> $\Delta\underset{\sim}{x}_r^t$ defined by:

$$\left.\begin{array}{l} \text{when } t > 0, \quad \Delta\underset{\sim}{x}_r^t(s) = \begin{cases} \underset{\sim}{x}^t(s) - \underset{\sim}{x}^0(0) = \underset{\sim}{x}(t-s) - \underset{\sim}{x}(0) & \text{for } s \in (0,t], \\ \underset{\sim}{0} \quad \text{for } s \in (t,\infty), \end{cases} \\ \\ \text{when } t \le 0, \quad \Delta\underset{\sim}{x}_r^t(s) = 0 \quad \text{for } s \in (0,\infty). \end{array}\right\} \quad (4.1)$$

Since $\underset{\sim}{x}(\cdot)$ is a solution, $\underset{\sim}{x}(\cdot)$ is admissible with respect to $\underline{\underline{V}}$, and hence each past history $\underset{\sim}{x}_r^t$ along $\underset{\sim}{x}(\cdot)$ is in $\underline{\underline{V}}_r$. Moreover, $|\underset{\sim}{x}_r^t(s) - \underset{\sim}{x}(0)| \ge |\Delta\underset{\sim}{x}_r^t(s)|$ for all s in $(0,\infty)$, and thus each truncated history $\Delta\underset{\sim}{x}_r^t$ is also in $\underline{\underline{V}}_r$. We denote by $\{\Delta\underset{\sim}{x}_r^t\}$ the set of all the truncated histories, with $-\infty < t < \infty$, occurring along a given solution $\underset{\sim}{x}(\cdot)$ of (3.1).

The set of all histories $\underset{\sim}{x}^t$ with $t \ge 0$ occurring along a preassigned solution of (3.1) is denoted by $\{\underset{\sim}{x}^t \mid t \ge 0\}$. Of course, $\{\underset{\sim}{x}^t \mid t \ge 0\}$ is a subset of $\underline{\underline{V}}$.

Lemma 4.1. If $\underset{\sim}{x}(\cdot)$ is a solution of (3.1) with $\{\underset{\sim}{x}^t \mid t \geq 0\} \subset \underline{S}(h)$, then the sets $\{\Delta \underset{\sim}{x}_r^t\}$ and $\{\underset{\sim}{x}^t \mid t \geq 0\}$ corresponding to $\underset{\sim}{x}(\cdot)$ are both precompact; that is

(1) $\{\Delta \underset{\sim}{x}_r^t\}$ is contained in a compact subset of $\underline{V}_r$, and

(2) $\{\underset{\sim}{x}^t \mid t \geq 0\}$ is contained in a compact subset of $\underline{V}$.

Proof. We first prove (1), employing the fact that for metric spaces compactness is equivalent to sequential compactness. Since it is assumed that

$$\|\underset{\sim}{x}^t\| < h \quad \text{for} \quad t \in [0,\infty) \tag{4.2}$$

and since $|\underset{\sim}{x}(t)| = |\underset{\sim}{x}^t(0)| \leq \|\underset{\sim}{x}^t\|$, we have

$$|\Delta \underset{\sim}{x}_r^t(s)| < 2h \quad \text{for} \quad t \in (-\infty,\infty),\ s \in (0,\infty). \tag{4.3}$$

Furthermore, (3.1), (4.2), and Property 2 of $\underset{\sim}{f}$ tell us that

$$\sup_{t \geq 0} |\dot{\underset{\sim}{x}}(t)| = \sup_{t \geq 0} |\underset{\sim}{f}(\underset{\sim}{x}^t)| \stackrel{\text{def}}{=} K < \infty,$$

i.e.#

#At s = t, (4.4) gives a bound on the left-hand derivative of $\Delta \dot{\underset{\sim}{x}}_r^t(\cdot)$ and the right-hand derivative is zero. Elsewhere $\Delta \underset{\sim}{x}^t(\cdot)$ is differentiable in the classical sense.

$$\left|\frac{d}{ds} \Delta \underset{\sim}{x}_r^t(s)\right| \leq K < \infty \quad \text{for} \quad t \in (-\infty,\infty),\ s \in (0,\infty). \tag{4.4}$$

Thus, $\{\Delta \underset{\sim}{x}_r^t\}$ is an equicontinuous, uniformly bounded, set of functions, and, by a standard argument involving diagonalization, each sequence $\Delta \underset{\sim}{x}_r^{t_n}$ composed of functions in $\{\Delta \underset{\sim}{x}_r^t\}$ has a subsequence $\Delta \underset{\sim}{x}_r^{t_m}$ which converges uniformly on every compact subset of $(0,\infty)$. The limit function $\underset{\sim}{\psi}$ of $\Delta \underset{\sim}{x}_r^{t_m}$ is continuous on $(0,\infty)$ and, by (4.3), obeys

$$|\underset{\sim}{\psi}(s)| \leq 2h \quad \text{for} \quad s \in (0,\infty). \tag{4.5}$$

Hence, by Properties A and C of $\underline{\underline{V}}_r$, $\underset{\sim}{\psi}$ is in $\underline{\underline{V}}_r$. To complete our proof of (1) we must show that

$$\lim_{m\to\infty} \|\Delta \underset{\sim}{x}_r^{t_m} - \underset{\sim}{\psi}\|_r = 0. \tag{4.6}$$

As we observed in Remark 2.1, the dominated convergence theorem holds in $\underline{\underline{V}}_r$, and it follows trivially from (4.3) and (4.5) that if $\underset{\sim}{e}\cdot\underset{\sim}{e} = 1$, then

$$|\Delta \underset{\sim}{x}^{t_m}(s) - \underset{\sim}{\psi}(s)| < |4h\underset{\sim}{e}_r^{\dagger}(s)| \quad \text{for all } s \in (0,\infty) \text{ and each m.}$$

Of course, by Property A of $\underline{\underline{V}}_r$, the function $4h\underset{\sim}{e}_r^{\dagger}$ is in $\underline{\underline{V}}_r$, and therefore, by Remark 2.1, the pointwise convergence of $\Delta x_r^{t_m}$ to $\underset{\sim}{\psi}$ does imply (4.6). Thus the closure of $\{\Delta \underset{\sim}{x}_r^t\}$ is compact in $\underline{\underline{V}}_r$.

To demonstrate statement (2) of the lemma we let $\underset{\sim}{x}^{t_n}$ be a sequence in $\{\underset{\sim}{x}^t \mid t \geq 0\}$. We wish to show that $\underset{\sim}{x}^{t_n}$ has a subsequence $\underset{\sim}{x}^{t_m}$ obeying

$$\lim_{m\to\infty} \|\underset{\sim}{x}^{t_m} - \underset{\sim}{\Phi}\| = 0 \tag{4.7}$$

for some function $\underset{\sim}{\Phi}$ in $\underline{\underline{V}}$. Since $|\underset{\sim}{x}^t(0)| < \|\underset{\sim}{x}^t\|$, it follows from (4.2) that the vectors $\underset{\sim}{x}^{t_n}(0) = \underset{\sim}{x}(t_n)$ form a bounded sequence in E^n, and, by the Bolzano-Weierstrass theorem, the sequence of numbers t_n has a subsequence t_ℓ such that

$$\lim_{\ell \to \infty} |\underset{\sim}{x}(t_\ell) - \underset{\sim}{a}| = 0$$

for some vector $\underset{\sim}{a}$ in E^n. Putting $\underset{\sim}{\Phi}(0)$ equal to $\underset{\sim}{a}$ we observe that, by (2.5), to establish (4.7) it suffices to show that t_ℓ has a subsequence t_m such that

$$\lim_{m \to \infty} \|\underset{\sim}{x}_r^{t_m} - \Phi_r\|_r = 0 \qquad (4.8)$$

for some function $\underset{\sim}{\Phi}_r$ in $\underline{\underline{V}}_r$. If the sequence t_ℓ is bounded above, then t_ℓ will have a subsequence t_m with a finite limit τ, i.e.

$$t_m \to \tau < \infty \quad \text{as} \quad m \to \infty,$$

and it will follow from Remark 2.2 that (4.8) holds with $\underset{\sim}{\Phi}_r = \underset{\sim}{x}_r^\tau$. Therefore, the only case to be examined is that in which t_ℓ is unbounded and hence has a subsequence t_m obeying

$$t_m \to \infty \quad \text{as} \quad m \to \infty. \qquad (4.9)$$

In this case we observe that (2.1) and (4.1) yield, for $t > 0$,

$$\underset{\sim}{x}_r^t = \Delta \underset{\sim}{x}_r^t + \underset{\sim}{x}(0)\chi_{(0,t]} + \underset{\sim}{I}^{(t)} \underset{\sim}{x}_r^0 \qquad (4.10)$$

with $\chi_{(0,t]}$ the characteristic function of (0,t . Clearly, we may choose the sequence t_m in (4.9) in such a way that (4.6) holds for some function $\underset{\sim}{\psi}$ in $\underline{\underline{V}}_r$. Employing the function $\underset{\sim}{\psi}$ so obtained, we define $\underset{\sim}{\Phi}_r$ as

$$\underset{\sim}{\Phi}_r = \underset{\sim}{\psi} + \underset{\sim}{x}(0)\chi_{(0,\infty)}. \tag{4.11}$$

Properties A and C of $\underline{\underline{V}}_r$ imply that $\underset{\sim}{x}(0)\chi_{(0,\infty)}$, and hence $\underset{\sim}{\Phi}_r$, is in $\underline{\underline{V}}_r$, while (4.10) and (4.11) imply

$$\|\underset{\sim}{x}_r^{t_m} - \underset{\sim}{\Phi}_r\|_r \leq \|\Delta\underset{\sim}{x}_r^{t_m} - \underset{\sim}{\psi}\|_r + \|\underset{\sim}{x}(0)\chi_{(0,t_m]} - \underset{\sim}{x}(0)\chi_{(0,\infty)}\|_r + \|\underset{\sim}{T}^{(t_m)}\underset{\sim}{x}_r^0\|_r . \tag{4.12}$$

By Property E of $\underline{\underline{V}}_r$, (4.9) yields

$$\lim_{m\to\infty}\|\underset{\sim}{T}^{(t_m)}\underset{\sim}{x}_r^0\|_r = 0, \tag{4.13}$$

and, because the dominated convergence theorem holds in $\underline{\underline{V}}_r$, (4.9) also yields

$$\lim_{m\to\infty}\|\underset{\sim}{x}(0)\chi_{(0,t_m]} - \underset{\sim}{x}(0)\chi_{(0,\infty)}\|_r = 0. \tag{4.14}$$

It is clear that (4.6) and (4.12)-(4.14) imply (4.8); q.e.d.

With each solution $\underset{\sim}{x}(\cdot)$ of (3.1) which can be extended to infinity we associate a set Ω (possibly empty) called the ω-<u>limit set</u> of $\underset{\sim}{x}(\cdot)$ and defined as follows: $\underset{\sim}{\Psi}$ is in Ω if and only if $\underset{\sim}{\Psi}$ is in $\underline{\underline{V}}$ and there exists a sequence of positive numbers t_m such that

$$t_m \to \infty \quad \text{and} \quad \|\underset{\sim}{x}^{t_m} - \underset{\sim}{\Psi}\| \to 0 \quad \text{as} \quad m \to \infty. \tag{4.15}$$

Lemma 4.2.# <u>If $\underset{\sim}{x}(\cdot)$ is a solution of (3.1) with $\{\underset{\sim}{x}^t \mid t \geq 0\} \subset \underline{\underline{S}}(h)$, then</u>

#Cf. Hale [1965, 1, Lemma 3].

<u>the ω-limit set Ω of $\underset{\sim}{x}(\cdot)$ is a non-empty compact subset of $\underline{\underline{V}}$, and</u>

$$\text{dist}(\underset{\sim}{x}^t, \Omega) \to 0 \quad \text{as} \quad t \to \infty. \tag{4.16}$$

Proof. It follows from Lemma 4.1 that if $\underset{\sim}{x}(\cdot)$ is a solution of (3.1) with $\{\underset{\sim}{x}^t \mid t \geq 0\}$ in $\underline{\underline{S}}(h)$, then each unbounded positive sequence t_n has a subsequence t_m obeying (4.15) for some element $\underset{\sim}{\Psi}$ of $\underline{\underline{V}}$. Hence, Ω is not empty. Since Ω is a subset of the closure $\underline{\underline{C}}$ of $\{\underset{\sim}{x}^t \mid t \geq 0\}$ in $\underline{\underline{V}}$, and since, by Lemma 4.1, $\underline{\underline{C}}$ is compact, to prove that Ω is compact it suffices to show that Ω is closed in $\underline{\underline{V}}$. To this end, suppose that $\underset{\sim}{\Psi}_n$ in Ω approaches $\underset{\sim}{\Psi}$ in $\underline{\underline{V}}$ as $n \to \infty$. There then exists, for each n, an unbounded sequence $t_\ell(n)$ of positive numbers such that

$$t_\ell(n) \to \infty \quad \text{and} \quad \|\underset{\sim}{x}^{t_\ell(n)} - \underset{\sim}{\Psi}_n\| \to 0 \quad \text{as} \quad \ell \to \infty,$$

and for any given integer m we can find an integer k such that

$$\|\underset{\sim}{\Psi}_k - \underset{\sim}{\Psi}\| < \frac{1}{2m}.$$

Moreover, by choosing $\ell_k \geq k$ sufficiently large we will have

$$t_{\ell_k}(k) > m \quad \text{and} \quad \left\|\underset{\sim}{x}^{t_{\ell_k}(k)} - \underset{\sim}{\Psi}_k\right\| < \frac{1}{2m},$$

and therefore

$$t_{\ell_k}(k) > m \quad \text{and} \quad \left\| \underset{\sim}{x}^{t_{\ell_k}(k)} - \underset{\sim}{\Psi} \right\| < \frac{1}{m} .$$

If we write t_m for $t_{\ell_k}(k)$, then the sequence t_m so obtained obviously obeys (4.15), which proves that $\underset{\sim}{\Psi}$ is in Ω, i.e. that Ω is closed and hence compact.

If (4.16) were not true there would exist a positive sequence t_n and a number $\gamma > 0$ such that

$$t_n \to \infty \quad \text{and} \quad \| \underset{\sim}{x}^{t_n} - \underset{\sim}{\Psi} \| \geq \gamma \tag{4.17}$$

for all $\underset{\sim}{\Psi}$ in Ω. But, since $\{x^t \mid t \geq 0\}$ is precompact, $\underset{\sim}{x}^{t_n}$ must have a subsequence $\underset{\sim}{x}^{t_m}$ such that, for some $\underset{\sim}{\Psi}$ in $\underline{\underline{V}}$,

$$\lim_{m \to \infty} \| \underset{\sim}{x}^{t_m} - \underset{\sim}{\Psi} \| = 0,$$

and the $\underset{\sim}{\Psi}$ so obtained is clearly in Ω. Thus, (4.17) is impossible, and (4.16) must hold; q.e.d.

One can easily show that the hypothesis of Lemma 4.2 implies, further, that Ω is a <u>connected</u> subset of $\underline{\underline{V}}$.

A set $\underline{\underline{T}}$ in $\underline{\underline{S}}(h)$ is called an <u>invariant set for</u> (3.1) if for each $\underset{\sim}{\Psi}$ in $\underline{\underline{T}}$ there exists a function $\underset{\sim}{y}(\cdot)$ on $(-\infty,\infty)$ such that

(a) $\underset{\sim}{y}^t$ is in $\underline{\underline{T}}$ for each t in $(-\infty,\infty)$;

(b) $\underset{\sim}{y}(\cdot)$ is continuously differentiable on $(-\infty,\infty)$;

(c) $\dot{\underset{\sim}{y}}(t) = \underset{\sim}{f}(\underset{\sim}{y}^t)$ for each t in $(-\infty,\infty)$;

(d) $\underset{\sim}{y}^0 = \underset{\sim}{\Psi}$.#

#<u>Note</u>: We do not require that <u>every</u> solution $\underset{\sim}{y}(\cdot)$ with $\underset{\sim}{y}^0$ in $\underline{\underline{T}}$ have the properties (a)-(c). Hence, what we call an <u>invariant set</u> others may prefer to call a "quasi-invariant set".

Lemma 4.3.# <u>If</u> Ω <u>is the</u> ω-<u>limit set of a solution</u> $\underset{\sim}{x}(\cdot)$ <u>of</u> (3.1) <u>having</u> $\{\underset{\sim}{x}^t \mid t \geq 0\} \subset \underline{\underline{S}}(h_1)$ <u>with</u> $h_1 < h$, <u>then</u> Ω <u>is contained in</u> $\underline{\underline{S}}(h)$ <u>and is an invariant set for</u> (3.1).

#Cf. Hale [1965, 1, Lemma 3].

Proof. If $\underset{\sim}{\Psi}$ is an element of Ω, then there must exist a positive sequence t_m obeying (4.15). Since we here assume that each $\underset{\sim}{x}^{t_m}$ in (4.15) is in $\underline{\underline{S}}(h_1)$, $\underset{\sim}{\Psi}$ must be in $\underline{\underline{S}}(h)$; hence Ω is contained in $\underline{\underline{S}}(h)$. Now, since $|\underset{\sim}{x}(t)| = |\underset{\sim}{x}^t(0)| \leq \|\underset{\sim}{x}^t\|$ and $\{\underset{\sim}{x}^t \mid t \geq 0\} \subset \underline{\underline{S}}(h_1) \subset \underline{\underline{S}}(h)$, we here have (4.3) holding and hence, for the sequence $\underset{\sim}{x}^{t_m}$ obeying (4.15),

$$|\Delta\underset{\sim}{x}_r^{t_m+\tau}(s)| < 2h \quad \text{for} \quad s \in (0,\infty),\ \tau \in (-\infty,\infty), \tag{4.18}$$

and, by (3.1), (4.1), and Property 2 of $\underset{\sim}{f}$,

$$\left|\frac{d}{ds}\,\Delta\underset{\sim}{x}_r^{t_m+\tau}(s)\right| \leq K < \infty \quad \text{for} \quad s \in (0,\infty),\ \tau \in (-\infty,\infty). \tag{4.19}$$

Employing Ascoli's theorem and a diagonalization argument, one easily proves that (4.18) and (4.19) imply the existence, for each integer N, of a subsequence $t_\ell = t_\ell(N)$ of t_m and a function $\underset{\sim}{\psi}^{(N)}(\cdot)$ such that as $\ell \to \infty$,

$$|\Delta\underset{\sim}{x}_r^{t_\ell+N}(s) - \underset{\sim}{\psi}^{(N)}(s)| \to 0,$$

uniformly in s for s in compact subsets of $(0,\infty)$. Clearly, since

$$\Delta\underset{\sim}{x}_r^{t_\ell+\tau}(s) = \Delta\underset{\sim}{x}_r^{t_\ell+N}(s+N-\tau), \tag{4.20}$$

the sequence t_ℓ has the additional property that as $\ell \to \infty$,

$$|\Delta \underset{\sim}{x}_r^{t_\ell+\tau}(s) - \underset{\sim}{\psi}^{(\tau)}(s)| \to 0,$$

uniformly in s and τ for s in compact subsets of $(0,\infty)$ and τ in $[-N,N]$; here $\underset{\sim}{\psi}^{(\tau)}(s) = \underset{\sim}{\psi}^{(N)}(s+N-\tau)$. We may now employ diagonalization relative to the integer N to find a subsequence t_k of t_m and a function $\underset{\sim}{\psi}^{(\cdot)}(\cdot)$ such that as $k \to \infty$, we have $t_k \to \infty$ and

$$|\Delta \underset{\sim}{x}_r^{t_k+\tau}(s) - \underset{\sim}{\psi}^{(\tau)}(s)| \to 0, \tag{4.21}$$

uniformly in s and τ for s in compact subsets of $(0,\infty)$ and τ in compact subsets of $(-\infty,\infty)$. Furthermore, by (4.18), we have $|\underset{\sim}{\psi}^{(\tau)}(s)| \leq 2h$ for all s in $(0,\infty)$ and all τ in $(-\infty,\infty)$. The continuity of $\Delta \underset{\sim}{x}_r^t(s)$, the relation (4.20), and the uniformity of the convergence in (4.21), imply that $\underset{\sim}{\psi}^{(\tau)}(s)$ is continuous in both s and τ. Thus, $\underset{\sim}{\psi}^{(\tau)}(\cdot)$ is in $\underset{=}{V}_r$ for each τ and, by Part b of Remark 2.1, we have

$$\lim_{k\to\infty} \|\Delta \underset{\sim}{x}_r^{t_k+\tau} - \underset{\sim}{\psi}^{(\tau)}\|_r = 0 \quad \text{for} \quad \tau \in (-\infty,\infty). \tag{4.22}$$

Furthermore, because $\underset{\sim}{\psi}^{(\tau)}(s) = \underset{\sim}{\psi}^{(N)}(s+N-\tau)$ there exists a continuous function $\underset{\sim}{z}(\cdot)$, on $(-\infty,\infty)$, such that $\underset{\sim}{z}^t = \underset{\sim}{\psi}^{(t)}$. Hence $\underset{\sim}{y} = \underset{\sim}{z} + \underset{\sim}{x}(0)$ is a continuous function on $(-\infty,\infty)$ obeying $\underset{\sim}{y}^\tau(s) = \underset{\sim}{\psi}^{(\tau)}(s) + \underset{\sim}{x}(0)$ for s in $(0,\infty)$; i.e.

$$\underset{\sim}{y}_r^\tau = \underset{\sim}{\psi}^{(\tau)} + \underset{\sim}{x}(0)\chi_{(0,\infty)} \quad \text{for} \quad \tau \in (-\infty,\infty), \tag{4.23}$$

where $\underset{\sim}{y}^\tau$ is the history and $\underset{\sim}{y}_r^\tau$ the past history of $\underset{\sim}{y}(\cdot)$ up to time τ.

Since $t_k \to \infty$, (4.21) and (4.1) imply that

$$|\underset{\sim}{x}(t_k+\tau-s) - \underset{\sim}{y}(\tau-s)| \to 0 \quad \text{pointwise, for} \quad s \in (0,\infty),\ \tau \in (-\infty,\infty),$$

as $k \to \infty$. This limit holds also for $s = 0$ and therefore implies that for τ in $(-\infty,\infty)$

$$|\underset{\sim}{x}(t_k+\tau) - \underset{\sim}{y}(\tau)| = |\underset{\sim}{x}^{t_k+\tau}(0) - \underset{\sim}{y}^{\tau}(0)| \to 0 \tag{4.24}$$

as $k \to \infty$. Employing (4.10) and (4.23) we may write, for $t_k > -\tau$,

$$\underset{\sim}{x}_r^{t_k+\tau} - \underset{\sim}{y}_r^{\tau} = \Delta\underset{\sim}{x}_r^{t_k+\tau} + \underset{\sim}{x}(0)\chi_{(0,t_k+\tau]} + \underset{\sim}{T}^{t_k+\tau}\underset{\sim}{x}_r^0 - \left(\underset{\sim}{\psi}^{(\tau)} + \underset{\sim}{x}(0)\chi_{(0,\infty)}\right);$$

thus

$$\|\underset{\sim}{x}_r^{t_k+\tau} - \underset{\sim}{y}_r^{\tau}\|_r \leq \|\Delta\underset{\sim}{x}_r^{t_k+\tau} - \underset{\sim}{\psi}^{(\tau)}\|_r + \|\underset{\sim}{x}(0)\chi_{(0,t_k+\tau]} - \underset{\sim}{x}(0)\chi_{(0,\infty)}\|_r + \|\underset{\sim}{T}^{t_k+\tau}\underset{\sim}{x}_r^0\|_r. \tag{4.25}$$

Since $t_k \to \infty$ as $k \to \infty$, it follows from (4.22), (4.25), (2.3), and Part b of Remark 2.1, that, for each $\tau \in (-\infty,\infty)$,

$$\lim_{k\to\infty}\|\underset{\sim}{x}_r^{t_k+\tau} - \underset{\sim}{y}_r^{\tau}\|_r = 0. \tag{4.26}$$

It is clear from (4.24) and (4.26) that we have produced a function $\underset{\sim}{y}(\cdot)$ on $(-\infty,\infty)$ with the property that for each τ in $(-\infty,\infty)$ the history $\underset{\sim}{y}^{\tau}$ of $\underset{\sim}{y}$ up to τ is in $\underline{\underline{V}}$ and

$$\lim_{t_k\to\infty}\|\underset{\sim}{x}^{t_k+\tau} - \underset{\sim}{y}^{\tau}\| = 0. \tag{4.27}$$

Hence, each of the functions $\underset{\sim}{y}^{\tau}$ is in Ω. If we can now show that $\underset{\sim}{y}(\cdot)$ has the Properties (b)-(c) listed on page 24, with $\underline{\underline{T}} = \Omega$ and $\underset{\sim}{\Psi}$ our original given element of Ω, then it will follow that Ω is an invariant set.

Since t_k is a subsequence of t_m, it follows from (4.15) and (4.27) that $\underset{\sim}{y}^0 = \underset{\sim}{\Psi}$, and hence $\underset{\sim}{y}(\cdot)$ has Property (d). Therefore, it remains only to verify that $\underset{\sim}{y}(\cdot)$ has Properties (b) and (c); i.e. that $\underset{\sim}{y}(\cdot)$ is continuously differentiable and obeys

$$\underset{\sim}{y}(t_b) - \underset{\sim}{y}(t_a) = \int_{t_a}^{t_b} \underset{\sim}{g}(\tau)d\tau, \quad \text{with} \quad \underset{\sim}{g}(\tau) \overset{\text{def}}{=} \underset{\sim}{f}(\underset{\sim}{y}^{\tau}), \tag{4.28}$$

for each pair of numbers t_a, t_b. Since $\underset{\sim}{f}$ is continuous over $\underline{\underline{S}}(h)$ and $\underset{\sim}{y}^{\tau}$ is in $\underline{\underline{S}}(h)$ for each τ, it is a consequence of Remark 2.2 that the function $\underset{\sim}{g}(\cdot)$ in (4.28) can suffer discontinuities at only those values of τ at which $\underset{\sim}{y}(\cdot)$ is discontinuous.# Thus, the continuity of $\underset{\sim}{y}(\cdot)$ insures that

#This property of equationsof the form $\underset{\sim}{g}(\tau) = \underset{\sim}{f}(\underset{\sim}{y}^{\tau})$, with $\underset{\sim}{f}$ continuous on a region in $\underline{\underline{V}}$, is called "conservation of regularity" and is discussed in detail by Coleman and Mizel [1968, 1, particularly Remark 3.3].

$\underset{\sim}{g}(\cdot)$ is continuous, and if we show that (4.28) holds then $\underset{\sim}{y}(\cdot)$ will automatically have not only Property (c) but also Property (b). Now, let a pair t_a, t_b be assigned, with, say, $t_b \geq t_a$. Noting that $t_k \to \infty$ as $k \to \infty$, we may pick an integer k^* so that $t_k + t_a$ is positive for all

$k \geq k^*$. Since $\underset{\sim}{x}(\cdot)$ is a solution of (3.1), we have, for $k \geq k^*$,

$$\underset{\sim}{x}(t_k+t_b) - \underset{\sim}{x}(t_k+t_a) = \int_{t_a}^{t_b} \underset{\sim}{g}_k(\tau)d\tau, \quad \text{with } \underset{\sim}{g}_k(\tau) \overset{\text{def}}{=\!=} \underset{\sim}{f}(\underset{\sim}{x}^{t_k+\tau}). \tag{4.29}$$

Clearly, (4.24) implies that, for each t, as $k \to \infty$ the left side, $\underset{\sim}{x}(t_k+t_b) - \underset{\sim}{x}(t_k+t_a)$, of (4.29) approaches the left side, $\underset{\sim}{y}(t_b) - \underset{\sim}{y}(t_a)$, of (4.28). Therefore, if we can show that

$$\lim_{k\to\infty} \int_{t_a}^{t_b} \underset{\sim}{g}_k(\tau)d\tau = \int_{t_a}^{t_b} \underset{\sim}{g}(\tau)d\tau, \tag{4.30}$$

then $\underset{\sim}{y}(\cdot)$ must obey (4.28). Since $\underset{\sim}{f}$ is bounded on $\underline{\underline{S}}(h)$ and $\{\underset{\sim}{x}^t \mid t \geq 0\}$ is a subset of $\underline{\underline{S}}(h)$, we have

$$\underset{\sim}{g}_k(\tau) \leq K < \infty, \tag{4.31}$$

for each $k \geq k^*$ and all τ in $[t_a, t_b]$; and since $\underset{\sim}{f}$ is continuous on $\underline{\underline{S}}(h)$, (4.27) yields

$$\underset{\sim}{g}_k(\tau) \to \underset{\sim}{g}(\tau), \quad \text{pointwise, as} \quad k \to \infty. \tag{4.32}$$

Furthermore, the continuity of $\underset{\sim}{x}(t)$ in t for $t \geq 0$ implies that each function $\underset{\sim}{g}_k(\tau)$, with $k > k^*$, is continuous in τ for τ in $[t_a, t_b]$; therefore, (4.31), (4.32), and a theorem of Arzelà#, tell us that (4.30)

#Or, if one prefers, the standard form of Lebesgue's theorem on dominated convergence.

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does indeed hold under our arbitrary assignment of t_a and t_b. Thus (4.28) holds for every pair t_a, t_b, and $\underset{\sim}{y}(\cdot)$ has the Properties (b) and (c); q.e.d.

As a direct consequence of the lemmas proven here we have

Theorem 4.1. If the zero solution of (3.1) is stable then there exists a $\zeta > 0$ such that each solution $x(\cdot)$ of (3.1) with its initial history x^0 in $S(\zeta)$ has the following properties:

(α) $\{x^t \mid t \geq 0\}$ is precompact in V;

(β) the ω-limit set of $x(\cdot)$ is a non-empty compact subset of $S(h)$ and is an invariant set for (3.1);

(γ) the ω-limit Ω set of $x(\cdot)$ "attracts $x(\cdot)$" in the sense that as $t \to \infty$, $\mathrm{dist}(x^t, \Omega) \to 0$.

Proof. Whenever the zero solution is stable it follows immediately from Lemma 2.1, Remark 3.1, and the definition of stability that there exists a $\zeta > 0$ such that each solution $x(\cdot)$ of (3.1) with x^0 in $S(\zeta)$ can be extended to $(-\infty, \infty)$ and has $\|x^t\| < h_1 < h$ for all $t \geq 0$. By Lemmas 4.1 - 4.3 each such solution $x(\cdot)$ has the Properties (α)-(γ); q.e.d.

5. Asymptotic Stability

We call a real-valued function Ξ on $\underset{\approx}{S}(h)$ a strictly dissipative free energy functional for (3.1) if Ξ has the Properties (i)-(iii) of free energy functionals and, in addition,

(iv) for each solution $\underset{\sim}{x}(\cdot)$ of (3.1) that is not identically equal to $\underset{\sim}{0}$, there exists a number $t^* > 0$ such that

$$\Xi(\underset{\sim}{x}^0) > \Xi(\underset{\sim}{x}^{t^*}). \qquad (5.1)$$

Thus, a free energy functional is strictly dissipative if it eventually decreases on each solution that differs at some time from the zero solution.

Our main result is

Theorem 5.1. *If (3.1) has a strictly dissipative free energy functional whose equilibrium response function Ξ° has a strict local minimum at $\underset{\sim}{0}$, then the zero solution of (3.1) is asymptotically stable.*

Proof. It is clear from Theorem 3.1 that under our present hypothesis the zero solution of (3.1) is stable. Therefore, there exists a $\zeta > 0$ such that each solution of (3.1) with its initial history in $\underset{\approx}{S}(\zeta)$ has the Properties $(\underset{\sim}{\alpha})$, $(\underset{\sim}{\beta})$, and $(\underset{\sim}{\gamma})$ listed in Theorem 4.1. Let $\underset{\sim}{x}(\cdot)$ be one such solution; we wish to show that

$$\lim_{t\to\infty} |\underset{\sim}{x}(t)| = 0. \tag{5.2}$$

Property (iii) of free energy functionals and our assumption that Ξ° has a local minimum at $\underset{\sim}{0}$ imply that $\Xi(\underset{\sim}{x}^t)$ is bounded below by the constant $\Xi^\circ(\underset{\sim}{0})$:

$$\Xi(\underset{\sim}{x}^t) \geq \Xi^\circ\big(\underset{\sim}{x}^t(0)\big) \geq \Xi^\circ(\underset{\sim}{0}).$$

Furthermore, by Property (ii) of Ξ, $\Xi(\underset{\sim}{x}^t)$ is non-increasing in t along the solution $\underset{\sim}{x}(\cdot)$. Hence, $\Xi(\underset{\sim}{x}^t)$ has a limit as $t \to \infty$:

$$\lim_{t\to\infty} \Xi(\underset{\sim}{x}^t) = \ell. \tag{5.3}$$

Of course, ℓ depends on the solution $\underset{\sim}{x}(\cdot)$ under consideration. Let Ω be the ω-limit set of $\underset{\sim}{x}(\cdot)$. By $(\underset{\sim}{\beta})$, Ω is a non-empty invariant set for (3.1) and is contained in the set $\underset{=}{S}(h)$ on which Ξ is continuous. If $\underset{\sim}{\Psi}$ is an arbitrary element of Ω, then there exists a sequence $\underset{\sim}{x}^{t_m}$ in $\{\underset{\sim}{x}^t \mid t \geq 0\}$ obeying (4.15). Therefore, by (5.3) and the continuity of Ξ,

$$\Xi(\underset{\sim}{\Psi}) = \ell \quad \text{for each} \quad \underset{\sim}{\Psi} \in \Omega. \tag{5.4}$$

Now, since Ω is an invariant set, for each $\underset{\sim}{\Psi}$ in Ω (and there is at least one $\underset{\sim}{\Psi}$ in Ω) there exists a solution $\underset{\sim}{y}(\cdot)$ of (3.1) with $\underset{\sim}{y}^0 = \underset{\sim}{\Psi}$ and $\{\underset{\sim}{y}^t \mid t \geq 0\} \subset \Omega$. On this solution $\underset{\sim}{y}(\cdot)$ we have, by (5.4),

$$\Xi(\underset{\sim}{y}^t) \equiv \ell \quad \text{for} \quad t \geq 0.$$

But, since Ξ is strictly dissipative, this is possible only if $\underset{\sim}{y}(\cdot)$ is the zero solution. Hence, $\underset{\sim}{0}^{\dagger}$ is the only element of Ω, and, by Property $(\underset{\sim}{\gamma})$ of Ω,

$$\lim_{t\to\infty} \|\underset{\sim}{x}^t\| = \lim_{t\to\infty} \|\underset{\sim}{x}^t - 0^{\dagger}\| = 0, \qquad (5.5)$$

which, in view of (2.5), implies (5.2); q.e.d.

The arguments used to prove Theorems 3.1 and 5.1 yield also the following generalization of these two propositions.#

#Cf. [1968, 2, Theorem 3].

Theorem 5.2. If there exist two real-valued functions V and v such that

(1) V is defined and continuous on $\underset{=}{S}(h)$ while v is defined and continuous on S_h,

(2) there is a $\delta_1 > 0$ such that $V(\underset{\sim}{x}^t)$ is a non-increasing function of t, for $t \geq 0$, along each solution $\underset{\sim}{x}(\cdot)$ of (3.1) with $\underset{\sim}{x}^0$ in $\underset{=}{S}(\delta_1)$,

(3) v has a strict local minimum at $\underset{\sim}{0}$,

(4) $V(\underset{\sim}{0}^{\dagger}) = v(\underset{\sim}{0})$, and for each $\underset{\sim}{\Psi}$ in $\underset{=}{S}(h)$ with $\underset{\sim}{\Psi}(0)$ in S_h

$$v\big(\underset{\sim}{\Psi}(0)\big) \leq V(\underset{\sim}{\Psi}),$$

then the zero solution of (3.1) is stable. If, in addition, we can find *a* $\delta_2 > 0$ *such that, for each solution* $\underset{\sim}{x}(\cdot)$ *that is not identically zero and has* $\underset{\sim}{x}^0$ *in* $\underset{=}{S}(\delta_2)$, *there exists a time* $t^* > 0$ *at which* $V(\underset{\sim}{x}^{t^*}) < V(\underset{\sim}{x}^0)$, *then the zero solution of* (3.1) *is asymptotically stable*.

6. Examples

a. The Dangling Spider

We suppose that a ball of mass M is hanging from a ceiling by a massless but extensible filament of length z. The forces acting on the ball are the tension T in the filament and a body force F in the z-direction. We take F to be derivable from a time-independent, twice continuously differentiable potential h:

$$F = F(z) = -\frac{dh(z)}{dz}. \tag{6.1}$$

In the special case in which the only long-range force acting on the ball is that of gravity, we have

$$h = -gMz \quad \text{and} \quad F = gM, \tag{6.2}$$

with g a constant. We may, however, seek greater generality and allow for the possibility that F varies with z. Since the filament is supposed massless, at each instant the tension and strain in it are spatially homogeneous. We assume that the filament is composed of a simple viscoelastic material with constant and uniform temperature. The value of the tension T at time t is therefore given by a function $\mathcal{t}$ (which may be non-linear) of the history of z up to t; i.e.

$$T(t) = \mathcal{t}(z^t), \quad \text{with} \quad z^t(s) = z(t-s), \quad s \in [0,\infty). \tag{6.3}$$

The constitutive functional $\mathcal{t}$ is assumed compatible with the "principle of fading memory"; that is, $\mathcal{t}$ is a continuous functional on a history space $\underline{\underline{V}}^{(1)}$ obeying the postulates laid down for $\underline{\underline{V}}$ in Section 2. [The superscript in the symbol $\underline{\underline{V}}^{(1)}$ serves to indicate that this Banach space is formed from functions mapping $[0,\infty)$ into E^1, i.e. the real numbers.] We assume that $\mathcal{t}$ is also locally Lipschitz continuous on $\underline{\underline{V}}^{(1)}$. We further assume that $\mathcal{t}$ is compatible with a recent formulation# of the

#Coleman [1964, 1]; Coleman & Mizel [1967, 3].

thermodynamics of materials with memory. Thus, at each time t the filament has a Helmholtz free energy $\psi(t)$ which may be regarded as a function of z^t,

$$\psi(t) = \mathcal{p}(z^t).$$

Equivalently, we may regard $\psi(t)$ as a function of the present value $z(t)$ and past history z_r^t of z [see the paragraph containing equation (2.5) and note that $z(t) = z^t(0)$]:

$$\psi(t) = \mathcal{p}(z^t) = \mathcal{p}\left(z(t), z_r^t\right). \tag{6.4}$$

The functional $\mathcal{p}$ is assumed to be continuously differentiable on $\underline{\underline{V}}^{(1)}$, in the sense of Fréchet. Thus, in particular, the "partial derivative" $D\mathcal{p}$ defined by

$$D\mathcal{p}(\Phi) = \frac{\partial}{\partial \gamma} \mathcal{p}(\gamma, \Phi_r)\Big|_{\gamma = \Phi(0)} \tag{6.5}$$

exists for each Φ in $\underline{\underline{V}}^{(1)}$. $D\mathfrak{p}$ is called the <u>instantaneous derivative of</u> $\mathfrak{p}$.#

#Coleman [1964, 1, eq. (9.5), p. 252].

In the thermodynamics of materials with memory there is a theorem## which

##[1964, 1, Theorem 1, p. 19]; see also [1967, 3, Theorem 1].

here implies that

(α) $\mathfrak{p}$ determines the tension through the "stress-relation", $\mathfrak{t} = D\mathfrak{p}$, i.e.

$$T(t) = D\mathfrak{p}(z^t); \tag{6.6}$$

(β) whenever the indicated time-derivatives exist, the number

$$\omega(t) \stackrel{\text{def}}{=} \dot{\psi}(t) - T(t)\dot{z}(t) = \frac{d}{dt}\mathfrak{p}(z^t) - \mathfrak{t}(z^t)\dot{z}(t) = \frac{d}{dt}\mathfrak{p}(z^t) - D\mathfrak{p}(z^t)\dot{z}(t) \tag{6.7}$$

obeys the following "internal dissipation inequality":

$$\omega(t) \leq 0. \tag{6.8}$$

It can be shown### that (α) and (β) together imply that the

###For details see [1964, 1, Theorem 3, p. 26 & Remark 11, p. 27] and [1967, 1, Theorems 2 & 4].

equilibrium response functions $\mathfrak{p}^{\circ}$ and $\mathfrak{t}^{\circ}$ corresponding to $\mathfrak{p}$ and $\mathfrak{t}$

[see (2.9)] obey the formula

$$\mathcal{t}^{o}(z) = \frac{d}{dz}\, \mathcal{p}^{o}(z), \quad \text{for each number } z, \tag{6.9}$$

and that

$$\mathcal{p}(\Phi) \geq \mathcal{p}^{o}(\Phi(0)) \quad \text{for each } \Phi \text{ in } \underline{\underline{V}}^{(1)}. \tag{6.10}$$

The equation of motion for the ball is

$$M\ddot{z} = F(z) - \mathcal{t}(z^t).$$

Putting $y = \dot{z}$ we may write this equation as a functional-differential equation of the type (3.1) with $\underset{\sim}{x}(t) = \left(\underset{\sim}{y}(t),\ z(t)\right)$:

$$\left.\begin{aligned} \dot{y} &= \frac{1}{M} F(z) - \frac{1}{M}\, \mathcal{t}(z^t) \\ \dot{z} &= y. \end{aligned}\right\} \tag{6.11}$$

If the origin $z = 0$ can, and has been chosen so that

$$\mathcal{t}^{o}(0) = F(0) \tag{6.12}$$

then the right-hand side of (6.11) has the Properties $(\underset{\sim}{1})$-$(\underset{\sim}{3})$ required of $\underset{\sim}{f}$ at the beginning of Section 3. Here, in line with the discussion of Sections 2-5, the domain of $\underset{\sim}{f}$ should be considered a history space formed from functions mapping $[0,\infty)$ in E^2. However, although $\underset{\sim}{f}$ depends on the past history z_r^t of the second component z of $\underset{\sim}{x} = (y,z)$, $\underset{\sim}{f}$ is independent of the past history of the first component. Hence we can alternatively take the direct sum $\underline{\underline{V}}^{(2)}$ of $E^{(1)}$ and $\underline{\underline{V}}^{(1)}$ for the domain of $\underset{\sim}{f}$. That is, if we take the norm $\|\underset{\sim}{x}^t\|$ of an element $\underset{\sim}{x}^t = (y^t, z^t)$ of $\underline{\underline{V}}^{(2)}$ to be

$$\|\underset{\sim}{x}^t\| = |y^t(0)| + |z^t(0)| + \|z_r^t\|_r, \tag{6.13}$$

where $\|\cdot\|_r$ is the norm on $\underline{\underline{V}}_r^{(1)}$, then $\underset{\sim}{f}$, as a functional on $\underline{\underline{V}}^{(2)}$, is locally Lipschitz continuous. The fact that $\underset{\sim}{f}$ is independent of y_r^t is important; it means that our initial data can be such that the velocity $y(t) = \dot{z}(t)$ does not exist for some $t \leq 0$. Of course, we must here extend slightly our concept of a "solution". By a <u>solution up to</u> A of (6.11) we mean a function pair $\underset{\sim}{x}(\cdot) = \big(y(\cdot), z(\cdot)\big)$, with $y(\cdot)$ defined on $[0,A)$ and $z(\cdot)$ on $(-\infty,A)$, such that z^t is in $\underline{\underline{V}}^{(1)}$ (i.e. $\underset{\sim}{x}^t$ is in $\underline{\underline{V}}^{(2)}$) for all t in $[0,A)$ and (6.11) holds for all t in $[0,A)$.# The equation (6.12) is equivalent

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#It is not difficult to verify that Theorems 3.1, 5.1, and 5.2 remain valid under this extended concept of solution, provided that $\underline{\underline{V}} = \underline{\underline{V}}^{(2)}$ and the norm on $\underline{\underline{V}}$ has the form (6.13).

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to asserting that $\underset{\sim}{x} \equiv \underset{\sim}{0}$ is a solution of (6.11). Hence (6.12) may be called the <u>equation of equilibrium</u>. By (6.13), the definition of stability given in Section 3 here reduces to the assertion that the zero solution $\underset{\sim}{x} \equiv \underset{\sim}{0}$ is <u>stable</u> if and only if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that if $\underset{\sim}{x}(\cdot) = \big(y(\cdot), z(\cdot)\big)$ is a solution up to A of (6.11) with

$$\|\underset{\sim}{x}^0\| = |y(0)| + |z(0)| + |z_r^0| < \delta$$

then $\underset{\sim}{x}(\cdot)$ can be extended until $A = \infty$, and for each $t \geq 0$,

$$|y(t)| + |z(t)| = |\dot{z}(t)| + |z(t)| < \epsilon.$$

If, in addition, there is a $\zeta > 0$ such that

$$|\dot{z}(t)| + |z(t)| \to 0 \quad \text{as} \quad t \to \infty$$

whenever $\|\underset{\sim}{x}^0\|$ is less than ζ, then the zero solution is asymptotically stable.

Let $\underset{\sim}{\Phi} = (\Phi_1, \Phi_2)$ be a generic element of $\underline{\underline{V}}^{(2)}$. The functional Ξ, defined by

$$\Xi(\underset{\sim}{\Phi}) = \wp(\Phi_2) + h\big(\Phi_2(0)\big) + \frac{1}{2} M\Phi_1(0)^2, \tag{6.14}$$

is clearly continuous over $\underline{\underline{V}}^{(2)}$; in fact, the differentiability of $\wp$ implies that Ξ has a Fréchet derivative at each point in $\underline{\underline{V}}^{(2)}$. Along solutions $\underset{\sim}{x}(\cdot) = \big(y(\cdot), z(\cdot)\big)$ of (6.11),

$$\Xi(\underset{\sim}{x}^t) = \wp(z^t) + h\big(z(t)\big) + \frac{1}{2} My(t)^2. \tag{6.15}$$

The value of Ξ is the Helmholtz free energy of the filament, plus the potential energy of the body force acting on the ball, plus the kinetic energy of the ball.

When Ξ is defined by (6.14), the function Ξ° of (2.9) is given by

$$\Xi^\circ(\underset{\sim}{x}) = \wp^\circ(z) + h(z) + \frac{1}{2} My^2, \qquad \underset{\sim}{x} = (y,z) \in E^2, \tag{6.16}$$

and, by (6.10), we have $\Xi(\underset{\sim}{\Phi}) \geq \Xi^\circ\big(\underset{\sim}{\Phi}(0)\big)$ for all $\underset{\sim}{\Phi}$ in $\underline{\underline{V}}^{(2)}$. In view of

(6.1), (6.11) yields immediately the "energy equation",

$$\frac{1}{2}\frac{d}{dt} My(t)^2 + \frac{d}{dt} h\big(z(t)\big) = -T(t)\dot{z}(t),$$

and, by (6.15), (6.7), and (6.8), we have, for $t \geq 0$,

$$\frac{d}{dt}\Xi(\underset{\sim}{x}^t) = \frac{d}{dt}\mathscr{P}(z^t) - \mathscr{t}(z^t)\dot{z}(t) = \omega(t) \leq 0, \qquad (6.17)$$

along those solutions $\underset{\sim}{x}(\cdot)$ of (6.11) for which $\frac{d}{dt}\mathscr{P}(z^t)$ exists. Thus, <u>the function Ξ defined in</u> (6.14) <u>is a free energy functional for the functional-differential equation</u> (6.11).#

#We assert, but we omit the proof, that monotonicity of $\Xi(\underset{\sim}{x}^t)$ for $t \geq 0$ on the solutions for which $\frac{d}{dt}\mathscr{P}(z^t)$ exists (for $t \geq 0$) implies the required monotonicity of $\Xi(\underset{\sim}{x}^t)$ on other solutions. The proof uses the fact that since $\mathscr{t}$ and F are locally Lipschitz continuous, so also is the right hand side $\underset{\sim}{f}$ of (6.11), and therefore $\underset{\sim}{x}^t$, for each $t \geq 0$, depends continuously on the initial data $\underset{\sim}{x}^0$.

The function $\mathscr{g}$ defined by

$$\mathscr{g}(z) = \mathscr{P}^{\circ}(z) + h(z) \qquad (6.18)$$

may be called the <u>equilibrium Gibbs function</u> for (6.14). Its value is the sum of the equilibrium free energy and the potential of the applied body force, both evaluated at position z of the ball. Since M is positive, the function Ξ° of (6.16) has a strict local minimum at $\underset{\sim}{0}$ in

E^2 if, and only if, $\mathcal{g}$ has a strict local minimum at 0 in E^1. Therefore, Theorem 3.1 here yields

Remark 6.1. Suppose that the functional $\mathcal{t}$ in (6.11) is locally Lipschitz continuous on a history space $\underline{\underline{V}}^{(1)}$ and obeys the restrictions which thermodynamics places on the response function for the stress in a material with memory. If the equilibrium Gibbs function (6.18) has a strict local minimum at zero, then $\underset{\sim}{x} \equiv \underset{\sim}{0}$ is a stable solution of (6.11).

We may note if $\mathcal{g}$ has a minimum at 0 then, $\mathcal{g}'(0) = 0$ and, by (6.1) and (6.9), the equation of equilibrium (6.12) is automatically satisfied. Hence, in Remark 6.1 we need not assume $\underset{\sim}{f}(\underset{\sim}{0}^{\dagger}) = \underset{\sim}{0}$, as a separate hypothesis, for it follows whenever the equilibrium Gibbs function has a minimum at zero.

We may also note that if (6.12) is assumed, a sufficient condition for the stability of the zero solution of (6.11) is that $\mathcal{g}''(0)$ be positive, i.e. that

$$\mathcal{t}^{\circ\prime}(0) \;>\; F'(0). \tag{6.19}$$

When (6.2) holds, (6.19) reduces to the condition that the equilibrium infinitesimal modulus $\mathcal{t}^{\circ\prime}$ be positive at zero, i.e. that

$$\mathcal{t}^{\circ\prime}(0) \;>\; 0. \tag{6.20}$$

The most elementary considerations in mechanics suggest that in the case (6.2) the condition $\mathcal{t}^{o\prime}(0) \geq 0$ is necessary for stability; we believe it interesting that under our present precise concept of the dynamical stability of equilibrium, the condition (6.20) is sufficient, even for a non-linear filament with memory.

b. Linear Filaments

In Remark 6.1 it is not assumed that the tension-functional $\mathcal{t}$ is in any way linear; a particularly interesting special case of the present theory arises, however, when (6.3) has the form found in the linear theory of viscoelasticity:

$$T(t) \quad = \quad G(0)z^t(0) + \int_0^\infty G'(s)z^t(s)ds \quad = \quad \mathcal{t}(z^t). \qquad (6.21)$$

Here G, with derivative G′, is a real-valued function on $[0,\infty)$ characterizing the material under consideration. We call G the relaxation function; G(0) is the instantaneous modulus; and the limit

$$G(\infty) \quad \stackrel{\text{def}}{=\!=} \quad \lim_{s\to\infty} G(s),$$

which we assume exists, is the equilibrium modulus. Let us assume, as is usual in linear viscoelasticity, that G can be written in the form

$$G(s) \quad = \int_0^\infty k(\tau)e^{-s/\tau}d\tau + G(\infty) \qquad (6.22)$$

with k a non-negative, measurable function, with bounded support and with $\frac{1}{\tau^3} k(\tau)$ summable:

$$k(\tau) \geq 0, \quad \text{while for some } N > 0, \int_0^N \frac{1}{\tau^n} k(\tau)d\tau = a_n < \infty \quad \text{for} \quad n = 1,2,3, \text{ and}$$
$$k(\tau) = 0 \quad \text{whenever} \quad \tau > N. \tag{6.23}$$

k is called the <u>relaxation spectrum</u>. It follows from (6.22) and (6.23) that $-G'(s)$ is a positive, bounded, decreasing, analytic function on $(0,\infty)$ dominated by the function

$$\ell(s) = a_1 e^{-s/N}, \tag{6.24}$$

which has Properties (I)-(IV) listed in Remark 2.4. If we let $\bar{\underline{\underline{V}}}^{(1)}$ be a space of real-valued functions for which the norm (2.11) is finite with $p = 1$ and with ℓ given by (6.24), then the functional $\mathcal{L}$ in (6.21), i.e.

$$\mathcal{L}(\Phi) = G(0)\Phi(0) + \int_0^\infty G'(s)\Phi(s)ds, \tag{6.25}$$

is clearly well defined for each Φ in $\bar{\underline{\underline{V}}}^{(1)}$ and is continuous over $\bar{\underline{\underline{V}}}^{(1)}$.

So as to be able to compute time-derivatives and perform integrations by parts we shall here assume that we are dealing with solutions of (6.11) whose initial histories z^0 are of bounded variation on $(0,\infty)$. Then equation (6.25), which by (6.22) may be written

$$\mathcal{L}(\Phi) = G(\infty)\Phi(0) + \int_0^\infty k(\tau)\left[\Phi(0) - \frac{1}{\tau}\int_0^\infty e^{-s/\tau}\Phi(s)ds\right]d\tau, \tag{6.26}$$

yields, for $t \geq 0$,

$$\mathscr{t}(z^t) = G(\infty)z^t(0) - \int_0^\infty k(\tau)\left[\int_0^\infty e^{-s/\tau} dz^t(s)\right] d\tau. \qquad (6.27)$$

A theory based on (6.21)-(6.23) is certainly compatible with the thermodynamics of materials with memory. Indeed, it is readily verified that if (6.22) and (6.23) hold, then the functional $\mathscr{p}$ defined by

$$\mathscr{p}(\Phi) = \frac{1}{2} G(\infty)\Phi(0)^2 + \frac{1}{2}\int_0^\infty k(\tau)\left[\Phi(0) - \frac{1}{\tau}\int_0^\infty e^{-s/\tau}\Phi(s) ds\right]^2 d\tau \qquad (6.28)$$

is continuous over the space $\bar{\underline{\underline{V}}}^{(1)}$ and is a Helmholtz free energy function for the functional $\mathscr{t}$ of (6.26) in the sense that $\mathscr{p}$ and $\mathscr{t}$ obey (6.6)-(6.8). The equilibrium response functions corresponding to $\mathscr{p}$ and $\mathscr{t}$ are

$$\mathscr{p}^\circ(z) = \frac{1}{2} G(\infty)z^2, \qquad \mathscr{t}^\circ(z) = G(\infty)z. \qquad (6.29)$$

These functions obviously obey (6.9) and (6.10). It follows from (6.23) that either $k \stackrel{\circ}{=} 0$, or

$$\int_0^\infty k(\tau)d\tau > 0. \qquad (6.30)$$

If $k \stackrel{\circ}{=} 0$, then $\mathscr{t}(\Phi) = \mathscr{t}^\circ(\Phi(0))$ and $\mathscr{p}(\Phi) = \mathscr{p}^\circ(\Phi(0))$; that is, our theory reduces to the linear theory of elasticity.

Direct calculation shows that when $\mathscr{p}$ has the form (6.28), the quantity $\omega(t)$, defined in (6.7) and occurring in (6.17), is given by

$$\omega(t) = -\int_0^\infty \frac{1}{\tau} k(\tau)\left[z^t(0) - \frac{1}{\tau}\int_0^\infty e^{-s/\tau} z^t(s)ds\right]^2 d\tau = -\int_0^\infty \frac{1}{\tau} k(\tau)\left[\int_0^\infty e^{-s/\tau} dz^t(s)\right]^2 d\tau. \tag{6.31}$$

Let Φ be a function of bounded variation in $\bar{\underline{V}}^{(1)}$ and consider

$$\mathscr{L}(\Phi,\tau) = \int_0^\infty e^{-s/\tau} d\Phi(s).$$

Clearly, $\mathscr{L}(\Phi,\tau)$ is an analytic function of τ for τ in $(0,\infty)$, and if $\mathscr{A}$ denotes the set of points τ in $(0,\infty)$ at which $\mathscr{L}(\Phi,\tau) = 0$, then $\mathscr{A}$ has an accumulation point only if $\Phi(s) = \Phi(0)$ for almost all s in $[0,\infty)$. Thus, (6.23), (6.30), and (6.31) imply that $\omega(t) = 0$ only if $z^t(s)$ is almost everywhere equal to $z^t(0)$, and this, by (6.17), implies that Ξ, defined in (6.15), is a <u>strictly dissipative</u> free energy functional for solutions of (6.11) whose initial histories lie in a sufficiently small neighborhood of zero. (We are assuming here that 0 is an isolated solution of the equation $F(z) - \mathscr{L}^{\circ}(z) = 0$.) Theorem 5.1 now yields

Remark 6.2. Suppose the functional $\mathscr{L}$ in (6.11) obeys (6.21)-(6.23) and also (6.30). The zero solution of (6.11) is then asymptotically stable whenever

$$G(\infty) > F'(0). \tag{6.32}$$

If, in particular, the only long range force acting on the ball is that due to gravity, then the zero solution is asymptotically stable whenever

$$G(\infty) > 0. \tag{6.33}$$

Appendix: On the Extent of Stability

Each type of evolution equation has a natural "state space" $\underline{\underline{R}}$. For ordinary differential equations $\underline{\underline{R}}$ is E^n; for functional-differential equations $\underline{\underline{R}}$ is a linear function space containing the domain of definition of $\underset{\sim}{f}$ in (1.1). A function V on $\underline{\underline{R}}$ that is bounded below and decreases on all solutions of an evolution equation may be called a Lyapunov function for the equation. We may denote by $\underline{\underline{U}}_b$ the set in $\underline{\underline{R}}$ on which V is less than b. For ordinary differential equations, La Salle [1960, 2] showed that if V is a Lyapunov function with $\underline{\underline{U}}_b$ bounded in $\underline{\underline{R}}$ and if $\underline{\underline{M}}$ is the largest invariant set in $\underline{\underline{U}}_b$ on which $\frac{d}{dt} V = 0$, then every solution with its trajectory in $\underline{\underline{U}}_b$ approaches $\underline{\underline{M}}$. This result was extended by Hale to certain types of functional-differential equations. In his first paper on this subject [1963, 1], Hale considered the case in which $\underline{\underline{R}}$ is the set of continuous functions mapping a finite interval [0,r] into E^n and he employed the uniform topology on $\underline{\underline{R}}$.# Later [1965, 1], he explored the case in which $\underline{\underline{R}}$ is endowed with the compact open topology and is the set of continuous functions mapping $[0,\infty)$ into E^n. Once the lemmas of Section 4 are in hand, it is easy to show that stability theorems of the type obtained by Hale hold when $\underline{\underline{R}}$ is a history space $\underline{\underline{V}}$ as defined in Section 2.

#See also the work of Krasovskii [1959, 1, §§27-34], and a recent essay by La Salle [1967, 4].

Henceforth, unless we state otherwise, we shall drop the assumption (3) about $\underset{\sim}{f}$, and replace the assumptions (1) and (2) by

(1)' $\underset{\sim}{f}$ is defined and continuous over $\underline{\underline{V}}$

(2)' $\underset{\sim}{f}$ maps bounded subsets of $\underline{\underline{V}}$ into bounded subsets of E^n.

We shall keep our other postulates unaltered. Clearly, Property (3) of $\underset{\sim}{f}$ is not needed for the Lemmas 4.1 - 4.3, and when (1)' and (2)' are assumed these lemmas are valid with h an arbitrary positive number.

If V is a continuous functional over an open region in $\underline{\underline{V}}$ and if $\underset{\sim}{\Phi}$ is in the domain of V, we write $\dot{V}(\underset{\sim}{\Phi})$ for the supremum of

$$\overline{\lim_{t \to 0^+}} \frac{1}{t}\left(V(\underset{\sim}{x}^t) - V(\underset{\sim}{x}^0)\right)$$

over all solutions $\underset{\sim}{x}(\cdot)$ of (3.1) with initial history $\underset{\sim}{\Phi}$.

Theorem 7.1[#]. *Let V be a continuous real-valued function on $\underline{\underline{V}}$, let b be a positive number, and let $\underline{\underline{U}}_b$ be the set of elements $\underset{\sim}{\Phi}$ in $\underline{\underline{V}}$ for which $V(\underset{\sim}{\Phi}) < b$. Suppose there exists a $\zeta > 0$ such that*

$$|\underset{\sim}{\Phi}(0)| < \zeta, \quad V(\underset{\sim}{\Phi}) \geq 0, \quad \text{and} \quad \dot{V}(\underset{\sim}{\Phi}) \leq 0, \tag{7.1}$$

for all $\underset{\sim}{\Phi}$ in $\underline{\underline{U}}_b$. If $\underline{\underline{M}}$ is the union of all the invariant sets in $\underline{\underline{U}}_b$ on which $\dot{V} = 0$, and if $\underset{\sim}{x}(\cdot)$ is a solution of (3.1) with $\underset{\sim}{x}^0$ in $\underline{\underline{U}}_b$, then $\underset{\sim}{x}^t$ approaches $\underline{\underline{M}}$ as $t \to \infty$.

[#]Cf. Hale [1963, 1, Theorem 1], [1965, 1, Theorem 1].

Proof. It follows from (7.1) that each solution $\underset{\sim}{x}(\cdot)$ of (3.1) with $\underset{\sim}{x}^0$ in $\underline{\underline{U}}_b$ has $\underset{\sim}{x}^t$ in $\underline{\underline{U}}_b$ and $|\underset{\sim}{x}(t)| < \zeta$ for $t \geq 0$. By Lemma 2.2, along each such solution $\|\underset{\sim}{x}^t\|$ is bounded for $t \geq 0$. Thus every solution which has its initial history in $\underline{\underline{U}}_b$ can be extended to $(-\infty,\infty)$ and obeys the conclusions of Lemmas 4.1 - 4.3. Hence the ω-limit set Ω of such a solution $\underset{\sim}{x}(\cdot)$ is a non-empty invariant set. It follows from $(7.1)_{2\&3}$ that $V(\underset{\sim}{x}^t)$ has a limit $b_o < b$ as $t \to \infty$; therefore,

$$V(\underset{\sim}{\Psi}) = b_o \quad \text{for each} \quad \underset{\sim}{\Psi} \in \Omega, \tag{7.2}$$

and Ω is contained in $\underline{\underline{U}}_b$. Since Ω is invariant, (7.2) implies that $\dot{V}(\underset{\sim}{\Psi}) = 0$ for each $\underset{\sim}{\Psi}$ in Ω. Thus, Ω is a subset of $\underline{\underline{M}}$, and (4.16) implies that

$$\text{dist}(\underset{\sim}{x}^t, \underline{\underline{M}}) \to 0 \quad \text{as} \quad t \to \infty;$$

q.e.d.

Theorem 7.2.# If there exists a functional V obeying the hypothesis of Theorem 7.1 and, in addition, such that $\dot{V}(\underset{\sim}{\Phi}) \neq 0$ for all $\underset{\sim}{\Phi}$ in $\underline{\underline{U}}_b$ with $\underset{\sim}{\Phi} \neq \underset{\sim}{0}^\dagger$, then $\underset{\sim}{f}(\underset{\sim}{0}^\dagger) = \underset{\sim}{0}$ and every solution of (3.1) that has its initial history in $\underline{\underline{U}}_b$ obeys

$$\lim_{t\to\infty} |\underset{\sim}{x}(t)| = 0. \tag{7.3}$$

#Cf. Hale [1963, 1, Corollary 1], [1965, 1, Corollary 1].

Proof. Lemmas 4.2 and 4.3 and the proof of Theorem 7.1 require that each solution with $\tilde{x}^0$ in $\underline{\underline{U}}_b$ have a non-empty invariant limit set Ω on which $\dot{V} = 0$. But, since we here have $\dot{V}(\tilde{\Phi}) < 0$ for $\tilde{\Phi} \neq \tilde{0}^\dagger$, the only possible element in Ω is $\tilde{0}^\dagger$. Of course, the singleton $\Omega = \{\tilde{0}^\dagger\}$ can be an invariant set only if $\tilde{f}(\tilde{0}^\dagger) = \tilde{0}$. Furthermore, when $\tilde{x}^0$ is in $\underline{\underline{U}}_b$, (4.16) here reduces to (5.5), which implies (7.3).

Theorem 7.3.# Let V be a continuous real-valued function on $\underline{\underline{V}}$ obeying $V(\tilde{\Phi}) \geq 0$ and $\dot{V}(\tilde{\Phi}) \leq 0$ for all $\tilde{\Phi}$ in $\underline{\underline{V}}$. If $\underline{\underline{M}}$ is the union of all the invariant sets in $\underline{\underline{V}}$ on which $\dot{V} = 0$, then every solution of (3.1) with $|\tilde{x}(t)|$ bounded for $t \geq 0$ approaches $\underline{\underline{M}}$ as $t \to \infty$.

If, in addition, there exists a continuous non-negative function u on $(0,\infty)$ such that $u(\sigma) \to \infty$ as $\sigma \to \infty$, and

$$u\left(|\tilde{\Phi}(0)|\right) \leq V(\tilde{\Phi}) \tag{7.4}$$

for all $\tilde{\Phi}$ in $\underline{\underline{V}}$, then all solutions of (3.1) have $|\tilde{x}(t)|$ bounded for $t \geq 0$.

Proof. The first part of this theorem was demonstrated in the proof of Theorem 7.1. The proof that (7.4) implies boundedness proceeds as follows. Let $\tilde{x}(\cdot)$ be a solution of (3.1). Since $u(\sigma) \to \infty$ as $\sigma \to \infty$, it

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#Cf. Hale [1965, 1, Theorem 3].

follows from (7.4) that for this solution $\underset{\sim}{x}(\cdot)$ there is a number m such that $V(\underset{\sim}{\Phi}) > V(\underset{\sim}{x}^0)$ whenever $|\underset{\sim}{\Phi}(0)| \geq m$. Therefore, since $V(\underset{\sim}{x}^t)$ is a non-increasing function of t for $t \geq 0$, $|\underset{\sim}{x}(t)|$ cannot exceed m at any $t \geq 0$; q.e.d.

Theorem 7.4. *Let V be a continuous real-valued function on $\underline{V}$ obeying $\dot{V}(\underset{\sim}{\Phi}) \leq 0$ for all $\underset{\sim}{\Phi}$ in $\underline{V}$ with equality holding only when $\underset{\sim}{\Phi} = \underset{\sim}{0}^{\dagger}$. If there exists a continuous non-negative function u on $(0,\infty)$ such that $u(\sigma) \to \infty$ as $\sigma \to \infty$ and $u\left(|\underset{\sim}{\Phi}(0)|\right) \leq V(\underset{\sim}{\Phi})$ for all $\underset{\sim}{\Phi}$ in $\underline{V}$, then*

(α) $\underset{\sim}{f}(\underset{\sim}{0}^{\dagger}) = \underset{\sim}{0}$,

(β) *all solutions of (3.1) have $|\underset{\sim}{x}(t)|$ bounded for $t \geq 0$, and*

(γ) *for every solution*

$$\lim_{t\to\infty} |\underset{\sim}{x}(t)| = 0.$$

Proof. It follows immediately from Theorem 7.3 that (β) holds here. Hence, by Lemma 2.2, $\|\underset{\sim}{x}^t\|$ is bounded, for $t \geq 0$, along each solution $\underset{\sim}{x}(\cdot)$ of (3.1), and each solution can be extended to $(-\infty,\infty)$. By Lemmas 4.2 and 4.3, the ω-limit set of $\underset{\sim}{x}(\cdot)$, Ω, is non-empty and invariant. As we saw in the proof of Theorem 7.1, this, along with the fact that $V(\underset{\sim}{x}^t)$ has a limit as $t \to \infty$, implies that $\dot{V} = 0$ on Ω. Therefore, since here $\dot{V}(\underset{\sim}{\Phi}) = 0$ only when $\underset{\sim}{\Phi} = \underset{\sim}{0}^{\dagger}$, Ω is the singleton $\{\underset{\sim}{0}^{\dagger}\}$, and, because Ω is invariant, (α) holds. Just as in theproof of Theorem 7.2, (4.16) here reduces to (5.5) which implies (γ).

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