ON THE STABILITY OF CERTAIN MOTIONS OF INCOMPRESSIBLE MATERIALS WITH MEMORY

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Report 68-20

February, 1968

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1. Introduction

For incompressible materials with sufficient symmetry there exist classes of dynamically admissible motions which can be completely characterized by giving the temporal dependence of a finite list of real variables. One example is the class of isochoric, homogeneous, circulation-preserving motions.# An example involving simpler surface

#See Coleman and Truesdell [1965, 1]. They show that in a given incompressible simple material of arbitrary symmetry, an isochoric homogeneous motion is dynamically possible if and only if it is circulation-preserving.

loads is the set $\underline{\underline{C}}$ of motions which describe isochoric radial expansions of an infinitely long hollow cylindrical tube; in 1960, Knowles## observed

##[1960, 3].

that each motion in this class is dynamically admissible in every isotropic, incompressible, elastic material and found conditions under which such motions can be expected to be periodic. It is easy to show that each motion in $\underline{\underline{C}}$ is also dynamically admissible in every transversely isotropic, incompressible, simple material with memory,### and may be

###Of course, the axis of symmetry for the transversely isotropic material must be parallel to the axis of the tube.

characterized completely by giving the dependence on time of the inner (or outer) radius of the tube. Another example with practical applications is the set $\underline{\underline{S}}$ of motions in which a spherical shell undergoes isochoric radially symmetric inflation; this class of motions, which was mentioned by Knowles [1960, 3], has been discussed for isotropic elastic materials by Guo Zhong-Heng & Solecki [1963, 2, 3], Wang [1965, 4], and Knowles & Jakub [1965, 3]. Each motion in $\underline{\underline{S}}$ is dynamically admissible in every isotropic, incompressible, simple material with memory, and may be described by giving the dependence on time of the inner radius of the shell. Both $\underline{\underline{C}}$ and $\underline{\underline{S}}$ are subclasses of the "quasi-equilibrated" motions which Truesdell [1962, 2] has shown to be dynamically possible in every isotropic, elastic, incompressible material. Carroll [1967, 1] has recently pointed out that many of Truesdell's results remain valid for isotropic materials with memory of integral type.

It appears to us that an important problem in the theory of motions of type $\underline{\underline{C}}$ or $\underline{\underline{S}}$ is the following: Given the history of the tube or shell up to some time, say $t = 0$, and given some rule for calculating the pressures on the bounding surfaces for times $t \geq 0$, what can one say about the qualitative properties of the motion for times $t \geq 0$? The qualitative properties that interest us the most here are <u>simple Lyapunov stability</u> and <u>asymptotic stability</u>. When the material under consideration is elastic, our problem reduces to one in the theory of ordinary differential equations; for a material obeying the principle of fading memory,[#] the

[#]That is, the fading memory postulate recently proposed by Coleman & Mizel [1967, 3] [1968, 2] in their generalization of the earlier work of Coleman & Noll [1960, 1] [1961, 1] [1964, 2], and Coleman [1964, 1].

problem becomes one of analyzing a functional-differential equation of the type recently studied by Coleman & Mizel [1968, 4].[##] In this essay

[##]For earlier studies of the stability of solutions of functional-differential equations see Krasovskii [1959, 2] and Hale [1963, 4], [1965, 2]. These authors employed hypotheses of smoothness different from those used by Coleman & Mizel in [1968, 4].

we study the problem for materials with memory, emphasizing the role that thermodynamics can play in its solution. The propositions we prove give some justification to the common practice in applied mechanics of declaring (without demonstration) that <u>states of stable equilibrium are those which minimize an appropriate "equilibrium free energy"</u>.

The present study continues a recent series of investigations of the relation of thermodynamic principles to criteria for dynamical stability.[###]

[###]Coleman & Greenberg [1967, 2], Coleman & Mizel [1968, 3,4].

2. Canonical Free Energy

Let a fixed reference configuration $\mathcal{R}$ be assigned for the body $\mathcal{B}$ under consideration, and identify each of the material points X of $\mathcal{B}$ with the position ξ that it occupies in $\mathcal{R}$. A process of $\mathcal{B}$ is a collection of functions of ξ and t compatible with the laws of balance of momentum and energy. At the level of generality which we seek, each process is characterized by eight functions: (1) the motion χ, with $x = \chi(\xi, t)$ called the position at time t of the material point located at ξ in $\mathcal{R}$, (2) the temperature $\theta > 0$, (3) the specific internal energy ϵ, (4) the specific entropy η, (5) the stress tensor $T = T^T$, (6) the heat flux q, (7) the specific body force b, and (8) the rate of heat supply ω. The laws of balance of momentum and energy assert that for each part $\mathcal{P}$ of $\mathcal{B}$ and each time t,

$$\frac{d}{dt}\int_{\mathcal{P}} \dot{x}\, dm = \int_{\mathcal{P}} b\, dm + \int_{\partial\mathcal{P}} Tn\, da, \tag{2.1}$$

$$\frac{d}{dt}\int_{\mathcal{P}} \left(\epsilon + \frac{1}{2}\dot{x}\cdot\dot{x}\right) dm = \int_{\mathcal{P}} (\dot{x}\cdot b + \omega)\, dm + \int_{\partial\mathcal{P}} (\dot{x}\cdot Tn - q\cdot n)\, da, \tag{2.2}$$

where dm is the element of mass in the body, $\partial\mathcal{P}$ is the surface of $\mathcal{P}$ in the configuration at time t, ds is the element of surface area, n is the exterior unit normal vector to $\partial\mathcal{P}$, and the superposed dots denote time-derivatives. When sufficient smoothness is granted, (2.1) and (2.2) together are equivalent to the field equations,

$$\rho \ddot{\underset{\sim}{x}} = \operatorname{div} \underset{\sim}{T} + \rho \underset{\sim}{b}, \tag{2.3}$$

$$\rho \dot{\varepsilon} = \operatorname{tr}\{\underset{\sim}{T}\underset{\sim}{D}\} - \operatorname{div} \underset{\sim}{q} + \rho\omega, \tag{2.4}$$

with ρ the mass density and $\underset{\sim}{D}$ the <u>stretching tensor</u>, i.e. the symmetric part of the velocity gradient:

$$\underset{\sim}{D} = \frac{1}{2}[\operatorname{grad} \dot{\underset{\sim}{x}} + (\operatorname{grad} \dot{\underset{\sim}{x}})^T]. \tag{2.5}$$

[The symbols grad and div refer to differentiations in which $\underset{\sim}{x}$ not $\underset{\sim}{\xi}$, is the independent variable. We shall use the symbol ∇ to indicate differentiation with respect to $\underset{\sim}{\xi}$.]

It is easily verified that (2.3) implies that

$$\frac{d}{dt}\frac{1}{2}\int_{\mathcal{B}} \dot{\underset{\sim}{x}}\cdot\dot{\underset{\sim}{x}}\, dm = \int_{\partial\mathcal{B}} \underset{\sim}{t}\cdot\dot{\underset{\sim}{x}}\, da + \int_{\mathcal{B}} \underset{\sim}{b}\cdot\dot{\underset{\sim}{x}}\, dm - \int_{\mathcal{B}} \frac{1}{\rho} \operatorname{tr}\{\underset{\sim}{T}\underset{\sim}{D}\} dm, \tag{2.6}$$

with $\underset{\sim}{t}$ the contact force per unit area which is applied to the surface $\partial\mathcal{B}$ of $\mathcal{B}$ at the instant under consideration. The number

$$W \overset{\text{def}}{=\!=} \int_{\partial\mathcal{B}} \underset{\sim}{t}\cdot\dot{\underset{\sim}{x}}\, da = \int_{\partial\mathcal{B}} \dot{\underset{\sim}{x}}\cdot\underset{\sim}{T}\underset{\sim}{n}\, da \tag{2.7}$$

is the rate of working of the contact forces applied to the surface of $\mathcal{B}$. In a given process $\mathcal{C}$, $W = W(t)$ is a function of time alone. Let the origin of the time axis be chosen for convenience, and put

$$w(t) = -\frac{1}{M}\int_0^t W(\tau)d\tau, \tag{2.8}$$

with M the mass of the body. We assume that the specific body force $\underset{\sim}{b}$ may be derived from a potential function h, which is a function of $\underset{\sim}{x}$ alone:

$$\underset{\sim}{b} = -\text{grad } h, \quad \text{i.e.} \quad \underset{\sim}{b}(\underset{\sim}{\xi}, t) = -\text{grad } h(\underset{\sim}{x})\Big|_{\underset{\sim}{x}=\underset{\sim}{\chi}(\xi,t)} . \tag{2.9}$$

Substitution of (2.8) and (2.9) into (2.6) yields

$$\frac{d}{dt}\int_{\mathcal{B}}\left(\frac{1}{2}\,\dot{\underset{\sim}{x}}\cdot\dot{\underset{\sim}{x}} + w + h\right) dm = -\int_{\mathcal{B}} \frac{1}{\rho}\, \text{tr}\, \underset{\sim\sim}{TD}\, dm; \tag{2.10}$$

we call

$$\int_{\mathcal{B}} (w+h)\, dm = \int_{\mathcal{B}} \left[w(t) + h\big(\underset{\sim}{\chi}(\underset{\sim}{\xi}, t)\big)\right] dm \tag{2.11}$$

the <u>mechanical potential</u> of $\mathcal{B}$ at time t.

The specific <u>Helmholtz free energy</u>, $\psi = \psi(\underset{\sim}{\xi}, t)$ is defined by

$$\psi = \epsilon - \theta\eta. \tag{2.12}$$

We call the integral

$$\Phi = \Phi(t) = \int_{\mathcal{B}} \left[\psi + \frac{1}{2}\,\dot{\underset{\sim}{x}}\cdot\dot{\underset{\sim}{x}} + w + h\right] dm \tag{2.13}$$

the <u>canonical free energy</u> of $\mathcal{B}$; it is the sum of the Helmholtz free energy, the kinetic energy, and the mechanical potential of $\mathcal{B}$ at time t.

The <u>specific rate of production of entropy</u> is a function γ of $\underset{\sim}{\xi}$ and t obeying the equation

$$\int_{\mathcal{P}} \gamma\, dm = \frac{d}{dt}\int_{\mathcal{P}} \eta\, dm - \int_{\mathcal{P}} \frac{1}{\theta}\, \omega\, dm + \int_{\partial\mathcal{P}} \frac{1}{\theta}\, \underset{\sim}{q}\cdot\underset{\sim}{n}\, da, \tag{2.14}$$

which holds at each time t and for all parts $\mathcal{P}$ of $\mathcal{B}$. Thus, under suitable assumptions of smoothness we have

$$\gamma = \dot{\eta} - \frac{\omega}{\theta} + \frac{1}{\rho}\,\mathrm{div}(\underset{\sim}{q}/\theta). \tag{2.15}$$

The <u>second law of thermodynamics</u> asserts that in each process of $\mathcal{B}$ the rate of production of entropy is non-negative in every part $\mathcal{P}$ of $\mathcal{B}$ at each time t. Hence

$$\gamma(\underset{\sim}{\xi}, t) \geq 0 \tag{2.16}$$

for all $\underset{\sim}{\xi}$ and t. Employing (2.4) and (2.12) we may cast (2.15) into the form

$$\dot{\psi} - \frac{1}{\rho}\,\mathrm{tr}\{\underset{\sim}{T}\underset{\sim}{D}\} + \eta\dot{\theta} + \frac{1}{\rho\theta}\,\underset{\sim}{q}\cdot\mathrm{grad}\,\theta = -\theta\gamma. \tag{2.17}$$

Let $\mathcal{C}$ be a process of $\mathcal{B}$. We say that $\mathcal{C}$ is <u>isothermal at time</u> t^* if

$$\dot{\theta}(\underset{\sim}{\xi}, t^*) = 0 \quad \text{and} \quad \nabla\theta(\underset{\sim}{\xi}, t^*) = \underset{\sim}{0} \tag{2.18}^{\#}$$

#We assume that, at each $\underset{\sim}{\xi}$, the fields θ and ψ are continuous and piecewise C^1 in t; the superposed dots in (2.18) - (2.20) represent right-hand derivatives.

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at each $\underset{\sim}{\xi}$ in $\mathcal{B}$. For such a process (2.17) implies that, when $t = t^*$,

$$\dot{\psi} - \frac{1}{\rho}\,\mathrm{tr}\{\underset{\sim}{T}\underset{\sim}{D}\} = -\theta\gamma, \tag{2.19}$$

and substitution of this into (2.13) yields, by (2.10) and (2.16),

$$\dot{\Phi}(t) = -\theta\int_{\mathcal{B}} \gamma\, dm \leq 0. \tag{2.20}$$

Thus, we can assert

Remark 2.1. *In a process which is isothermal at all times, the canonical free energy Φ of $\mathcal{B}$ never increases.*

3. Constitutive Equations

The deformation gradient at $\underset{\sim}{\xi}$ at time t is the invertible tensor $\underset{\sim}{F}(\underset{\sim}{\xi},t) = \nabla\underset{\sim}{\chi}(\underset{\sim}{\xi},t)$. The histories up to time t of the deformation gradient and temperature at $\underset{\sim}{\xi}$ are functions on $[0,\infty)$ defined by

$$\underset{\sim}{F}^t(s) = \underset{\sim}{F}(\underset{\sim}{\xi},t-s), \qquad \theta^t(s) = \theta(\underset{\sim}{\xi},t-s), \qquad 0 \leq s < \infty. \tag{3.1}$$

Let us suppose that $\mathcal{B}$ is an incompressible perfect conductor of heat; that is, each process admissible in $\mathcal{B}$ obeys the following constitutive relations:

$$\left.\begin{aligned} \psi &= \underline{\underline{p}}(\underset{\sim}{F}^t,\theta^t), \\ \underset{\sim}{T} &= -p\underset{\sim}{1} + \underline{\underline{T}}(\underset{\sim}{F}^t,\theta^t), \\ \eta &= \underline{\underline{h}}(\underset{\sim}{F}^t,\theta^t), \\ |\det \underset{\sim}{F}| &= 1, \\ \nabla\theta &= \underset{\sim}{0}. \end{aligned}\right\} \tag{3.2}$$

Here p is an arbitrary function of $\underset{\sim}{\xi}$ and t, and $\underline{\underline{p}}$, $\underline{\underline{T}}$, and $\underline{\underline{h}}$ are functionals which are specified in advance and characterize the particular material comprising $\mathcal{B}$. It is customary to use the normalization $\mathrm{tr}\,\underline{\underline{T}} = 0$. The condition $|\det \underset{\sim}{F}| = 1$ expresses the assumption that all deformations of an incompressible material are isochoric, while the equation $\nabla\theta = \underset{\sim}{0}$ asserts that in a perfect conductor of heat the temperature field must be uniform throughout the body.# In general, the form of $\underline{\underline{p}}$, $\underline{\underline{T}}$, and $\underline{\underline{h}}$ can depend on

#For our incompressible perfect conductor, instead of constitutive equations for p and $\tilde{q}$ we have the constraints $|\det \tilde{F}| = 1$ and $\nabla\theta = \tilde{0}$. Of course, when constructing processes we must choose $p = p(\tilde{\xi}, t)$ and $\tilde{q} = \tilde{q}(\tilde{\xi}, t)$ so that the laws of balance of momentum and energy hold.

$\tilde{\xi}$ as a parameter. $\mathcal{B}$ is <u>materially homogeneous</u> if there exists a reference configuration, called a <u>homogeneous reference</u>, such that $\underline{\underline{p}}$, $\underline{\underline{T}}$, and $\underline{\underline{h}}$ are independent of $\tilde{\xi}$. We here consider only homogeneous materials. Without saying so again, we shall always take the reference configuration to be homogeneous and assume that the mass density ρ is constant in it and hence constant in space and time forever.

Although it is not necessary for us to do so, let us simplify matters by confining attention to those processes in which the temperature of the surface of $\mathcal{B}$ is held constant in time, at all times. Then, by $(3.2)_5$, the temperature field obeys $\dot{\theta} = 0$ throughout $\mathcal{B}$ at all times; i.e. each process we consider is isothermal at every instant, and there is no longer any reason to exhibit explicitly the history θ^t in $(3.2)_{1,2,3}$. It suffices to remember that $\underline{\underline{p}}$, $\underline{\underline{T}}$, and $\underline{\underline{h}}$ depend on θ as a parameter which remains constant throughout our discussion. Thus, we replace

(3.2) by

$$\left.\begin{aligned}
\psi &= \underline{\underline{p}}(\mathbf{F}^t), \\
\mathbf{T} &= -p\mathbf{1} + \underline{\underline{T}}(\mathbf{F}^t), \\
\eta &= \underline{\underline{h}}(\mathbf{F}^t), \\
|\det \mathbf{F}| &= 1, \\
\nabla\theta &= \mathbf{0}, \\
\dot{\theta} &= 0.
\end{aligned}\right\} \tag{3.3}$$

Let $\mathcal{H}$ be the set of all unimodular tensors and $\mathcal{O}$ the set of all orthogonal tensors:

$$\mathcal{H} = \left\{\mathbf{H} \;\middle|\; |\det \mathbf{H}| = 1\right\}, \qquad \mathcal{O} = \left\{\mathbf{Q} \;\middle|\; \mathbf{Q}^T\mathbf{Q} = \mathbf{1}\right\}.$$

We refer to $\mathcal{H}$ as the <u>unimodular group</u> and to $\mathcal{O}$ as the <u>orthogonal group</u>. If $\mathcal{F}$ is a set, we denote by $\mathcal{F}^*$ the set of all functions mapping $[0,\infty)$ into $\mathcal{F}$. We write $\mathcal{D}$ for the domain of definition of the functionals $\underline{\underline{p}}$, $\underline{\underline{T}}$, and $\underline{\underline{h}}$ in (3.3). It follows from $(3.3)_4$ that $\mathcal{D}$ is a subset of $\mathcal{H}^*$. The principle of material frame indifference here requires that $\underline{\underline{p}}$, $\underline{\underline{T}}$, and $\underline{\underline{h}}$ obey the identities

$$\underline{\underline{p}}(\mathbf{Q}^*\mathbf{F}^*) = \underline{\underline{p}}(\mathbf{F}^*), \qquad \underline{\underline{T}}(\mathbf{Q}^*\mathbf{F}^*) = \mathbf{Q}^*(0)\underline{\underline{T}}(\mathbf{F}^*)\mathbf{Q}^*(0)^{-1}, \qquad \underline{\underline{h}}(\mathbf{Q}^*\mathbf{F}^*) = \underline{\underline{h}}(\mathbf{F}^*) \tag{3.4}$$

for all $\mathbf{F}^*$ in $\mathcal{D}$ and all $\mathbf{Q}^*$ in $\mathcal{O}^*$. The <u>symmetry group</u>[#] $\mathcal{S}_{\mathcal{R}}$ of a material

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[#]See Noll [1958, 1], who called it the "isotropy group". The concepts of <u>solid</u>, <u>fluid</u>, and <u>undistorted reference configuration</u> which we employ here are those of Noll.

obeying (3.3) is the set of tensors $\mathbf{H}$ in $\mathcal{H}$ such that

$$\underline{\underline{p}}(\mathbf{F}^*\mathbf{H}) = \underline{\underline{p}}(\mathbf{F}^*), \qquad \underline{\underline{T}}(\mathbf{F}^*\mathbf{H}) = \underline{\underline{T}}(\mathbf{F}^*), \qquad \underline{\underline{h}}(\mathbf{F}^*\mathbf{H}) = \underline{\underline{h}}(\mathbf{F}^*) \tag{3.5}$$

for all $\mathbf{F}^*$ in $\mathcal{D}$. It is easily verified that $\mathcal{S}_{\mathcal{R}}$ is a group: if $\mathbf{H}_1$ and $\mathbf{H}_2$ are in $\mathcal{S}_{\mathcal{R}}$ then so also are $\mathbf{H}_1\mathbf{H}_2$ and $\mathbf{H}_2\mathbf{H}_1$. Of course, $\mathcal{S}_{\mathcal{R}}$ depends on the choice of the reference configuration $\mathcal{R}$. If there is an $\mathcal{R}$ such that $\mathcal{S}_{\mathcal{R}}$ contains the orthogonal group $\mathcal{O}$, then we say that the material under consideration is <u>isotropic</u>, and $\mathcal{R}$ is called an <u>undistorted</u> reference configuration. If, for some $\mathcal{R}$, $\mathcal{S}_{\mathcal{R}}$ is contained in $\mathcal{O}$, then the material is a <u>solid</u>, and we again call $\mathcal{R}$ undistorted. Hence, for an undistorted reference configuration $\mathcal{R}$ of an <u>isotropic</u> <u>solid</u>, $\mathcal{S}_{\mathcal{R}} = \mathcal{O}$. In general, we say that a configuration $\mathcal{R}$ is <u>undistorted</u> if $\mathcal{S}_{\mathcal{R}}$ is comparable to $\mathcal{O}$, that is, if $\mathcal{S}_{\mathcal{R}}$ either is itself a subgroup of $\mathcal{O}$ or contains $\mathcal{O}$ as a subgroup. Without saying so again, whenever the material under consideration is such that the class $\underline{\underline{C}}$ of its undistorted configurations is non-empty, we shall assume that the reference configuration $\mathcal{R}$ we are employing is in $\underline{\underline{C}}$. A <u>fluid</u> is a material for which $\mathcal{S}_{\mathcal{R}}$ is the unimodular group $\mathcal{U}$. If $\mathcal{S}_{\mathcal{R}} = \mathcal{U}$ for one reference configuration $\mathcal{R}$ then $\mathcal{S}_{\mathcal{R}} = \mathcal{U}$ for all. It is clear that every fluid is isotropic and has the property that all of its reference configurations are undistorted.

Let $\mathbf{k}$ be a unit vector and let $\mathcal{O}_{\mathbf{k}}$ be the group of all orthogonal tensors with $\mathbf{k}$ as a proper vector:

$$\mathcal{O}_{\mathbf{k}} = \left\{\mathbf{R} \mid \mathbf{R}\mathbf{R}^T = \mathbf{1},\ \mathbf{R}\mathbf{k} = \pm\mathbf{k}\right\}.$$

If a material is such that for some reference configuration $\mathcal{R}$, $\mathcal{S}_{\mathcal{R}}$ contains $\mathcal{O}_{\underset{\sim}{k}}$, then the material is said to be transversely isotropic with $\underset{\sim}{k}$ as the axis of symmetry.

We assume that the constitutive functionals $\underline{\underline{p}}$, $\underline{\underline{T}}$, and $\underline{\underline{h}}$ possess the properties of smoothness employed in Coleman and Mizel's recent formulation# of the general theory of materials with gradually fading memory. The requirement that (2.16) hold for all smooth processes obeying (3.3) places restrictions on $\underline{\underline{p}}$, $\underline{\underline{T}}$, and $\underline{\underline{h}}$.## These restrictions may be read off immediately from Coleman's discussion### of this problem for compressible materials with fading memory.#### We do not list all these

#[1967, 3] [1968, 2]. See also their study of $\mathcal{L}_p$-spaces, [1966, 1], and the earlier articles of Coleman & Noll, [1960, 1] [1961, 1].

##Coleman & Noll [1963, 1].

###[1964, 1].

####No difficulty is caused by the fact that we consider perfect conductors while Coleman [1964, 1] and Coleman & Mizel [1967, 3] focus attention on materials for which the heat flux $\underset{\sim}{q}$ is given by a constitutive equation of the form $\underset{\sim}{q} = \hat{\underset{\sim}{q}}(\underset{\sim}{F}^t, \theta^t; \text{grad}\,\theta)$. Furthermore, the proofs employed by these authors for compressible materials are easily modified to cover the incompressible case.

restrictions here, for we shall emphasize only one of them in our subsequent discussion. To state this one we need some definitions: If $\mathbf{F}$ is a tensor we denote by $\mathbf{F}^{\dagger}$ the (constant) function on $[0,\infty)$ whose value is $\mathbf{F}$ for all s; i.e.

$$\mathbf{F}^{\dagger}(s) \equiv \mathbf{F}, \qquad 0 \leq s < \infty. \tag{3.6}$$

When given a functional such as $\mathbf{p}$ with $\mathcal{D}$ for its domain, we may define a function $\mathbf{p}^{\circ}$ on a region in $\mathcal{H}$ by the relation

$$\mathbf{p}^{\circ}(\mathbf{F}) = \mathbf{p}(\mathbf{F}^{\dagger}), \qquad \text{for } \mathbf{F}^{\dagger} \text{ in } \mathcal{D}\,. \tag{3.7}$$

$\mathbf{p}^{\circ}$ is called the *equilibrium response function corresponding to* $\mathbf{p}$. The second law of thermodynamics requires that the functional $\mathbf{p}$ in (3.3) have the following property:

Remark 3.1. *For each function* $\mathbf{F}^{*}$ *in the domain* $\mathcal{D}$ *of* $\mathbf{p}$

$$\mathbf{p}^{\circ}\big(\mathbf{F}^{*}(0)\big) \leq \mathbf{p}(\mathbf{F}^{*}). \tag{3.8}$$

Let $\mathcal{L}$ be the subset of $\mathcal{D}$ consisting of those functions $\mathbf{F}^{t}$ which can occur as the history up to some time t of the deformation gradient in an irrotational motion. It follows from a result of Coleman and Truesdell# that to prove that (3.8) holds for all $\mathbf{F}^{*}$ in $\mathcal{D}$ it suffices

#[1965, 1]. See the proof of their Proposition 4.

to show that (3.8) holds for all $\mathbf{F}^*$ in $\mathcal{S}$. To do this, let $\mathbf{F}^*$ be in $\mathcal{S}$ and define, for each $\delta \geq 0$, the function $\mathbf{F}^{*(\delta)}$ by

$$\mathbf{F}^{*(\delta)}(s) = \begin{cases} \mathbf{F}^*(0), & 0 \leq s \leq \delta, \\ \mathbf{F}(s-\delta), & s > \delta; \end{cases} \tag{3.9}$$

$\mathbf{F}^{*(\delta)}$ is the <u>static continuation of $\mathbf{F}^*$ by amount</u> δ.# We are assuming that

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#Cf. Coleman & Noll [1964, 2].

$\underline{\underline{p}}$ is continuous over $\mathcal{D}$, and that $\mathcal{D}$ is contained in a history space of the type discussed by Coleman and Mizel [1967, 3]; the norm $\|\cdot\|$ on such a history space has the following <u>relaxation property</u>:

$$\lim_{\delta\to\infty} \|\mathbf{F}^{*(\delta)} - \mathbf{F}^*(0)^\dagger\| = 0.$$

Therefore,

$$\lim_{\delta\to\infty} \underline{\underline{p}}(\mathbf{F}^{*(\delta)}) = \underline{\underline{p}}\big(\mathbf{F}^*(0)^\dagger\big) = \underline{\underline{p}}^\circ\big(\mathbf{F}^*(0)\big). \tag{3.10}$$

Our assumption that $\mathbf{F}^*$ is in $\mathcal{S}$ implies that $\mathbf{F}^{*(\delta)}$ is also in $\mathcal{S}$ for each $\delta \geq 0$. Furthermore, there is a process $\mathcal{C}$ of $\mathcal{B}$ which obeys (3.3) and is such that the motion χ of $\mathcal{B}$ is a homogeneous irrotational motion with the history $\mathbf{F}^t$ of the deformation gradient at each point of $\mathcal{B}$ obeying

$$\mathbf{F}^t = \mathbf{F}^{*(t)} \quad \text{for} \quad t \geq 0. \tag{3.11}$$

[To construct such a process $\mathcal{C}$, we first take any homogeneous motion χ obeying (3.11). We then note that when $\mathbf{F}^*$ is in $\mathcal{S}$, χ is automatically isochoric and irrotational, and whenever $\underset{\sim}{b}$ obeys (2.9), a homogeneous, isochoric, irrotational motion in a material of the type (3.3) automatically satisfies the dynamical equation (2.3) for a suitable choice of the pressure field p.# Finally, we pick $\underset{\sim}{q}$ so that the energy balance equation

#See Corollary 2 to Proposition 7 of Coleman & Truesdell [1965, 1].

(2.4) holds with ω set equal to zero and $\dot{\epsilon}$ and $\mathrm{tr}\{\underset{\sim}{T}\underset{\sim}{D}\}$ determined by (3.3), (3.11), and (2.5).] In the process $\mathcal{C}$ we have, at each point of $\mathcal{B}$, $\underset{\sim}{D} = \underset{\sim}{0}$ for $t \geq 0$, and, since $\mathcal{C}$ is isothermal for all t, (2.19) and (2.16) yield

$$\dot{\psi} = -\theta\gamma \leq 0 \quad \text{for} \quad t \geq 0.$$

Hence, by $(3.3)_1$ and (3.10),

$$\underline{\underline{p}}(\underset{\sim}{F}^*) \geq \underline{\underline{p}}(\underset{\sim}{F}^{*(t)}) \quad \text{for} \quad t \geq 0. \tag{3.12}$$

It follows immediately from (3.10) and (3.12) that (3.8) must hold for each $\underset{\sim}{F}^*$ in $\mathcal{L}$. Therefore, (3.8) holds for each $\underset{\sim}{F}^*$ in $\mathcal{D}$.

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4. Motions of Isochoric Extension

The right and left stretch tensors, $\mathbf{U}$ and $\mathbf{V}$, are positive definite symmetric tensors defined by the polar decompositions

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \qquad \mathbf{R}\mathbf{R}^T = \mathbf{1}, \tag{4.1}$$

with $\mathbf{F}$ the deformation gradient. Since $\mathbf{U} = \mathbf{R}^T\mathbf{V}\mathbf{R}$ with $\mathbf{R}$ orthogonal, $\mathbf{U}$ and $\mathbf{V}$ have the same proper numbers α_i, $i = 1,2,3$. These positive numbers α_i are called principal stretch ratios. Clearly, $|\det \mathbf{F}| = 1$ if and only if $\alpha_1\alpha_2\alpha_3 = 1$. The proper vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ of $\mathbf{U}$ and $\mathbf{V}$ are called, respectively, the right and left principal directions of stretch; they are related through the formulae

$$\mathbf{v}_i = \mathbf{R}\mathbf{u}_i, \qquad \mathbf{u}_i = \mathbf{R}^T\mathbf{v}_i. \tag{4.2}$$

The proper numbers σ_i and the proper vectors $\mathbf{s}_i$ of the stress tensor $\mathbf{T}$ are called principal stresses and principal axes of stress.

A motion $\boldsymbol{\chi}$ of $\mathcal{B}$ is called an isochoric extension if for each $\boldsymbol{\xi}$ there exists an orthonormal basis $\mathbf{u}_i(\boldsymbol{\xi})$, independent of t, such that the matrix of the components of $\mathbf{U}(\boldsymbol{\xi},t)$ with respect to $\mathbf{u}_i(\boldsymbol{\xi})$ has the form

$$[\mathbf{U}(\boldsymbol{\xi},t)] = \begin{bmatrix} \alpha_1(\boldsymbol{\xi},t) & 0 & 0 \\ \cdot & \alpha_2(\boldsymbol{\xi},t) & 0 \\ \cdot & \cdot & \alpha_3(\boldsymbol{\xi},t) \end{bmatrix}, \qquad \alpha_1\alpha_2\alpha_3 = 1, \tag{4.3}$$

for all t, $-\infty < t < \infty$. In words: a motion of isochoric extension is a density preserving motion in which the right principal directions of stretch $\underset{\sim}{u}_i$ remain constant in time at each material point, although these directions of stretch may vary from point to point and the stretch ratios α_i may vary in time.

It follows immediately from results obtained by Coleman# for

#[1968, 1] Theorems 4 and 5.

general isotropic materials that we can here make the following assertion.

Remark 4.1. *If the motion of an incompressible isotropic material is an isochoric extension, the principal directions of stress are given by*

$$\underset{\sim}{s}_i(\underset{\sim}{\xi},t) = \underset{\sim}{v}_i(\underset{\sim}{\xi},t) = \underset{\sim}{R}(\underset{\sim}{\xi},t)\underset{\sim}{u}_i(\underset{\sim}{\xi}), \tag{4.4}$$

where $\underset{\sim}{u}_i(\underset{\sim}{\xi})$, $i = 1,2,3$, *is the orthonormal basis relative to which* (4.3) *holds and* $\underset{\sim}{R}$ *is the orthogonal tensor in* (4.1); *furthermore, the principal stresses* $\sigma_i = \sigma_i(\underset{\sim}{\xi},t)$ *and the specific Helmholtz free energy* $\psi = \psi(\underset{\sim}{\xi},t)$ *are related as follows to the histories* α_j^t *of the principal stretch ratios*:

$$\left.\begin{aligned} \sigma_2 - \sigma_1 = \mathcal{f}(\alpha_2^t,\alpha_1^t), \qquad \sigma_3 - \sigma_1 = \mathcal{f}(\alpha_3^t,\alpha_1^t), \\ \psi = \mathcal{g}(\alpha_2^t,\alpha_3^t). \end{aligned}\right\} \tag{4.5}$$

Here $\alpha_j^t(s) = \alpha_j(\underset{\sim}{\xi},t-s)$ *for* $0 \leq s < \infty$, *and* $\mathcal{f}$ *and* $\mathcal{g}$ *are scalar-valued* functionals obeying the identities

$$\mathscr{f}(\alpha^*,\beta^*) = -\mathscr{f}(\beta^*,\alpha^*), \qquad \mathscr{g}(\alpha^*,\beta^*) = \mathscr{g}(\beta^*,\alpha^*), \tag{4.6}$$

$$\mathscr{f}(\alpha^*,\beta^*) = \mathscr{f}(\alpha^*,\gamma^*) - \mathscr{f}(\beta^*,\gamma^*), \qquad \mathscr{g}(\alpha^*,\beta^*) = \mathscr{g}(\alpha^*,\gamma^*) = \mathscr{g}(\gamma^*,\beta^*) \tag{4.7}$$

for positive functions α^*, β^*, γ^* on $[0,\infty)$ with $\alpha^*(s)\beta^*(s)\gamma^*(s) \equiv 1$. Of course, $\mathscr{f}$ and $\mathscr{g}$ are determined when the functionals $\underline{\underline{T}}$ and $\underline{\underline{p}}$ are specified.

We are here considering perfect conductors obeying (3.3). An elementary calculation shows that

$$\mathrm{tr}\{\underset{\sim}{T}\underset{\sim}{D}\} = \mathrm{tr}\{\underset{\sim}{R}^{-1}\underset{\sim}{T}\underset{\sim}{R}\dot{\underset{\sim}{U}}\underset{\sim}{U}^{-1}\},$$

and, therefore, (4.3)-(4.5) yield

$$\mathrm{tr}\,\underset{\sim}{T}\underset{\sim}{D} = \sum_{i=1}^{3} \frac{\sigma_i\dot{\alpha}_i}{\alpha_i} = \frac{\sigma_2-\sigma_1}{\alpha_2}\dot{\alpha}_2 + \frac{\sigma_3-\sigma_1}{\alpha_3}\dot{\alpha}_3 = \mathscr{f}(\alpha_2^t,\alpha_1^t)\dot{\overline{\ln\alpha_2}} + \mathscr{f}(\alpha_3^t,\alpha_1^t)\dot{\overline{\ln\alpha_3}}\,.$$

Substitution of this expression and $(4.5)_3$ into (2.19) yields the following equation for the specific rate γ of production of entropy in an incompressible isotropic material obeying (3.3) and undergoing a motion of isochoric extension:

$$-\theta\gamma = \frac{d}{dt}\mathscr{g}(\alpha_2^t,\alpha_3^t) - \frac{1}{\rho}\mathscr{f}(\alpha_2^t,\alpha_1^t)\frac{d}{dt}\ln\alpha_2^t(0) - \frac{1}{\rho}\mathscr{f}(\alpha_3^t,\alpha_1^t)\frac{d}{dt}\ln\alpha_3^t(0). \tag{4.8}$$

An isochoric extension with $\alpha_1 \equiv 1$ is called a planar isochoric extension with $\underset{\sim}{u}_1$ the neutral direction.# For such a motion the condition

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#It can be called also a "pure shear".

$\alpha_1\alpha_2\alpha_3 = 1$ implies that a single function λ^t determines α_1^t, α_2^t, α_3^t; i.e. if we put $\lambda^t(s) = \alpha_2(\underset{\sim}{\xi}, t-s)$, then at $\underset{\sim}{\xi}$,

$$\alpha_1^t(s) = 1, \qquad \alpha_2^t(s) = \lambda^t(s), \qquad \alpha_3^t(s) = \left(\lambda^t(s)\right)^{-1}, \qquad 0 \leq s < \infty. \tag{4.9}$$

In this special case (4.5) may be written

$$\left.\begin{aligned} \sigma_2 - \sigma_3 &= \mathscr{m}(\lambda^t), \\ \sigma_2 - \sigma_1 &= \mathscr{n}(\lambda^t), \\ \psi &= \mathscr{p}(\lambda^t), \end{aligned}\right\} \tag{4.10}$$

with $\mathscr{m}$, $\mathscr{n}$, and $\mathscr{p}$ scalar-valued functionals determined as follows by $\mathscr{f}$ and $\mathscr{g}$:

$$\left.\begin{aligned} \mathscr{m}(\alpha^*) &= \mathscr{f}(\alpha^*, 1^\dagger) - \mathscr{f}\left((\alpha^*)^{-1}, 1^\dagger\right), \\ \mathscr{n}(\alpha^*) &= \mathscr{f}(\alpha^*, 1^\dagger), \\ \mathscr{p}(\alpha^*) &= \mathscr{g}\left(\alpha^*, (\alpha^*)^{-1}\right). \end{aligned}\right\} \tag{4.11}$$

Here α^* is an arbitrary positive function on $[0,\infty)$, $(\alpha^*)^{-1}$ is defined by $(\alpha^*)^{-1}(s) = \left(\alpha^*(s)\right)^{-1}$, and $1^\dagger(s) = 1$ for all s in $[0,\infty)$. The identities (4.7) imply that $(4.10)_1$ and $(4.10)_3$ are equivalent to

$$\mathscr{m}(\alpha^*) = \mathscr{f}\left(\alpha^*, (\alpha^*)^{-1}\right), \qquad \mathscr{p}(\alpha^*) = \mathscr{g}(1^\dagger, \alpha^*). \tag{4.12}$$

When (4.9) holds, (4.8) reduces to

$$-\theta\gamma = \frac{d}{dt}\mathscr{p}(\lambda^t) - \frac{1}{\rho}\mathscr{m}(\lambda^t)\frac{d}{dt}\ln\lambda^t(0). \tag{4.13}$$

The arguments employed by Coleman [1968, 1] for isotropic materials may be used to prove also the following assertion which generalizes Remark 4.1.

Remark 4.2. *Suppose an incompressible transversely isotropic material with $\underset{\sim}{k}$ for its axis of symmetry is undergoing a motion $\underset{\sim}{\chi}$ of isochoric extension with $\underset{\sim}{u}_1(\underset{\sim}{\xi}) = \pm\underset{\sim}{k}$. Then (4.4), (4.5), (4.6), and (4.8) hold again, but the functionals $\mathscr{f}$ and $\mathscr{g}$ need not obey (4.7). If the isochoric extension is planar with the neutral direction along $\underset{\sim}{k}$, i.e. if (4.9) holds for some λ^t, then (4.10) and (4.13) hold with $\mathscr{m}$, $\mathscr{n}$, and $\mathscr{p}$ given by (4.11), but (4.12) may not hold.*

By (2.16), $\mathscr{p}$ and $\mathscr{m}$ must be such that the left side of (4.8) is not positive. If, in addition, we have

$$\rho\frac{d}{dt}\mathscr{p}(\lambda^t) - \mathscr{m}(\lambda^t)\frac{d}{dt}\ln\lambda^t(0) < 0 \tag{4.14}$$

whenever the indicated derivatives exist and the function λ^t is not constant on $[0,\infty)$, then we say that the material under consideration is *strictly dissipative in motions of planar isochoric extension*. Similarly, if

$$\rho\frac{d}{dt}\mathscr{g}(\alpha_2^t,\alpha_3^t) - \mathscr{f}(\alpha_2^t,\alpha_1^t)\frac{d}{dt}\ln\alpha_2^t(0) - \mathscr{f}(\alpha_3^t,\alpha_1^t)\frac{d}{dt}\ln\alpha_3^t(0) < 0 \tag{4.15}$$

whenever one of the functions α_1^t, α_2^t, α_3^t is not constant on $[0,\infty)$, then the material is said to be <u>strictly dissipative in motions of general isochoric extension</u>.

5. Inflation of Circular Tube

Let us now suppose that in its reference configuration $\mathcal{B}$ has the form of a hollow circular tube with inner radius R_I and outer radius R_O. Employing a single, fixed, cylindrical coordinate system with the z-axis along the common axis of the cylinders which bound the tube, we assume that $\mathcal{B}$ is undergoing a motion of the form

$$z = Z, \qquad r = r(R,t), \qquad \theta = \Theta, \tag{5.1}$$

where z, r, θ are the coordinates at time t of the material point which has the coordinates Z, R, Θ in the reference configuration. In such a motion, $\mathcal{B}$ remains a circular tube at all times; its inner and outer radii at time t are

$$r_I(t) = r(R_I,t), \qquad r_O = r(R_O,t). \tag{5.2}$$

Each of the unit vectors $\underset{\sim}{e}_z$, $\underset{\sim}{e}_r$, $\underset{\sim}{e}_\theta$ along coordinate lines is both a right and a left principal axis of stretch; the stretch ratios are[#]

[#] Cf. Coleman [1968, 1, §5].

$$\alpha_z(\underset{\sim}{\xi},t) \equiv 1, \qquad \alpha_r(\underset{\sim}{\xi},t) = \frac{\partial}{\partial R} r(R,t), \qquad \alpha_\theta(\underset{\sim}{\xi},t) = \frac{1}{R} r(R,t). \tag{5.3}$$

If we suppose the motion is isochoric, then it is clearly a planar isochoric extension with $\underset{\sim}{e}_z$ the neutral direction. The condition $\alpha_1\alpha_2\alpha_3 = 1$, when combined with (5.3), yields#

#For a thorough discussion of such motions in isotropic elastic materials see the papers of Knowles [1960, 3] [1962, 1].

$$r(R,t)^2 = R^2 + \beta(t). \tag{5.4}$$

Thus a single scalar function $\beta(\cdot)$ completely determines the motion. In this particular isochoric extension, two of the principal axes of stretch, $\underset{\sim}{e}_r$ and $\underset{\sim}{e}_\theta$, vary from point to point, albeit they are constant in time at each point.

We assume that the body is composed of a transversely isotropic material with $\underset{\sim}{e}_z$ its axis of symmetry. By Remark 4.2, $\underset{\sim}{e}_z$, $\underset{\sim}{e}_\theta$, and $\underset{\sim}{e}_r$ are then principal axes of stress, and the corresponding principal stresses, which we may call σ_z, σ_θ, σ_r, obey the equations

$$\sigma_r - \sigma_\theta = \mathscr{m}(\lambda^t), \qquad \sigma_r - \sigma_z = \mathscr{n}(\lambda^t). \tag{5.5}$$

Here

$$\lambda^t(s) = \alpha_r(\underset{\sim}{\xi}, t-s), \qquad 0 \leq s < \infty;$$

and, by (5.3) and (5.4),

$$\lambda^t(s) = [1 + R^{-2}\beta(t-s)]^{-1/2} = \left[1 + \frac{\beta(t-s)}{r^2 - \beta(t)}\right]^{-1/2}, \qquad 0 \leq s < \infty. \tag{5.6}$$

If we employ the history β^t of β up to time t, i.e. the function β^t on

$[0,\infty)$ defined by $\beta^t(s) = \beta(t-s)$, then the equation $(5.5)_1$ can be written, in the spatial description, as

$$\sigma_r(r,t) - \sigma_\theta(r,t) = \mathscr{m}\left(\left[1 + \frac{\beta^t}{r^2 - \beta(t)}\right]^{-1/2}\right). \tag{5.7}$$

Similarly, for the specific Helmholtz free energy we have

$$\psi(r,t) = \mathscr{p}\left(\left[1 + \frac{\beta^t}{r^2 - \beta(t)}\right]^{-1/2}\right),$$

or

$$\psi(R,t) = \mathscr{p}\left([1 + R^{-2}\beta^t]^{-1/2}\right). \tag{5.8}$$

We assume that the rate of heat supply ω and the potential h of the body force both vanish at all times. Since the material under consideration is a perfect conductor, $\underset{\sim}{q}$ assumes the value necessary to make (2.4) hold with θ at its preassigned constant value. It follows from (5.1), (5.5), and (5.6) that of the three scalar equations embodied in the vectorial dynamical equation (2.3), two reduce to 0 = 0, and the remaining one becomes#

#Cf. [1968, 1, Eq. (5.10)] which is the analogous equation for the case in which the tube is made of an isotropic compressible material.

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial r}, \tag{5.9}$$

where σ_r and σ_θ are considered functions of r and t, and

$$v = \frac{\partial}{\partial t}\, r(R,t) = v(r,t).$$

By (5.4),

$$v(r,t) = \frac{1}{2}\frac{1}{r}\,\dot\beta(t), \tag{5.10}$$

and therefore (5.9) can be written

$$\frac{\partial}{\partial r}\,\sigma_r(r,t) + \frac{\sigma_r(r,t) - \sigma_\theta(r,t)}{r} = \rho\left[\frac{\ddot\beta(t)}{2r} - \frac{\dot\beta(t)^2}{4r^3}\right]. \tag{5.11}$$

When β^t is specified, the term $(\sigma_r - \sigma_\theta)/r$ in (5.11) is completely determined by (5.7), i.e. by the functional $\underline{\underline{T}}$ in $(3.2)_2$, but β^t determines σ_r through $(3.2)_2$ only to within an arbitrary pressure $p(r,t)$. For every choice of the function $\beta(\cdot)$ on $(-\infty,\infty)$, with $\beta > -R_I^2$, there exists a pressure function $p(\cdot,\cdot)$ on $[r_I, r_O] \times (-\infty,\infty)$ such that the dynamical equation (2.3) holds for the motion defined by (5.1) and (5.4); according to (5.11), $p(\cdot,\cdot)$ gives to σ_r the following dependence on r and t:

$$\sigma_r(r,t) = \sigma_r(r_I,t) - \int_{r_I}^{r} \frac{\sigma_r(\zeta,t) - \sigma_\theta(\zeta,t)}{\zeta}\, d\zeta + \ddot\beta(t)\frac{\rho}{2}\ln\frac{r}{r_I} + \dot\beta(t)^2\,\frac{\rho}{8}[r^{-2} - r_I^{-2}]. \tag{5.12}$$

Let us put

$$\mathscr{l}(\beta^t) \;\underline{\underline{\mathrm{def}}}\; 2\pi\rho\int_{r_I}^{r_O} r\,\mathscr{p}\left(\left[1^\dagger + \frac{\beta^t}{r^2+\beta^t(0)}\right]^{-1/2}\right)dr = 2\pi\rho\int_{R_I}^{R_O} R\,\mathscr{p}\left([1^\dagger + R^{-2}\beta^t]^{-1/2}\right)dR, \tag{5.13}$$

and

$$\mathscr{h}(\beta^t) \;\underline{\underline{\mathrm{def}}}\; \int_{r_I}^{r_O} \frac{1}{r}\,\mathscr{m}\left(\left[1^\dagger + \frac{\beta^t}{r^2+\beta^t(0)}\right]^{-1/2}\right)dr = \int_{R_I}^{R_O} \frac{R\,\mathscr{m}\left([1^\dagger + R^{-2}\beta^t]^{-1/2}\right)}{R^2 + \beta^t(0)}\,dR. \tag{5.14}$$

It is clear from (5.8) that $\mathscr{L}(\beta^t)$ is the total Helmholtz free energy of a unit length of the tube. We may easily relate $\mathscr{h}(\beta^t)$ to the difference in the external pressures $P_I(t)$ and $P_0(t)$ applied to the inner and outer bounding surfaces of the tube. Indeed, since

$$\Delta P(t) = P_I(t) - P_0(t) = \sigma_r(r_0, t) - \sigma_r(r_I, t), \tag{5.15}$$

on putting $r = r_0$ in (5.12) we obtain

$$\Delta P(t) = -\mathscr{h}(\beta^t) + \Omega_1 \ddot{\beta}(t) - \Omega_2 \dot{\beta}(t)^2, \tag{5.16}$$

with

$$\left.\begin{aligned} \Omega_1 &= \Omega_1\big(\beta(t)\big) = \frac{\rho}{2}\ln\frac{r_0}{r_I} = \frac{\rho}{4}\ln\frac{R_0^2+\beta(t)}{R_I^2+\beta(t)} > 0, \\ \Omega_2 &= \Omega_2\big(\beta(t)\big) = \frac{\rho}{8}\left[\frac{1}{r_I^2} - \frac{1}{r_0^2}\right] = \frac{\rho}{8}\left[\frac{1}{R_I^2+\beta(t)} - \frac{1}{R_0^2+\beta(t)}\right] > 0. \end{aligned}\right\} \tag{5.17}$$

A problem with obvious physical applications is the following: Suppose $\mathscr{m}$ is assigned. Given the history of the motion up to time zero, find the motion for times after zero, assuming that the pressures on the bounding surfaces of the tube are known. That is, given $\beta(t)$ for $t \leq 0$ and $\Delta P(t)$ for all $t \geq 0$, find $\beta(t)$ for $t \geq 0$. This problem is one of solving the non-linear functional-differential equation (5.16).

Writing υ for $\dot{\beta}$, we can express (5.16) in the form

$$\left.\begin{aligned} \dot{\beta}(t) &= \upsilon(t), \\ \dot{\upsilon}(t) &= \frac{\Omega_2}{\Omega_1}\upsilon(t)^2 + \frac{1}{\Omega_1}\mathscr{h}(\beta^t) + \frac{1}{\Omega_1}\Delta P. \end{aligned}\right\} \tag{5.18}$$

Let us assume that ΔP is held constant for $t \geq 0$. Our problem is now the following: Given a number b and a function g on $[0,\infty)$, find a function pair $\beta(\cdot), \upsilon(\cdot)$, with β defined on $(-\infty,\infty)$ and υ defined on $[0,\infty)$,#

#Along a solution of (5.18), $\beta(t)$ is automatically differentiable for $t > 0$ and has a right-hand derivative at $t = 0$. However, we do not require that $\beta(t)$ be differentiable for $t < 0$, i.e. that g be differentiable on $[0,\infty)$; hence, $\upsilon(t)$ may not exist for $t < 0$.

such that (5.18) holds for all $t \geq 0$, and $\beta^0 = g$, $\upsilon(0) = b$, where β^0 is the history of β up to time 0. The pair $\beta(\cdot), \upsilon(\cdot)$ is called the <u>solution of</u> (5.15) <u>with initial velocity</u> b <u>and initial history</u> g.## Since we are

##Here, by the "initial velocity" we mean $\dot{\beta}(0)$. According to (5.10),

$$\dot{r}(R,0) = \frac{\dot{\beta}(0)}{2r(R,0)} = \frac{b}{2\sqrt{R^2 + g(0)}} .$$

assuming that the functionals $\underline{\underline{p}}$ and $\underline{\underline{T}}$ obey the smoothness postulates employed in Coleman and Mizel's [1967, 3] formulation of the thermodynamics of materials with memory, the present functionals $\mathscr{l}$ and $\mathscr{h}$ are continuous on a history space $\underline{\underline{V}}^{(1)}$, with norm $\|\cdot\|^{(1)}$, of the type used in Coleman and Mizel's theory of the stability of solutions of functional-differential equations.### If we further assume that $\mathscr{h}$ is locally Lipschitz continuous

###See [1968, 4, §2 & 3]. [The superscript (1) in $\underline{\underline{V}}^{(1)}$ serves to indicate that this Banach function space is formed from functions mapping $[0,\infty)$ in E^1, i.e. the real axis. The basic properties of such a history space are listed in the Appendix.]

on $\underline{\underline{V}}^{(1)}$,# then (5.18) becomes a functional-differential equation of the

#This condition is met, for example, when $\mathscr{h}$ has a continuous Fréchet derivative at each point in $\underline{\underline{V}}^{(1)}$.

form $\dot{\tilde{y}}(t) = \tilde{f}(\tilde{y}^t)$ with $\tilde{f}$ a locally Lipschitzian functional on a region in the Banach space $\underline{\underline{V}}^{(2)} = E^1 \oplus \underline{\underline{V}}^{(1)}$; the norm $\|\cdot\|^{(2)}$ on $\underline{\underline{V}}^{(2)}$ is given by

$$\|\tilde{y}^t\|^{(2)} = \|\beta^t, \upsilon(t)\|^{(2)} = \|\beta^t\| + |\upsilon(t)|. \tag{5.19}$$

If α is a number, we denote by $\alpha^\dagger$ the constant function in $\underline{\underline{V}}^{(1)}$ with value α. The equilibrium response functions $\mathscr{p}^\circ$, $\mathscr{m}^\circ$, $\mathscr{l}^\circ$, and $\mathscr{h}^\circ$ corresponding to $\mathscr{p}$, $\mathscr{m}$, $\mathscr{l}$, and $\mathscr{h}$ are defined by

$$\mathscr{p}^\circ(\alpha) = \mathscr{p}(\alpha^\dagger), \qquad \mathscr{m}^\circ(\alpha) = \mathscr{m}(\alpha^\dagger), \qquad \mathscr{l}^\circ(\alpha) = \mathscr{l}(\alpha^\dagger), \qquad \mathscr{h}^\circ(\alpha) = \mathscr{h}(\alpha^\dagger).$$

Clearly, for each number $\beta > -R_I^2$

$$\left.\begin{aligned} \mathscr{l}^\circ(\beta) &= 2\pi\rho\int_{R_I}^{R_O} R\,\mathscr{p}^\circ\!\left([1+R^{-2}\beta]^{-1/2}\right)dR, \\ \mathscr{h}^\circ(\beta) &= \int_{R_I}^{R_O} \frac{R\,\mathscr{m}^\circ\!\left([1+R^{-2}\beta]^{-1/2}\right)}{R^2+\beta}\,dR. \end{aligned}\right\} \tag{5.20}$$

If there exists a number $\beta_e > -R_I^2$ such that

$$\mathscr{h}^\circ(\beta_e) + \Delta P = 0, \tag{5.21}$$

then (5.18) has the solution

$$\beta(t) \equiv \beta_e, \qquad \upsilon(t) \equiv 0, \qquad -\infty < t < \infty; \tag{5.22}$$

for this solution $\beta^0(s) \equiv \beta_e$. We call (5.22) the equilibrium solution corresponding to the pressure difference ΔP, while (5.21) is called the equation of equilibrium.

The equilibrium solution (5.22) of (5.18) (with a fixed value of ΔP) is called stable if, given any $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that for every pair b, g, with b a number, g in $\underline{\underline{V}}^{(1)}$, and

$$|b| + \|g - \beta_e^{\dagger}\| < \delta,$$

(5.18) has a unique# solution $\beta(\cdot), \upsilon(\cdot)$ obeying

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#The definition of stability employed by Coleman & Mizel [1968, 4] does not, in general, require that the solution $\beta(\cdot), \upsilon(\cdot)$ obeying the initial condition $\beta^0 = g$, $\upsilon(0) = b$ be unique. However, since in the present application we assume that $\mathcal{h}$ is locally Lipschitz continuous, if the other conditions of the definition are met this one is here fulfilled automatically. The example discussed in §6a of [1968, 4] is very similar to the present.

$$\beta^0 = g, \qquad \upsilon(0) = b,$$

and along this solution

$$|\beta(t) - \beta_e| + |\upsilon(t)| = |\beta(t) - \beta_e| + |\dot{\beta}(t)| < \epsilon$$

for all $t \geq 0$. If, in addition, there exists a $\zeta > 0$ such that for each solution $\beta(\cdot), \upsilon(\cdot)$ with

$$\|\beta^0 + \beta_e^\dagger\| + |\upsilon(0)| < \zeta, \tag{5.23}$$

both $\dot{\beta}(t) \to 0$ and $\beta(t) \to \beta_e$ as $t \to \infty$, then we say that the equilibrium solution (5.22) is asymptotically stable.

When (5.22) gives a stable solution of (5.18), the configuration,

$$z = Z, \qquad r = \sqrt{R^2 + \beta_e}, \qquad \theta = \Theta, \tag{5.24}$$

is an equilibrium configuration of the tube that is stable against those perturbing motions of the form (5.1) which preserve the fixed pressure difference ΔP. Of course, stability of (5.22) as a solution of (5.18) does not, by any means, imply that the configuration (5.24) is stable against perturbing motions which do not have the form (5.1). In terms more suggestive than precise, we can assert that asymptotic stability of the solution (5.22) of (5.18) implies that in any motion of the form (5.1) which preserves the given value of ΔP for $t \geq 0$, the velocity approaches zero and the configuration approaches (5.24) as $t \to \infty$, provided that the motion has small initial velocity and an initial history β^t not too far, in $\underline{\underline{V}}^{(1)}$, from $\beta_e^\dagger$.

We now seek to express the canonical free energy (2.13), per unit length of the tube, as a function $\tilde{\Phi}$ of the history of β^t and the present value of $\dot{\beta}$; that is, we try to cast the equation

$$\Phi(t) \stackrel{\text{def}}{=} \int_{r_I}^{r_O} \left[\psi + \frac{1}{2} v^2 + w\right] \rho 2\pi r dr, \tag{5.25}$$

into the form

$$\Phi(t) = \tilde{\Phi}\big(\beta^t, \dot{\beta}(t)\big) = \tilde{\Phi}\big(\beta^t, \upsilon(t)\big), \tag{5.26}$$

with $\tilde{\Phi}$ a functional. By (5.10) and (5.4), the kinetic energy of a unit length of the tube is

$$\int_{r_I}^{r_0} \frac{1}{2}\rho v^2 2\pi r dr = \frac{\pi}{4}\rho\dot{\beta}(t)^2 \ln\frac{r_0}{r_I} = \frac{\pi}{8}\rho\dot{\beta}(t)^2 \ln\frac{R_0^2 + \beta(t)}{R_I^2 + \beta(t)} . \tag{5.27}$$

For the rate of working of the contact forces applied to the bounding surfaces of a unit length of the tube we have

$$W(t) = 2\pi r_0\sigma_r(r_0,t)v(r_0,t) - 2\pi r_I\sigma_r(r_I,t)v(r_I,t) = \pi\sigma_r(r_0,t)\dot{\beta}(t) - \pi\sigma_r(r_I,t)\dot{\beta}(t) = \pi\dot{\beta}(t)\Delta P,$$

and, since we are assuming that ΔP is constant for $t \geq 0$,

$$W(t) = \frac{d}{dt}\pi\beta(t)\Delta P \qquad \text{for} \quad t \geq 0.$$

Thus, to within an additive constant, w(t) in (2.8) here obeys

$$\int_{r_I}^{r_0} w(t)\rho 2\pi r dr = -\pi\beta(t)\Delta P \qquad \text{for} \quad t \geq 0. \tag{5.28}$$

It is clear from (5.8), (5.13), (5.27), and (5.28) that, for $t \geq 0$, (5.25) can indeed be written in the form (5.26); in fact, $\tilde{\Phi}$ is given by

$$\tilde{\Phi}\big(\beta^t, \upsilon(t)\big) = \mathscr{L}(\beta^t) + \frac{\pi}{8}\rho\upsilon(t)^2 \ln\frac{R_0^2 + \beta^t(0)}{R_I^2 + \beta^t(0)} - \pi\beta^t(0)\Delta P, \tag{5.29}$$

and is continuous when regarded as a function on a neighborhood $\underline{\underline{S}}$ of

$\beta_e^\dagger, 0$ in $\underline{\underline{V}}^{(2)}$. Let us define the function $\tilde{\Phi}^\circ$, of two numbers β and υ, by

$$\tilde{\Phi}^\circ(\beta,\upsilon) = \tilde{\Phi}(\beta^\dagger,\upsilon) = \mathcal{L}^\circ(\beta) + \frac{\pi}{8}\rho\upsilon^2 \ln \frac{R_O^2+\beta}{R_I^2+\beta} - \pi\beta\Delta P. \qquad (5.30)$$

It follows from Remark 3.1 that $\mathcal{P}^\circ(\lambda^t(0)) \leq \mathcal{P}(\lambda^t)$, and, by (5.13) and (5.20),

$$\mathcal{L}^\circ(\beta^t(0)) \leq \mathcal{L}(\beta^t). \qquad (5.31)$$

Therefore, for each pair β^t, υ in $\underline{\underline{S}}$

$$\tilde{\Phi}^\circ(\beta^t(0),\upsilon) \leq \tilde{\Phi}(\beta^t,\upsilon). \qquad (5.32)$$

Remark 2.1 here tells us that the second law of thermodynamics requires the functionals $\mathcal{P}$ and $\mathcal{M}$ to be such that the canonical free energy $\Phi(t)$ never increases in a motion of the form (5.1), (5.4). That is, <u>on each solution of</u> (5.18), $\tilde{\Phi}(\beta^t,\upsilon(t))$ <u>is a non-increasing function of</u> t <u>for</u> $t \geq 0$. This observation, when combined with (5.32), implies

Remark 5.1. <u>The functional</u> $\tilde{\Phi}$ <u>defined in</u> (5.29) <u>is a free energy functional, in the sense of Coleman and Mizel</u> [1968, 3, 4], <u>for the functional-differential equation</u> (5.18).

The function $\mathcal{G}$ defined by

$$\mathcal{G}(\beta) = \tilde{\Phi}^\circ(\beta,0) = \mathcal{L}^\circ(\beta) - \pi\beta\Delta P \qquad (5.33)$$

may be called the <u>Gibbs function</u> for the tube. Its value is the sum of equilibrium Helmholtz free energy and the mechanical potential of a unit length of the tube. It is clear from (5.30), that $\tilde{\Phi}^{\circ}(\beta,\upsilon)$ <u>has a strict local minimum</u>[#] <u>at the point</u> $\beta_e, 0$ <u>in</u> E^2 <u>if and only if</u> $\mathcal{G}(\beta)$ <u>has a strict local minimum at</u> β_e. In view of this and Remark 5.1, from Coleman and Mizel's Theorem 3.1 [1968, 4] we may read off

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[#] A real-valued function ϕ on a vector space E^n is said to have a <u>strict local minimum</u> at a point $\underset{\sim}{z}$ in E^n if, for some $\delta > 0$, $0 < |\underset{\sim}{y}-\underset{\sim}{z}| < \delta \implies \phi(\underset{\sim}{y}) > \phi(\underset{\sim}{z})$

<u>Remark</u> 5.2. <u>If the Gibbs function</u> $\mathcal{G}$ <u>of</u> (5.33) <u>has a strict local minimum at</u> β_e, <u>then</u> (5.22) <u>gives a stable solution of</u> (5.18).

In both the thermodynamics of materials with memory and the classical theories of thermostatics, arguments are given to show that the equilibrium response function for the free energy determines the equilibrium stress-strain function through a formula called the "equilibrium stress relation". Here the equilibrium stress relation yields

$$\mathcal{m}^{\circ}(\lambda) = \rho\lambda \frac{d}{d\lambda} \mathcal{p}^{\circ}(\lambda) \qquad \text{for all } \lambda > 0, \tag{5.34}$$

and, therefore, by (5.20), we have

$$\mathcal{A}^{\circ}(\beta) = -\frac{1}{\pi}\frac{d}{d\beta} \mathcal{L}^{\circ}(\beta) \qquad \text{for all } \beta > -R_I^2. \tag{5.35}$$

Now, if $\mathcal{G}$ has a minimum at β_e, then

$$\frac{d}{d\beta}\mathcal{G}(\beta)\Big|_{\beta=\beta_e} = 0, \tag{5.36}$$

and, by (5.33),

$$\frac{d}{d\beta}\ell^{o}(\beta)\Big|_{\beta=\beta_e} = \pi\Delta P.$$

But, in view of (5.35), this equation is the same as (5.21). Hence, if the Gibbs function $\mathcal{G}$ has a minimum at β_e, then β_e automatically satisfies the equation of equilibrium (5.21). Furthermore, if β_e obeys (5.21), i.e. if (5.22) is known to be a solution of (5.18), then, by Remark 5.2, a sufficient condition for the stability of this equilibrium solution is that

$$\frac{d^2}{d\beta^2}\mathcal{G}(\beta)\Big|_{\beta=\beta_e} > 0. \tag{5.37}$$

By (5.33), the condition (5.37) can be written

$$\frac{d^2}{d\beta^2}\ell^{o}(\beta)\Big|_{\beta=\beta_e} > 0, \tag{5.38}$$

and by (5.35) this, in turn, is equivalent to

$$\frac{d}{d\beta}h^{o}(\beta)\Big|_{\beta=\beta_e} < 0. \tag{5.39}$$

It is clear from (5.20) that a sufficient (but by no means necessary) condition for (5.39) is that $m^{o}(\lambda)$ be a positive, non-decreasing function of λ throughout the range $[1+R_0^{-2}\beta_e]^{-1/2} < \lambda < [1+R_I^{-2}\beta_e]^{-1/2}$.

Let us now suppose that β_e is an isolated solution of (5.21) and that the material under consideration is strictly dissipative in planar isochoric extensions. Since (2.20) and (4.13) here yield

$$\dot{\Phi}(t) = 2\pi\rho\int_{R_I}^{R_O} R\left[\frac{d}{dt}\,p(\lambda^t) - m(\lambda^t)\frac{d}{dt}\ln\,\lambda^t(0)\right]dR,$$

we have, by (4.14), $\dot{\Phi}(t) < 0$ whenever the function λ^t of (5.6) is not constant on $[0,\infty)$. Furthermore, there exists a neighborhood $\underline{\underline{S}}$ of $\beta_e^{\dagger},0$ in $\underline{\underline{V}}^{(2)}$ such that of all solutions $\beta(\cdot),\upsilon(\cdot)$ of (5.18) with initial data $\beta^0,\upsilon(0)$ in $\underline{\underline{S}}$, (5.22) is the only one for which λ^t is constant on $[0,\infty)$ for each $t \geq 0$. Thus, when the initial data is not $\beta_e^{\dagger},0$, we have $\Phi(t)$ strictly monotone decreasing for $t \geq 0$. That is, $\tilde{\Phi}$ is a "strictly dissipative free energy functional" for (5.18) in the sense in which the term is used by Coleman & Mizel [1968, 4, §5], and their Theorem 5.1 here yields

Remark 5.3. *Suppose the material comprising the tube is strictly dissipative in motions of planar isochoric extensions. If β_e is an isolated solution of the equation $h^{\circ}(\beta_e) + \Delta P = 0$, and if $\mathcal{G}$ has a strict local minimum at β_e, then (5.22) gives an asymptotically stable solution of (5.18).*

It follows from (5.33) and (5.35) that if (5.36) and (5.38) both hold then $\mathcal{G}$ does have a strict local minimum at β_e, and, furthermore,

β_e is an isolated solution of the equation $h^\circ(\beta_e) + \Delta P = 0$. Hence we can assert

Remark 5.4. If the material is strictly dissipative in planar isochoric extensions and if

$$\frac{d}{d\beta}\ell^\circ(\beta)\Big|_{\beta=\beta_e} = \pi\Delta P, \qquad \frac{d^2}{d\beta^2}\ell^\circ(\beta)\Big|_{\beta=\beta_e} > 0, \tag{5.40}$$

then (5.22) gives an asymptotically stable solution of (5.18).

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An Example

Of course, the results we have obtained for transversely isotropic materials hold if the material is isotropic. Let us here consider an isotropic material for which the equilibrium response function $\underline{\underline{p}}^\circ$, corresponding to the free energy functional $\underline{\underline{p}}$, has the form associated with a "Mooney material":

$$\underline{\underline{p}}^\circ(\tilde{F}) = b_0 + b_1(I_1-3) + b_2(I_2-3), \qquad b_1 > 0, \quad b_2 \geq 0. \tag{5.41}$$

Here b_i, $i = 0,1,2$, are functions of the temperature alone, while I_1 and I_2 are the principal invariants of the Cauchy-Green tensor $\tilde{B} = \tilde{F}\tilde{F}^T = \tilde{V}^2$:

$$I_1 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2, \qquad I_2 = \alpha_1^2\alpha_2^2 + \alpha_2^2\alpha_3^2 + \alpha_1^2\alpha_3^2.$$

For such a material

$$\mathscr{P}^{o}(\lambda) = (b_1 + b_2)(\lambda^2 + \lambda^{-2}) + b_0 - 2(b_1 + b_2),$$

and (5.20) yields

$$\mathscr{L}^{o}(\beta) = \pi\rho(b_1 + b_2)\left[2\beta \ln\frac{R_0}{R_I} + \beta \ln\frac{R_I^2 + \beta}{R_0^2 + \beta}\right] + b_3, \qquad (5.42)$$

with b_3 a function of θ alone (for given values of R_0 and R_I). Hence

$$\frac{d}{d\beta}\mathscr{L}^{o}(\beta) = \pi\rho(b_1 + b_2)\left[2\ln\frac{R_0}{R_I} + \ln\frac{R_I^2 + \beta}{R_0^2 + \beta} + \beta\left[\frac{1}{R_I^2 + \beta} - \frac{1}{R_0^2 + \beta}\right]\right], \qquad (5.43)$$

and

$$\frac{d^2}{d\beta^2}\mathscr{L}^{o}(\beta) = \pi\rho(b_1 + b_2)\left\{\left[\frac{2}{R_I^2 + \beta} - \frac{2}{R_0^2 + \beta}\right] - \beta\left[\frac{1}{(R_I^2 + \beta)^2} - \frac{1}{(R_0^2 + \beta)^2}\right]\right\}. \qquad (5.44)$$

It follows from (5.35) that the equation of equilibrium (5.21) can be written in the form $(5.40)_1$, and here that equation becomes

$$\rho(b_1 + b_2)\left[\ln\frac{R_0^2(R_I^2 + \beta_e)}{R_I^2(R_0^2 + \beta_e)} + \left(\frac{\beta_e}{R_I^2 + \beta_e} - \frac{\beta_e}{R_0^2 + \beta_e}\right)\right] = \Delta P. \qquad (5.45)$$

For each value of ΔP in the range

$$-\infty < \Delta P < \Delta P_c, \qquad (5.46)$$

with

$$\Delta P_c \overset{\text{def}}{=\!=} 2(b_1 + b_2)\ln\frac{R_0}{R_I}, \qquad (5.47)$$

(5.45) has a unique solution β_e obeying

$$-R_I^2 < \beta_e < \infty, \qquad (5.48)$$

and for this value of β_e, (5.44) yields

$$\left.\frac{d^2}{d\beta^2}\mathscr{L}^\circ(\beta)\right|_{\beta=\beta_e} > 0. \tag{5.49}$$

Thus, by Remark 5.4, <u>when the material has an equilibrium response of the Mooney type</u> (5.41) <u>and is strictly dissipative in planar extensions, for each applied pressure difference</u> ΔP <u>less than</u> ΔP_c <u>the tube has a unique equilibrium configuration of the form</u> (5.24), <u>and there exists a</u> $\zeta > 0$ <u>such that in every perturbing motion of the form</u> (5.1), (5.4), <u>which preserves this pressure difference for</u> $t \geq 0$ <u>and obeys</u> (5.23), <u>both</u> $\beta(t) \to \beta_e$ <u>and</u> $\dot{\beta}(t) \to 0$ <u>as</u> $t \to \infty$.

It is worth noting that for values of ΔP greater than ΔP_c there is no root β_e of the equation (5.45) determining the static equilibrium of the tube.#

#This fact was observed by Knowles [1962, 1], who showed that if the material comprising the tube is not strictly dissipative, but instead perfectly elastic, i.e. if $\underline{\underline{p}}(\underset{\sim}{F}^t) = \underline{\underline{p}}^\circ\left(\underset{\sim}{F}^t(0)\right)$ for all $\underset{\sim}{F}^t$, with $\underline{\underline{p}}^\circ$ given by (5.41), then, for values of ΔP less than ΔP_c, those motions of the form (5.1), (5.4) which preserve ΔP must be periodic.

6. Inflation of a Spherical Shell

Employing a fixed spherical coordinate system, we here assume that the body is comprised of an incompressible isotropic material and is undergoing a motion of the form#

#For discussions of such motions in elastic materials see Truesdell [1962, 2], Guo Zhong-Heng & Solecki [1963, 2 & 3], Knowles & Jakub [1965, 3], and Wang [1965, 4].

$$\theta = \Theta, \qquad \phi = \Phi, \qquad r = r(R,t) = \sqrt[3]{R^3 + \beta(t)}, \tag{6.1}$$

with θ, ϕ, r the coordinates at time t of the material point which has coordinates Θ, Φ, R in the reference configuration $\mathcal{R}$, which, of course, is chosen to be undistorted. We suppose that in the configuration $\mathcal{R}$ the body $\mathcal{B}$ is a spherical shell with inner radius R_I, outer radius R_O, and center of curvature at $R = 0$. It follows from (6.1) that $\mathcal{B}$ is then a spherical shell at each time t with radii

$$r_I(t) = r(R_I,t), \qquad r_O = r(R_O,t). \tag{6.2}$$

In such a motion the right and left principal axes of stretch coincide at each material point and are given by the vectors $\underset{\sim}{e}_\theta$, $\underset{\sim}{e}_\phi$, $\underset{\sim}{e}_r$ tangent to the coordinate lines. Thus the motion is an isochoric extension.##

##Cf. Coleman [1968, 1, §6]. The special form of the function $r(R,t)$ in (6.1) is a consequence of our present requirement that the motion be isochoric.

The principal stretch ratios are

$$\left.\begin{aligned} \alpha_\theta(\underset{\sim}{\xi},t) &= \alpha_\phi(\underset{\sim}{\xi},t) = \frac{1}{R}\, r(R,t) = [1+R^{-3}\beta(t)]^{1/3}, \\ \alpha_r(\underset{\sim}{\xi},t) &= \frac{\partial}{\partial R}\, r(R,t) = [1+R^{-3}\beta(t)]^{-2/3}, \end{aligned}\right\} \tag{6.3}$$

and by Remark 4.1 the principal stresses σ_θ, σ_ϕ, σ_r and the specific Helmholtz free energy ψ obey

$$\left.\begin{aligned} \sigma_r - \sigma_\theta &= \sigma_r - \sigma_\phi = \mathscr{f}(\alpha_r^t,\alpha_\theta^t), \\ \psi &= \mathscr{g}(\alpha_r^t,\alpha_\theta^t), \end{aligned}\right\} \tag{6.4}$$

with

$$\left.\begin{aligned} \alpha_r^t(s) &= [1+R^{-3}\beta(t-s)]^{-2/3} = \left[1 + \frac{\beta(t-s)}{r^3-\beta(t)}\right]^{-2/3}, \\ \alpha_\theta^t(s) &= \alpha_\phi^t(s) = [\alpha_r^t(s)]^{-1/2}, \qquad 0 \le s < \infty. \end{aligned}\right\} \tag{6.5}$$

Let us define the functionals $\mathscr{s}$ and $\mathscr{u}$ by

$$\mathscr{s}(\alpha^*) = \mathscr{f}\big(\alpha^*,(\alpha^*)^{-1/2}\big), \qquad \mathscr{u}(\alpha^*) = \mathscr{g}\big(\alpha^*,(\alpha^*)^{-1/2}\big); \tag{6.6}$$

of course, (4.6) yields

$$\mathscr{s}(1^\dagger) = 0. \tag{6.7}$$

Employing (6.5) and (6.6), we obtain from (6.4),

$$\sigma_r - \sigma_\theta = \mathscr{s}(\alpha_r^t) = \mathscr{s}\left(\left[1^\dagger + \frac{\beta^t}{r^3-\beta^t(0)}\right]^{-2/3}\right) = \mathscr{s}\Big([1^\dagger + R^{-3}\beta^t]^{-2/3}\Big), \tag{6.8}$$

and

$$\psi = \mathscr{u}(\alpha_r^t) = \mathscr{u}\left([1^{\dagger} + R^{-3}\beta^t]^{-2/3}\right). \tag{6.9}$$

Let us now put

$$a(\beta^t) \stackrel{\text{def}}{=} \rho\int_{R_I}^{R_O} R^2 \mathscr{u}\left([1 + R^{-3}\beta^t]^{-2/3}\right) dR, \tag{6.10}$$

$$b(\beta^t) \stackrel{\text{def}}{=} 2\int_{R_I}^{R_O} \frac{R^2 s\left([1 + R^{-3}\beta^t]^{-2/3}\right)}{R^3 + \beta^t(0)} dR. \tag{6.11}$$

The value $a(\beta^t)$ of the functional a is clearly the Helmholtz free energy of the shell, per unit solid angle. The significance of $b(\beta^t)$ will be clear shortly.

We assume that the heat supply ω and the potential h of the body force both vanish. Since the material is a perfect conductor, $\underset{\sim}{q}$ assumes the values necessary to make (2.4) hold, and (2.3) is here equivalent to the single scalar equation#

#Cf. Coleman [1968, 1, Eq. (6.7)].

$$\frac{\partial \sigma_r}{\partial r} + \frac{2}{r}[\sigma_r - \sigma_\theta] = \rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial r}, \tag{6.12}$$

with

$$v = \frac{\partial}{\partial t} r(R,t) = \frac{1}{3}[R^3 + \beta(t)]^{-2/3}\dot{\beta}(t) = \frac{1}{3} r^{-2}\dot{\beta}(t). \tag{6.13}$$

We may write (6.12) in the form

$$\frac{\partial}{\partial r} \sigma_r(r,t) + \frac{2}{r}[\sigma_r(r,t) - \sigma_\theta(r,t)] = \rho\left[\frac{\ddot{\beta}(t)}{3r^2} - \frac{2\dot{\beta}(t)^2}{9r^5}\right]. \tag{6.14}$$

For each choice of the function $\beta(\cdot)$ on $(-\infty,\infty)$ with $\beta > -R_I^3$, there exists a pressure function $\mathscr{p}(\cdot,\cdot)$ on $[r_I, r_0] \times (-\infty,\infty)$ such that the dynamical equation (2.3) holds for the motion (6.1); according to (6.14), $\mathscr{p}(\cdot,\cdot)$ gives to σ_r the following dependence on r and t:

$$\sigma_r(r,t) = \sigma_r(r_I,t) - \int_{r_I}^{r} \frac{\sigma_r(\zeta,t) - \sigma_\theta(\zeta,t)}{\zeta}\, d\zeta + \ddot{\beta}(t)\frac{\rho}{3}\left[\frac{1}{r_I} - \frac{1}{r}\right] + \dot{\beta}(t)^2\frac{\rho}{18}\left[\frac{1}{r^4} - \frac{1}{r_I^4}\right]. \tag{6.15}$$

If we let P_I and P_0 be the pressures applied to the inner and outer spherical boundaries, i.e.

$$P_0 = -\sigma_r(r_0,t), \qquad P_I = -\sigma_r(R_I,t), \tag{6.16}$$

then from (6.8), (6.11), and (6.15) we obtain

$$\Delta P = -\mathscr{b}(\beta^t) + \Omega_1\ddot{\beta}(t) - \Omega_2\dot{\beta}(t)^2, \tag{6.17}$$

with

$$\left.\begin{aligned} \Omega_1 &= \Omega_1\big(\beta(t)\big) = \frac{\rho}{3}\Big([R_I^3+\beta(t)]^{-1/3} - [R_0^3+\beta(t)]^{-1/3}\Big) > 0, \\ \Omega_2 &= \Omega_2\big(\beta(t)\big) = \frac{\rho}{18}\Big([R_I^3+\beta(t)]^{-4/3} - [R_0^3+\beta(t)]^{-4/3}\Big) > 0, \\ \Delta P &= P_I - P_0. \end{aligned}\right\} \tag{6.18}$$

When ΔP is known as a function of t for $t \geq 0$ the equation (6.17) is of the same type as (5.16), and in the case in which ΔP is held constant for $t \geq 0$ a theory of the stability of equilibrium solutions of (6.17) is easily developed along the lines explored in Section 5.

Here we assume that $P_I - P_O$ is given by a prescribed function Q of β for $t \geq 0$: $P_I - P_O = Q(\beta)$.

For example, if the shell completely encloses the spherical region $\mathcal{S} = \{r \mid r \leq r_I\}$ (instead of just covering a spherical segment), and if the region $\mathcal{S}$ is filled with an ideal gas obeying Boyle's law, then in each deformed state

$$P_I = P_{\mathcal{R}}\left(\frac{R_I}{r_I}\right)^3 = \frac{P_{\mathcal{R}} R_I^3}{R_I^3 + \beta} . \tag{6.19}$$

Here $P_{\mathcal{R}}$ is a positive constant equal to the pressure on the inner surface when the shell is in its reference configuration. If such a closed shell is immersed in an atmosphere at constant pressure P_O, we have

$$Q(\beta) = \frac{P_{\mathcal{R}} R_I^3}{R_I^3 + \beta} - P_O. \tag{6.20}$$

Of course, situations in which $P_I - P_O$ is constant for $t \geq 0$ also fall as special cases of our present assumption that Q is a function of β for $t \geq 0$.

Writing υ for $\dot{\beta}$, we cast (6.17) into the form

$$\left.\begin{aligned} \dot{\beta}(t) &= \upsilon(t) \\ \dot{\upsilon}(t) &= \frac{\Omega_2}{\Omega_1}\,\upsilon(t)^2 + \frac{1}{\Omega_1}\,\mathscr{b}(\beta^t) + \frac{1}{\Omega_1}\,Q\big(\beta(t)\big) . \end{aligned}\right\} \tag{6.21}$$

We consider the following problem: First, let the function $Q(\cdot)$ be prescribed; then, given an element g of $\underline{\underline{V}}^{(1)}$ and a number $b > -R_I^3$ find a function pair $\beta(\cdot), \upsilon(\cdot)$, with β defined on $(-\infty,\infty)$ and υ defined on $[0,\infty)$, such that (6.21) holds for all $t \geq 0$ and $\beta^0 = g$, $\upsilon(0) = b$.

Let $\mathscr{s}^\circ$, $\mathscr{u}^\circ$, $\mathscr{a}^\circ$, and $\mathscr{b}^\circ$ be the equilibrium response functions corresponding to the functionals $\mathscr{s}$, $\mathscr{u}$, $\mathscr{a}$, and $\mathscr{b}$. By (6.10) and (6.11), for each number $\beta > -R_I^3$,

$$\left.\begin{aligned} \mathscr{a}^\circ(\beta) &= \rho\int_{R_I}^{R_O} R^2 \mathscr{u}^\circ\left([1+R^{-3}\beta]^{-2/3}\right) dR, \\ \mathscr{b}^\circ(\beta) &= 2\int_{R_I}^{R_O} \frac{R^2 \mathscr{s}^\circ\left([1+R^{-3}\beta]^{-2/3}\right)}{R^3+\beta}\, dR. \end{aligned}\right\} \tag{6.22}$$

If there is a number $\beta_e > -R_I^3$ such that

$$\mathscr{b}^\circ(\beta_e) + Q(\beta_e) = 0, \tag{6.23}$$

then (6.21) has the <u>equilibrium solution</u>

$$\beta(t) \equiv \beta_e, \qquad \upsilon(t) \equiv 0, \qquad -\infty < t < \infty. \tag{6.24}$$

Thus, (6.23) is the <u>equation of equilibrium for a spherical shell</u>.

We assume that Q is a continuously differentiable function of β in a neighborhood of β_e and that the functional $\mathscr{b}$ is Lipschitz continuous on a neighborhood of $\beta_e^\dagger$ in $\underline{\underline{V}}^{(1)}$. The equilibrium solution (6.24) of (6.21) is called <u>stable under the pressure relation</u> Q if for each $\epsilon > 0$

there is $\delta = \delta(\epsilon) > 0$ such that to each pair $\beta^0, \upsilon(0)$ obeying $\|\beta^0 - \beta_e^\dagger\| + |\upsilon(0)| < \delta$ there corresponds a unique solution $\beta(\cdot), \upsilon(\cdot)$ of (6.21) and this solution has $|\beta(t) - \beta_e| + |\upsilon(t)| < \epsilon$ for all $t \geq 0$. If, in addition, there is a $\zeta > 0$ such that for each solution of (6.21) with $\|\beta^0 - \beta_e^\dagger\| + |\upsilon(0)| < \zeta$ we have both $\dot{\beta}(t) \to 0$ and $\beta(t) \to \beta_e$ as $t \to \infty$, then we say that the equilibrium solution (6.24) is <u>asymptotically stable under the pressure relation</u> Q.

When (6.24) gives a stable solution of (6.21), the configuration

$$\theta = \Theta, \qquad \phi = \Phi, \qquad r = \sqrt[3]{R^3 + \beta_e} \tag{6.25}$$

is an equilibrium configuration of the shell that is stable against those perturbing motions of the form (6.1) in which $P_I - P_0$ equals $Q(\beta(t))$ for $t \geq 0$. If the solution (6.24) is asymptotically stable, then such perturbing motions have the property that $r(R,t) \to \sqrt[3]{R^3 + \beta_e}$ and $v \to 0$ as $t \to \infty$ provided that the initial velocity is small and the initial history β^t is not too far, in $\underline{\underline{V}}^{(1)}$, from $\beta_e^\dagger$.

Let A be the total solid angle subtended from the origin by the shell.# By (6.9) and (6.10), the Helmholtz free energy of the shell is

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#If the shell encloses a complete spherical region, $\mathcal{S} = \{r \mid r \leq r_I\}$, then $A = 4\pi$.

$$A\int_{r_I}^{r_0} \rho\psi r^2 dr = A\mathcal{A}(\beta^t). \tag{6.26}$$

By (6.13), for the total kinetic energy of the shell we have

$$A\int_{r_I}^{r_0} \frac{1}{2}\rho v^2 r^2 dr = \frac{A}{18}\rho\dot{\beta}^2\left[\frac{1}{r_I}-\frac{1}{r_0}\right] = \frac{A}{18}\rho\dot{\beta}(t)^2\Big([R_I^3+\beta(t)]^{-1/3} - [R_0^3+\beta(t)]^{-1/3}\Big). \quad (6.27)$$

The rate of working of the contact forces applied to the bounding surfaces of the shell is, by (6.13) and (6.16),

$$\begin{aligned} W(t) &= Ar_0^2\sigma_r(r_0,t)v(r_0,t) - Ar_I^2\sigma_r(r_I,t)v(r_I,t) \\ &= \frac{A}{3}\sigma_r(r_0,t)\dot{\beta} - \frac{A}{3}\sigma_r(r_I,t)\dot{\beta} \\ &= \frac{A}{3}\dot{\beta}(t)(P_I - P_0). \end{aligned} \quad (6.28)$$

Since we assume that $P_I - P_0 = Q\big(\beta(t)\big)$ for $t \geq 0$, if we put

$$Y(\beta) \overset{\text{def}}{=\!=} \int_0^{\beta} Q(\sigma)d\sigma, \quad (6.29)$$

then (6.28) yields

$$W(t) = \frac{A}{3}\frac{d}{dt}Y\big(\beta(t)\big) \qquad \text{for} \quad t \geq 0. \quad (6.30)$$

Thus, w(t) in (2.8) here obeys

$$A\int_{r_I}^{r_0} w(t)\rho r^2 dr = -\frac{A}{3}Y\big(\beta(t)\big). \quad (6.31)$$

It follows from (6.26), (6.27), and (6.31) that for the canonical free energy (2.13) of the shell,

$$\Phi = A\int_{r_I}^{r_0}\left[\psi + \frac{1}{2}v^2 + w\right]\rho r^2 dr, \quad (6.32)$$

we have the equation

$$\Phi(t) = \hat{\Phi}\big(\beta^t, \dot{\beta}(t)\big) = \hat{\Phi}\big(\beta^t, \upsilon(t)\big),$$

where the functional $\hat{\Phi}$ is defined by

$$\frac{1}{A}\,\hat{\Phi}(\beta^t,\upsilon) = \mathscr{a}(\beta^t) + \frac{\rho\upsilon^2}{18}\Big([R_I^3+\beta^t(0)]^{-1/3} - [R_0^3+\beta^t(0)]^{-1/3}\Big) - \frac{1}{3}\,Y\big(\beta^t(0)\big)\,, \qquad (6.33)$$

and is continuous when regarded as a function on a neighborhood $\underline{\underline{S}}$ of $\beta_e^{\dagger}, 0$ in $\underline{\underline{V}}^{(2)}$. Remark 2.1 here implies that $\hat{\Phi}\big(\beta^t, \upsilon(t)\big)$ _is a non-increasing function of_ t, _for_ $t \geq 0$, _on each solution_ of (6.21). Furthermore, since it follows from Remark 3.1 that

$$\mathscr{u}^{\circ}\big(\beta^t(0)\big) \leq \mathscr{u}(\beta^t) \quad \text{and} \quad \mathscr{a}^{\circ}\big(\beta^t(0)\big) \leq \mathscr{a}(\beta^t)$$

if we define the function $\hat{\Phi}^{\circ}$ by

$$\frac{1}{A}\,\hat{\Phi}^{\circ}(\beta,\upsilon) = \frac{1}{A}\,\hat{\Phi}(\beta^{\dagger},\upsilon) = \mathscr{a}^{\circ}(\beta) + \frac{\rho\upsilon^2}{18}\Big([R_I^3+\beta]^{-1/3} - [R_0^3+\beta]^{-1/3}\Big) - \frac{1}{3}\,Y(\beta), \qquad (6.34)$$

then for each pair β^t, υ in $\underline{\underline{S}}$ we have

$$\hat{\Phi}^{\circ}\big(\beta^t(0),\upsilon\big) \leq \hat{\Phi}(\beta^t,\upsilon). \qquad (6.35)$$

Therefore, we can assert

Remark 6.1. _The functional_ $\hat{\Phi}$ _defined in_ (6.33), _with_ $Y(\cdot)$ _given by_ (6.29), _is a free energy functional, in the sense of Coleman and Mizel_ [1968, 3, 4], _for the functional-differential equation_ (6.21).

The Gibbs function for the shell is defined by

$$\mathcal{G}(\beta) = \hat{\Phi}^\circ(\beta,0) = A\mathcal{a}^\circ(\beta) - \frac{1}{3}AY(\beta); \tag{6.36}$$

its value is just the equilibrium Helmholtz free energy of the shell plus the mechanical potential of the shell. It is clear from (6.34) that $\hat{\Phi}^\circ(\beta,\upsilon)$ has a strict local minimum at β°,υ in E^2 if and only if $\mathcal{G}(\beta)$ has a strict local minimum at β_e, and therefore Coleman and Mizel's Theorem 3.1 [1968, 4] here yields

Remark 6.2. If the function $\mathcal{G}$ of (6.36) has a strict local minimum at β_e, then (6.24) gives a stable solution of (6.21).

For an isotropic material the "equilibrium stress relation" of classical thermostatics yields the following formula connecting the equilibrium response functions $\mathfrak{f}^\circ$ and $\mathfrak{g}^\circ$ which correspond to the functionals $\mathfrak{f}$ and $\mathfrak{g}$ in Remark 4.1:

$$\mathfrak{f}^\circ(\alpha_i,\alpha_j) = \rho\alpha_i \frac{\partial}{\partial\alpha_i}\mathfrak{g}^\circ(\alpha_i,\alpha_j) - \rho\alpha_j \frac{\partial}{\partial\alpha_j}\mathfrak{g}^\circ(\alpha_i,\alpha_j), \quad i \neq j, \quad i,j = 1,2,3 \text{ (no summation)}. \tag{6.37}$$

In view of this general formula, the definitions (6.6), and the identities (4.6), (4.7), we have the following relation between the equilibrium response functions $\mathcal{s}^\circ$ and $\mathcal{u}^\circ$ corresponding to $\mathcal{s}$ and $\mathcal{u}$:

$$\mathcal{s}^\circ(\alpha) = \rho\alpha \frac{d}{d\alpha}\mathcal{u}^\circ(\alpha) \quad \text{for all } \alpha > 0. \tag{6.38}$$

Therefore, by (6.22),

$$\frac{d}{d\beta}\,\mathcal{a}^{\circ}(\beta) \;=\; -\frac{1}{3}\,\mathcal{b}^{\circ}(\beta) \quad \text{for all } \beta > -R_I^3. \tag{6.39}$$

If $\mathcal{Y}$ has a minimum at β_e, then $d\mathcal{Y}(\beta)/d\beta = 0$ at β_e, and, by (6.36) and (6.29),

$$\frac{d}{d\beta}\,\mathcal{a}^{\circ}(\beta)\Big|_{\beta_e} \;=\; \frac{1}{3}\,Q(\beta_e). \tag{6.40}$$

In view of (6.39), this equation is the same as (6.23). Hence, as expected, <u>whenever</u> $\mathcal{Y}$ <u>has a minimum at</u> β_e, (6.24) <u>is automatically a solution of</u> (6.21). <u>Remark</u> 6.2 <u>tells us that, in addition, for</u> (6.24) <u>to be a stable solution of</u> (6.21) <u>it suffices that</u>

$$\frac{d}{d\beta}\,\mathcal{Y}(\beta)\Big|_{\beta=\beta_e} = 0 \quad\text{and}\quad \frac{d^2}{d\beta^2}\,\mathcal{Y}(\beta)\Big|_{\beta=\beta_e} > 0. \tag{6.41}$$

<u>By</u> (6.36), (6.39), <u>and</u> (6.29), <u>the condition</u> $(6.41)_2$ <u>can be written in the following two equivalent forms</u>

$$3\,\frac{d^2}{d\beta^2}\,\mathcal{a}^{\circ}(\beta)\Big|_{\beta=\beta_e} > \frac{d}{d\beta}\,Q(\beta)\Big|_{\beta=\beta_e}, \tag{6.42}$$

$$\frac{d}{d\beta}\,\mathcal{b}^{\circ}(\beta)\Big|_{\beta=\beta_e} < -\frac{d}{d\beta}\,Q(\beta)\Big|_{\beta=\beta_e}. \tag{6.43}$$

In the special case in which P_I and P_0 are both held constant for $t \geq 0$, (6.43) reduces to the inequality

$$\frac{d}{d\beta}\,\mathcal{b}^{\circ}(\beta)\Big|_{\beta=\beta_e} < 0, \tag{6.44}$$

which is the analogue for a spherical shell of the inequality (5.39) which suffices for the dynamical stability of an equilibrium configuration of a cylindrical tube.

An argument completely analogous to that which led to Remark 5.3 here yields:

Remark 6.3. *Suppose the material comprising the shell is strictly dissipative in motions of general isochoric extension. If* β_e *is an isolated solution of the equation* $\overset{o}{\mathscr{b}}(\beta_e) + Q(\beta_e) = 0$, *and if the function* $\mathscr{Y}$ *in* (6.36) *has a strict local minimum at* β_e, *then* (6.24) *gives an asymptotically stable solution of* (6.21).

Furthermore, in analogy to Remark 5.4 we have

Remark 6.4. *If the material is strictly dissipative in motions of general isochoric extension and if* (6.40) *and* (6.42) *hold, then* (6.24) *gives an asymptotically stable solution of* (6.21).

It is easy to verify that in Remarks 6.3 and 6.4 the assumption that the material is strictly dissipative in motions of general isochoric extension can be replaced by the slightly weaker assertion that the inequality

$$\rho \frac{d}{dt} \mathscr{U}(\alpha^t) - \mathscr{s}(\alpha^t) \frac{d}{dt} \ln \alpha^t(0) < 0 \qquad (6.45)$$

holds whenever the indicated derivatives exist and the function α^t is not constant on $[0,\infty)$.

For an incompressible fluid the equilibrium Helmholtz free energy is a function of θ alone, and the equilibrium stress is a hydrostatic pressure. Therefore, for such a material we have

$$\mu^\circ(\alpha) = \text{const.}, \qquad a^\circ(\beta) = \text{const.}, \qquad s^\circ(\beta) = b^\circ(\beta) = 0, \qquad (6.46)$$

and Remarks 6.2 and 6.3 yield

Remark 6.5. Suppose the material comprising the shell is a fluid. If $Y(\beta)$, defined in (6.29), has a strict local maximum at β_e, or, equivalently, if $Q(\beta_e) = 0$ and $Q(\beta)$ is a strictly decreasing function of β in a neighborhood of β_e, then (6.24) gives a stable solution of (6.21). If, in addition, the material is strictly dissipative in motions of general isochoric extension, then the solution (6.24) is asymptotically stable.

Suppose we have a large mass of an incompressible viscoelastic fluid containing a bubble of radius $r_I = r(R_I,t) = [R_I^3 + \beta(t)]^{1/3}$ which is filled with an ideal gas obeying Boyle's law. If conditions are isothermal, if surface tension can be neglected, and if the viscoelastic fluid is subject to the boundary condition that the stress at $r = \infty$ is a constant, positive, hydrostatic pressure P_0, then the evolution of the radius of the bubble as a function of time is governed by the present theory with Q given by (6.20) and $R_0 = \infty$. Since (6.46) holds here, the equation (6.23) reduces to $Q(\beta_e) = 0$; i.e.

$$\beta_e = R_I^3\left(\frac{P_{\mathcal{R}}}{P_0} - 1\right), \tag{6.47}$$

and since $P_{\mathcal{R}}$ and R_I are positive constants, the corresponding equilibrium solution of (6.21) is, by Remark 6.5, stable; if the fluid is strictly dissipative in the sense of (6.45), then this solution is asymptotically stable. Of course, (6.47) states that the equilibrium radius of the bubble is

$$r_e = R_I\sqrt[3]{P_{\mathcal{R}}/P_0}\ . \tag{6.48}$$

What is new here is our conclusion that equilibrium configuration is stable against perturbations of the form (6.1). If, more generally, the gas entrapped in the bubble obeys an equation of state of the form $P_I = f(V)$ with V the volume of the bubble, then the equilibrium radius of the bubble is the root of the equation

$$P_0 = f\left(\frac{4}{3}\pi r_e^3\right), \tag{6.49}$$

and the equilibrium configuration is stable, in the sense we are considering, whenever f is a strictly decreasing function of V in a neighborhood of $V_e = \frac{4}{3}\pi r_e^3$. If, in addition, (6.45) holds, then there exists a $\zeta > 0$ such that, in every perturbing motion of the form (6.1) which preserves the pressure P_0 at infinity and obeys (5.23), both $r_I \to r_e$ and $\dot{r}_I \to 0$ as $t \to \infty$.

We may consider now a spherical shell comprised of an isotropic solid material for which the equilibrium response function $\underline{\underline{p}}^{\circ}$ has the form (5.41) associated with a Mooney material. In this case

$$\underline{\underline{p}}^{\circ}(\underset{\sim}{F}) = b_0 + b_1[\alpha_r^2+\alpha_\theta^2+\alpha_\phi^2-3] + b_2[\alpha_r^2\alpha_\theta^2+\alpha_\theta^2\alpha_\phi^2 + \alpha_\phi^2\alpha_r^2-3],$$

and since we here have $\alpha_\theta = \alpha_\phi = \alpha_r^{-1/2}$,

$$\underline{\underline{p}}^{\circ}(\underset{\sim}{F}) = b_0 + b_1(\alpha_r^2+2\alpha_r^{-1}-3) + b_2(2\alpha_r+\alpha_r^{-2}) = \mu^{\circ}(\alpha_r).$$

Hence (6.38) yields

$$s^{\circ}(\alpha_r) = 2\rho b_1(\alpha_r^2-\alpha_r^{-1}) + 2\rho b_2(\alpha_r-\alpha_r^{-2}), \qquad (6.50)$$

and $(6.22)_2$ becomes

$$\ell^{c}(\beta) = 4\rho\int_{R_I}^{R_0} \frac{R^2\{b_1(\lambda^2-\lambda^{-1}) + b_2(\lambda-\lambda^{-2})\}}{R^3+\beta}\, dR \qquad (6.51)$$

with

$$\lambda = [1+R^{-3}\beta]^{-2/3} = \alpha_r . \qquad (6.52)$$

Differentiating (6.52) we find that

$$\frac{R^2 dR}{R^3+\beta} = \frac{d\lambda}{2(\lambda^{-1/2}-\lambda)},$$

and therefore (6.51) can be written

$$\frac{1}{\rho}\,\mathscr{b}^{c}(\beta) = 2\int_{\lambda_I}^{\lambda_O} \frac{b_1(\lambda^2-\lambda^{-1}) + b_2(\lambda-\lambda^{-2})}{\lambda^{-1/2}-\lambda}\,d\lambda$$

$$= -b_1\left[\lambda_O^2 - \lambda_I^2 + 4(\lambda_O^{1/2}-\lambda_I^{1/2})\right] - 2b_2\left[\lambda_O - \lambda_I - 2(\lambda_O^{-1/2} - \lambda_I^{-1/2})\right], \quad (6.53)$$

with

$$\lambda_O = [1+R_O^{-3}\beta]^{-2/3}, \qquad \lambda_I = [1+R_I^{-3}\beta]^{-2/3}. \quad (6.54)$$

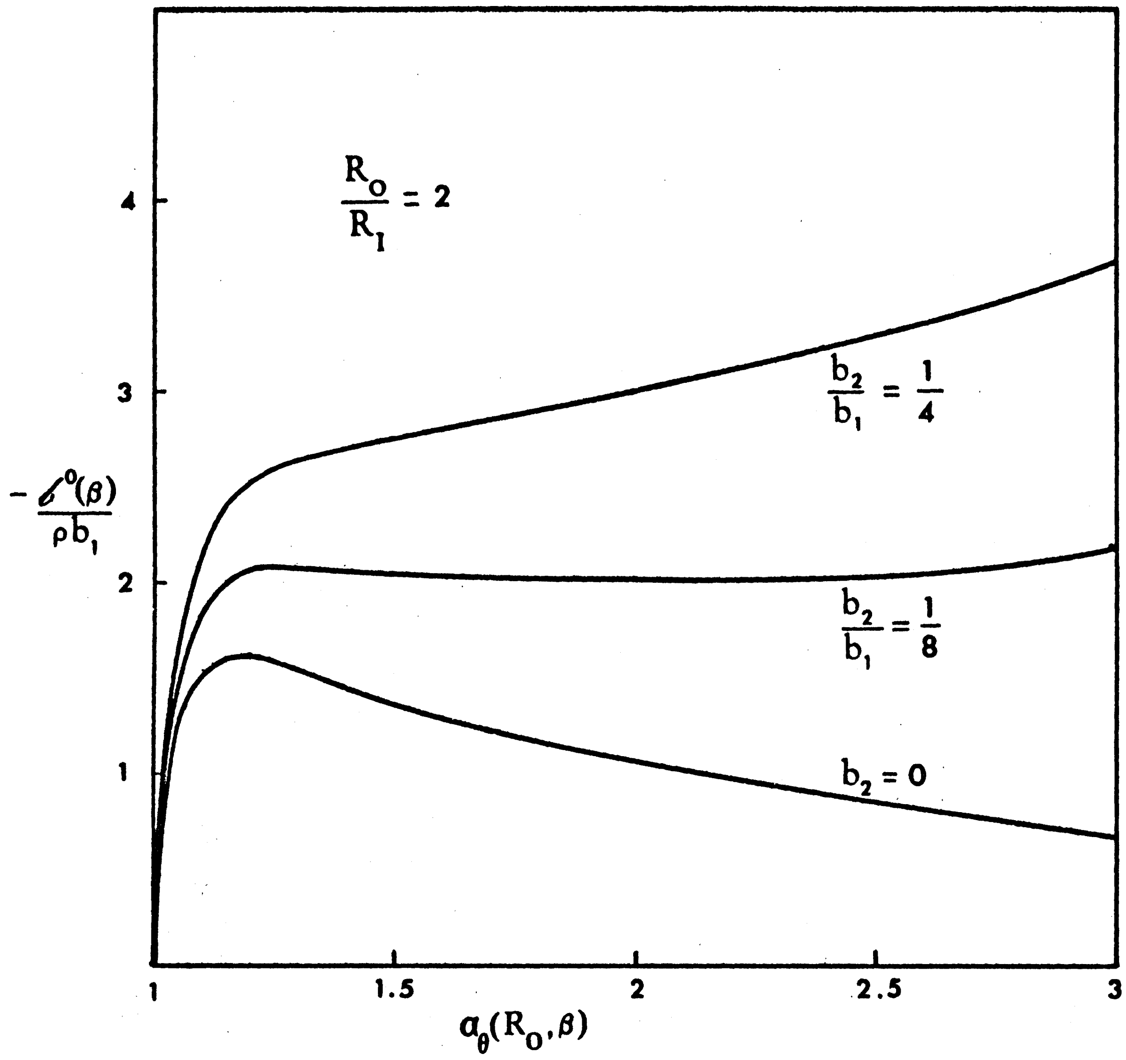
Thus, <u>for a spherical shell comprised of a material with an equilibrium response of the Mooney type, the equation of equilibrium</u> (6.23) <u>has the form</u>

$$\frac{1}{\rho}Q(\beta_e) = b_1 f(R_O,\beta_e)^4 + 4b_1 f(R_O,\beta_e) + 2b_2 f(R_O,\beta_e)^2 - 4b_2 f(R_O,\beta_e)^{-1}$$
$$- b_1 f(R_I,\beta_e)^4 - 4b_1 f(R_I,\beta_e) - 2b_2 f(R_I,\beta_e)^2 + 4b_2 f(R_I,\beta_e)^{-1}, \quad (6.55)$$

where

$$f(R,\beta)^3 = \frac{R^3}{R^3+\beta} = \alpha_\theta(R,\beta)^{-3}.$$

The relation (6.55) and the limiting case of a thin shell have been given by Green and Zerna [1954, 1, §3.10]. The accompanying graph shows the function $\mathscr{b}^{o}$ for the case $R_O/R_I = 2$ and several values of b_2/b_1. A similar graph for the special case of thin shells is given by Green and Adkins [1960, 2, §4.13].



The function $\mathscr{b}^{\circ}$ for materials with equilibrium response of the Mooney type, assuming $R_O = 2R_I$.

By (6.53), the condition (6.43) for dynamical stability becomes

$$\left.\frac{dQ(\beta)}{d\beta}\right|_{\beta=\beta_e} < \frac{4\rho}{3R_I^3}\Big[b_1 f(R_I,\beta_e)^7 + b_1 f(R_I,\beta_e)^4 + b_2 f(R_I,\beta_e)^5 + b_2 f(R_I,\beta_e)^2\Big]$$

$$- \frac{4\rho}{3R_0^3}\Big[b_1 f(R_0,\beta_e)^7 + b_1 f(R_0,\beta_e)^4 + b_2 f(R_0,\beta_e)^5 + b_2 f(R_0,\beta_e)^2\Big]. \qquad (6.56)$$

We assume, of course, $b_1 > 0$ and $b_2 \geq 0$. Consider the case for which $\Delta P = P_I - P_0$ is held constant. Then for $b_2 > 0$, (6.55) has at least one solution β_e, in the range $-R_I^3 < \beta_e < \infty$, for each value of $Q = \Delta P$. For $b_2 = 0$, there is a critical value of ΔP above which (6.55) has no solution. As the accompanying graph shows, some solutions β_e of (6.55) do not obey the condition (6.56).

For a spherical cavity in an infinite medium, $R_0 = \infty$, and we have[#]

[#] See Gent & Lingley [1959, 1] for the case $b_2 = 0$, for related calculations with other free energy functions, and for an interesting application.

$$\mathscr{W}^{\circ}(\beta) = \rho b_1(\lambda_I^2 + 4\lambda_I^{1/2} - 5) + 2\rho b_2(\lambda_I - 2\lambda_I^{-1/2} + 1) \qquad (6.57)$$

and

$$\frac{d\mathscr{W}^{\circ}(\beta)}{d\beta} = -\frac{4\rho b_1}{3R_I^3}(\lambda_I^{7/2} + \lambda_I^2) - \frac{4\rho b_2}{3R_I^3}(\lambda_I^{5/2} + \lambda_I). \qquad (6.58)$$

Thus

$$\frac{d\mathscr{W}^{\circ}(\beta)}{d\beta} < 0. \qquad (6.59)$$

Let us assume that the pressure P_0 at infinity is held constant. Then, if the pressure P_I in the cavity is constant, the equation of equilibrium (6.23) has at most one solution for each value of $\Delta P = P_I - P_0$. If

$b_2 = 0$ there is a critical value of ΔP above which the equation of equilibrium (6.23) has no solution; if $b_2 > 0$, then the equation of equilibrium has exactly one solution for each value of ΔP. In each case it follows from (6.44) and (6.59) that the equilibrium solution for the constant pressure difference ΔP is stable. If $P_I = f(V)$ with V the volume of the cavity, and if f is a decreasing function in the neighborhood of an equilibrium value of V, then, by (6.43), the corresponding equilibrium solution is again stable. In particular, equilibrium solutions are stable when P_I is given by (6.19), i.e. when the cavity is filled with an ideal gas.

Appendix: On History Spaces

Let $\underline{\underline{V}}_r^{(1)}$ be a Banach function space formed from functions mapping $(0,\infty)$ into the real axis E^1, and suppose that $\underline{\underline{V}}^{(1)}$ has the following properties:

(1) The norm $\|\cdot\|_r$ on $\underline{\underline{V}}_r^{(1)}$ is compatible with the usual partial ordering of functions on $(0,\infty)$; that is, if ζ is in $\underline{\underline{V}}_r^{(1)}$ and if ξ is a measurable function mapping $(0,\infty)$ into E^1 with $|\xi(s)| \leq |\zeta(s)|$ a.e., then ξ is in $\underline{\underline{V}}_r^{(1)}$ and $\|\xi\|_r \leq \|\zeta\|_r$. Furthermore, if $\|\cdot\|_r$ is not identically zero, then $\|\zeta\|_r = 0$ only if $|\zeta(s)| = 0$ a.e.

(2) $\underline{\underline{V}}_r^{(1)}$ has the following Fatou property: If $\zeta_1,\ldots,\zeta_n,\ldots$ are in $\underline{\underline{V}}_r^{(1)}$, if $\|\zeta_n\|_r \leq K < \infty$, and if $|\zeta_n(s)| \uparrow |\xi(s)|$ a.e., with ξ measurable, then ξ is in $\underline{\underline{V}}_r$ and

$$\lim_{n\to\infty} \|\zeta_n\|_r = \|\xi\|_r \leq K.$$

(3) $\underline{\underline{V}}_r^{(1)}$ contains all right and left translates of its elements; that is, if ζ is in $\underline{\underline{V}}_r^{(1)}$ then $T^{(\sigma)}\zeta$ and $T_{(\sigma)}\zeta$ defined by

$$T^{(\sigma)}\zeta(s) = \begin{cases} 0 & \text{for } s \in (0,\sigma], \\ \zeta(s-\sigma) & \text{for } s \in (\sigma,\infty), \end{cases}$$

and

$$T_{(\sigma)}\zeta(s) = \zeta(s+\sigma) \quad \text{for } s \in (0,\infty),$$

are in $\underline{\underline{V}}_r^{(1)}$ for each $\sigma \geq 0$.

(4) $\underline{\underline{V}}^{(1)}$ contains the constant function $1_r^\dagger$ defined by $1_r^\dagger(s) \equiv 1$ for s in $(0,\infty)$.

(5) $\underline{\underline{V}}_r^{(1)}$ has the relaxation property[#]; that is, for each ζ in $\underline{\underline{V}}_r^{(1)}$

[#]Cf. [1966, 1, §6] [1968, 1, §4,5].

$$\lim_{\sigma\to\infty} \|T^{(\sigma)}\zeta\|_r = 0.$$

(6) $\underline{\underline{V}}_r^{(1)}$ is a <u>separable</u> Banach space.

A Banach function space with the properties (1)-(6) is called a <u>past history space</u>.[##]

[##]See Coleman & Mizel [1967, 3] [1968, 2].

Given any past history space $\underline{\underline{V}}_r^{(1)}$, we may consider the set $\underline{\underline{V}}^{(1)}$ of measurable functions g which map $[0,\infty)$ into E^1 and satisfy $\|g_r\|_r < \infty$, where g_r, called the <u>past history of</u> g, is the restriction of g to $(0,\infty)$. The function $\|\cdot\|$ given by

$$\|g\| = |g(0)| + \|g_r\|_r \qquad (A.1)$$

is a well defined semi-norm on $\underline{\underline{V}}^{(1)}$. If we identify, in the usual way, functions g, f in $\underline{\underline{V}}^{(1)}$ obeying $\|g-f\| = 0$, then $\underline{\underline{V}}^{(1)}$ becomes a Banach space with $\|\cdot\|$ its norm. A Banach space so constructed is called a

<u>history space</u>.# The elements of $\underline{\underline{V}}^{(1)}$ are called <u>histories</u>; their independent

#Cf. [1967, 2, §3] [1968, 2, §3]. See also [1968, 4, §2] where E^1 is replaced by E^n and instead of $\underline{\underline{V}}^{(1)}$ the symbol $\underline{\underline{V}}$ is used. In [1968, 2] Coleman & Mizel give a motivation for this method of defining history spaces; in that essay the values of the functions in a history space are taken to be vectors in an arbitrary separable Banach space.

variable s is called the <u>elapsed time</u>. It follows from (A.1) that, even after identification, each history g has a well defined value g(0) at s = 0; g(0) is called the <u>present value</u> of g. It is clear that a continuous functional over $\underline{\underline{V}}^{(1)}$ must have a "special dependence" on the present values of the histories in $\underline{\underline{V}}^{(1)}$.

The form of the principal of fading memory# which we use in

#I.e. that proposed in [1967, 3].

this essay implies that the functionals $\mathfrak{m}$ and $\mathfrak{n}$ of (5.5) and $\mathfrak{s}$ of (6.8) are continuous functions on a history space $\underline{\underline{V}}^{(1)}$ while the functionals $\mathfrak{p}$ of (5.8) and μ of (6.9) are not only continuous, but have continuous Fréchet derivatives on $\underline{\underline{V}}^{(1)}$.

This research was supported in part by the Air Force Office of Scientific Research under Grant AFOSR 728-66 and by the National Aeronautics and Space Administration under Research Grant NsG-401.

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