ON THE SYMMETRY OF THE CONDUCTIVITY TENSOR AND OTHER RESTRICTIONS IN THE NONLINEAR THEORY OF HEAT CONDUCTION

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Report 68-21

July, 1968

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Introduction.

In continuum thermodynamics a rigid heat conductor is defined by constitutive relations giving the internal energy, entropy, and heat flux as functions of the coldness† and coldness gradient. COLEMAN and MIZEL [1963] have shown that the second law of thermodynamics implies certain restrictions on these constitutive relations. However there still remain two important restrictions which physicists believe to be true but which are not provable consequences of the second law. These are the symmetry of the conductivity tensor and the positivity of the heat capacity.

We here introduce a new notion, based on the requirement that a certain functional have a weak relative minimum at equilibrium, which yields as consequences the above restrictions. We conjecture that this requirement is intimately connected with the notion of stability.

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†The coldness is the reciprocal of the absolute temperature.

An interesting feature of our study is that we nowhere introduce the concept of entropy.

JL. Admissibility.

Throughout this paper the inner product space associated with euclidean point space is denoted by the symbol V .

We consider a rigid homogeneous heat conductor which occupies a compact regular' region G in euclidean space. A field on the body is any function defined at all pairs (x,t) e G x [0,oo) > where we interpret t as the 'time'. As examples of fields we have the (real-valued) internal energy e per unit volume and the (vector-valued) heat flux q . If these fields are of class C^1 , they determine the heat supply r through the energy balance equation

$$\dot{e} = -\mathrm{div}\, q + r \, , \qquad (1)$$

where • denotes d/dt . The coldness field 9 , whose values are strictly positive, is assumed to be of class C and we write

$$g = \mathrm{grad}\ 6 \qquad (2)$$

for the coldness gradient.

For the materials considered here the coldness field determines the internal energy and the heat flux in the following way: there are class C^1 constitutive functions e: (0,oo) X V -• (-00,00) and q: (0,oo) X \s -• \s such that

$$e(x,t) = \bar{e}(6(x,t),g(x,t)) \, ,$$

$$q(x,t) = \bar{q}(9(x,t),g(x,t)) \, . \qquad (3)$$

'We use the term regular in the sense of KELLOGG [1929].

Equation (1) then defines the corresponding heat supply r(x,t) . For brevity we refer to the ordered pair $(\bar{e}, \bar{q})$ as the <u>material</u>* It should be noted that in terms of the coldness the <u>heat capacity</u> is given by

$$c(e) = -e^2 \bar{e}_{fl}(e,o) \ , \qquad (4)$$

while the <u>conductivity tensor</u> equals

$$K(9) = 9^2 \bar{q} \ (9, 0) \ . \qquad (5)$$

Here and in the sequel the subscripts 0 and g denote partial differentiation with respect to these variables•

Let $6_o > 0$ be any given coldness. Our aim is to determine the restrictions imposed on the material by an assumption about the behaviour of a certain functional on coldness fields [1]close[1] to the constant field 8_o . To formulate the assumption we introduce the collection $0(9_o)$ of class C functions ꝏ: G X [0,oo) - (-00,00) with the properties (i) 9_Q + co > 0 , (ii) co(*,0) = 0 , (iii) there is a number T $\geq$ 0 , depending on ꝏ , such that co(#,t) = co(*,T) for all t $\geq$ T . Clearly the constant function 0 *e* &(9_o) , and the set £2(9_o) has the property that if co *e* 0(9_o) then there is a positive number 6 , depending on co , such that Aꝏ *e* *Cl*(d_o) for any A in -6 < X $\leq$ 1 . We define a real-valued functional F on $\Omega(\theta_o)$ by setting

$$F(\omega) = \int_0^{\infty} \int_{SG} 9q\text{-}ndAdt - \int_0^{\infty} \int_G 9rdVdt \qquad (6)$$

where

$$0 = 6_0 + ꝏ \ , \qquad (7)$$

$\underset{\sim}{q}$ and r are given by (1) - (3), and $\underset{\sim}{n}$ is the unit outward normal to ∂G. By the first and second variations of F we mean the real-valued functionals δF and $\delta^2 F$ defined on $\Omega(\theta_o)$ by

$$\begin{aligned} \delta F(\omega) &= \frac{d}{d\lambda}F(\lambda\omega)\Big|_{\lambda=0} \\ \delta^2 F(\omega) &= \frac{1}{2}\,\frac{d^2}{d\lambda^2}F(\lambda\omega)\Big|_{\lambda=0} \end{aligned} \tag{8}$$

The concept of admissibility of the material is phrased in terms of the functional F : we say that the <u>material is admissible at the coldness</u> θ_o if and only if

$$\begin{aligned} \delta F(\omega) &= 0\ , \\ \delta^2 F(\omega) &\geq 0\ , \end{aligned} \tag{9}$$

for every $\omega \in \Omega(\theta_o)$. The following theorem characterizes admissibility.

Theorem. *The material is admissible at the coldness* θ_o *if and only if the results* (I), (II), *and* (III) *hold:*

(I) *the conductivity tensor* $K(\theta_o)$ *is symmetric and positive semi-definite;*

(II) *the heat capacity* $c(\theta_o) \geq 0$;

(III)† $\overline{\underset{\approx}{q}}(\theta_o,\underset{\sim}{0}) = \overline{\underset{\approx}{q}}_\theta(\theta_o,\underset{\sim}{0}) = \overline{e}_{\underset{\sim}{g}}(\theta_o,\underset{\approx}{0}) = 0$.

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†The restriction (III) and the positive semi-definiteness of $K(\theta_o)$ are consequences of the second law as has been shown by COLEMAN AND MIZEL [1963].

6F<40) « q(0^,O)-f ∞ f grad co dVdt (11)
~ ° J0 JG

and

6^2F(co) = f ∞ f {-$\bar{e}_Q$(d ,O)co^2 - e (0 ,0)-co grad cb
ψ\ ψn U O f ~go**

(12)

+ $\bar{g}_n$(0^^O)'co grad co + grad co-q (0 ,,0)grad ω} dVdt
z^a o f* f5g o *'**
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We establish the necessity of the conditions (I) - (III) by making special choices of functions co *e* 0(0 .) . If veU is any vector the compactness of G enables us to choose a point x , depending on v ,, such that v-(x - x) > 0 for each xeG . Now define co € O(0) by co(x^t) = f(t)v(x - x) where f is any C^2 function with f > 0 , f(0) = 0 and f(t) = 1 for every t > 1 . Then (11) and the assumed admissibility of the material imply

0 = 6F(co) = volume(G)vq(6 .0) .

But VGV is arbitrary and so q(0^.,O) = 0 which proves the first part of (III) .

Next consider elements co e *Q(d*$_o$) of the form

co(x, t) = f(t)exp(v(x - x)) ,

where x is any fixed point, v is any vector in V , f(t) > 0 and f(0) = 0 . For an element of this type grad co = *ox* and (12) becomes

$$\delta^2 F(\omega) = -\left(\int_0^\infty \dot{f}^2(t)\,dt\right)\left(\int_G \exp(2\underset{\sim}{v}\cdot(\underset{\sim}{x}-\underset{\sim}{x}_o))\,dV\right)(\bar{e}_\theta(\theta_o,\underset{\sim}{0}) + \bar{e}_{\underset{\sim}{g}}(\theta_o,\underset{\sim}{0})\cdot\underset{\sim}{v})$$
$$+ \left(\int_0^\infty f(t)\dot{f}(t)\,dt\right)\left(\int_G \exp(2\underset{\sim}{v}\cdot(\underset{\sim}{x}-\underset{\sim}{x}_o))\,dV\right)(\bar{\underset{\sim}{q}}_\theta(\theta_o,\underset{\sim}{0})\cdot\underset{\sim}{v} + \underset{\sim}{v}\cdot\bar{\underset{\sim}{q}}_{\underset{\sim}{g}}(\theta_o,\underset{\sim}{0})\underset{\sim}{v}).$$

Since

$$\int_G \exp(2\underset{\sim}{v}\cdot(\underset{\sim}{x}-\underset{\sim}{x}_o))\,dV > 0 ,$$

admissibility requires that the inequality

$$\begin{aligned}&-\left(\int_0^\infty \dot{f}(t)^2 dt\right)(\bar{e}_\theta(\theta_o,\underset{\sim}{0}) + \bar{e}_{\underset{\sim}{g}}(\theta_o,\underset{\sim}{0})\cdot\underset{\sim}{v})\\ &+\left(\int_0^\infty f(t)\dot{f}(t)\,dt\right)(\bar{\underset{\sim}{q}}_\theta(\theta_o,\underset{\sim}{0})\cdot\underset{\sim}{v} + \underset{\sim}{v}\cdot\bar{\underset{\sim}{q}}_{\underset{\sim}{g}}(\theta_o,\underset{\sim}{0})\underset{\sim}{v}) \geq 0\end{aligned} \tag{13}$$

hold. Choosing f to be any non-negative C^2 function with $f(0) = 0$ and $f(t) = 0$ for $t \geq 1$ but which is not identically zero we find that

$$\bar{e}_\theta(\theta_o,\underset{\sim}{0}) + \bar{e}_{\underset{\sim}{g}}(\theta_o,\underset{\sim}{0})\cdot\underset{\sim}{v} \leq 0$$

for every vector $\underset{\sim}{v}$. Thus $\bar{e}_{\underset{\sim}{g}}(\theta_o,\underset{\sim}{0}) = 0$ and $c(\theta_o) = -\theta_o^2\bar{e}_\theta(\theta_o,\underset{\sim}{0}) \geq 0$.

Again, if we choose f with $f(t) = \tau^3 - (t-\tau)^3$ for $0 \leq t \leq \tau$ and $f(t) = \tau^3$ for $t > \tau$ then f is of class C^2 and the inequality (13) becomes

$$-\tfrac{9}{5}\tau^5\bar{e}_\theta(\theta_o,\underset{\sim}{0}) + \tfrac{1}{2}\tau^6(\bar{\underset{\sim}{q}}_\theta(\theta_o,\underset{\sim}{0})\cdot\underset{\sim}{v} + \underset{\sim}{v}\cdot\bar{\underset{\sim}{q}}_{\underset{\sim}{g}}(\theta_o,\underset{\sim}{0})\underset{\sim}{v}) \geq 0$$

holding for all numbers $\tau > 0$. Dividing throughout by $\frac{1}{2}\tau^6$ and letting $\tau \to \infty$ gives the inequality

$$\bar{\underset{\approx}{q}}_\theta(\theta_o, \underset{\sim}{0}) \cdot \underset{\sim}{v} + \underset{\sim}{v} \cdot \bar{\underset{\approx}{q}}_{\underset{\approx}{g}}(\theta_o, \underset{\sim}{0}) \underset{\sim}{v} \geq 0 ,$$

which can hold only if the conductivity tensor $K(\theta_o) = \theta_o^2 \bar{\underset{\approx}{q}}_{\underset{\approx}{g}}(\theta_o, \underset{\sim}{0})$ is positive semi-definite and $\bar{\underset{\approx}{q}}_\theta(\theta_o, \underset{\sim}{0}) = \underset{\sim}{0}$.

It remains to be shown that $K(\theta_o)$ is symmetric. To do this let $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$ be any vectors, $\underset{\sim}{x}_o$ any fixed point, and let f, h be real-valued C^2 functions on $[0, \infty)$ with $f(0) = h(0) = 0$, $f(t) = h(t) = 0$ for $t \geq 1$ and $\int_0^\infty f(t)\dot{h}(t)dt = 1$. Consider the sequence of functions

$$\omega_n(\underset{\sim}{x}, t) = \epsilon(f(\tfrac{t}{n})\underset{\sim}{u} + h(\tfrac{t}{n})\underset{\sim}{v}) \cdot (\underset{\sim}{x} - \underset{\sim}{x}_o) .$$

Taking ϵ sufficiently small guarantees that $\omega_n \in \Omega(\theta_o)$ and then a straightforward computation using (12) and the results already proved yields

$$\delta^2 F(\omega_n) \to \frac{\epsilon^2}{\theta_o^2}(\underset{\sim}{v} \cdot K(\theta_o)\underset{\sim}{u} - \underset{\sim}{u} \cdot K(\theta_o)\underset{\sim}{v}) , \text{ as } n \to \infty .$$

Thus the admissibility of the material implies that for all vectors $\underset{\sim}{u}, \underset{\sim}{v}$

$$\underset{\sim}{v} \cdot K(\theta_o)\underset{\sim}{u} \geq \underset{\sim}{u} \cdot K(\theta_o)\underset{\sim}{v} .$$

On interchanging $\underset{\sim}{u}$ and $\underset{\sim}{v}$ we find that equality must hold; i.e. the conductivity tensor $K(\theta_o)$ is symmetric and the necessity of the conditions (I) - (III) is established.

The sufficiency of these conditions follows immediately on noting that if they do hold then, for any $\omega \in \mathcal{X}(\theta_o)$,

$$\operatorname{grad}\omega \cdot K(\theta_o)\operatorname{grad}\omega = \tfrac{1}{2}\overline{[\operatorname{grad}\omega \cdot K(\theta_o)\operatorname{grad}\omega]}$$

and so

$$\delta^2 F(\omega) = \int_0^\infty \int_B \frac{\partial^2 \bar{\varepsilon}}{\partial\theta^2}(\theta_o, 0)\,\omega^2\, dV dt + \frac{1}{2}\int_B \frac{1}{\theta_o}\operatorname{grad}\omega(x,\infty)\cdot K(\theta_o)\operatorname{grad}\omega(x,\infty)\,dV \geq 0 ,$$

which completes the proof.

The result (III) of the theorem show that whenever the material is admissible at coldness θ_o

$$\bar{\varepsilon}(\theta, g) = \varepsilon(\theta_o, 0) - \frac{1}{\theta_o^2}\, c(\theta_o)(\theta-\theta_o) + o(|\theta-\theta_o| + |g|) ,$$

(14)

$$\bar{q}(\theta, g) = \frac{1}{\theta_o^2} K(\theta_o) g + o(|\theta-\theta_o| + |g|) .$$

If we consider coldness fields close to the constant coldness θ_o in the sense that $|\theta-\theta_o| + |g|$ is small then equations (14) tell us that effects due to the coldness and the coldness gradient uncouple; i.e. to within terms of order $o(|\theta-\theta_o| + |g|)$ the internal energy ε depends only on the coldness increment $\theta-\theta_o$ and not its gradient g, whereas the heat flux q depends only on g and not on $\theta-\theta_o$. In fact the second of (14) is, to within higher order terms, Fourier's law of heat conduction. Another important consequence of the first of (III) is that heat can flow at coldness 6_o only in the presence of a non-zero coldness gradient.

§2. The Linear Material.

The linearized theory corresponding to the theory discussed in section 1 results on considering the material with

$$\bar{\sigma}(\theta, g) = \sigma_0 - \frac{c}{2\theta_0^2}(\theta - \theta_0)^2 ,$$

$$\bar{g}(\theta, g) = \frac{1}{\theta_0^2} K g , \qquad (15)$$

where θ_0, c are constant scalars and K is a constant tensor. In this case it follows from (11) and (12) that $\delta F = 0$ and $F = \delta^2 F$ and we deduce that the material defined in (15) is admissible at some, and hence every, coldness θ_0 if and only if $F \geq 0$ on $\Omega(\theta_0)$. One way of stating this conclusion is

Theorem 2. For the linear material the heat capacity c is <u>non-negative and the conductivity tensor</u> K <u>is symmetric and positive semi-definite if and only if</u> F <u>has a minimum at</u> $0 \in \Omega(\theta_0)$.

The admissibility of a linear material can be characterized in another way if we introduce the concept of a <u>conduction potential</u>, by which we mean any class C^1 function $\varphi: V \to (-\infty, 0]$ with $\varphi(0) = 0$.

Theorem 3. A linear material is admissible if and only if there exists a conduction potential ϕ such that, for every $\omega \in \Omega(\theta_o)$,

$$\int_G \phi(\underset{\approx}{g})\,dV \geq -\int_{\partial G} \dot{\theta}\underset{\approx}{q}\cdot\underset{\approx}{n}\,dA + \int_G \dot{\theta} r\,dV , \qquad (16)$$

where $\theta = \theta_o + \omega$ and $\underset{\approx}{q}$ and r are given by (1) - (3) .

Proof. If ϕ is a conduction potential then we conclude from (6) that

$$0 \leq -\int_G \phi(\underset{\approx}{g})\,dV \leq F(\omega)$$

and hence the material is admissible. Conversely suppose the material is admissible. Then

$$-\int_{\partial G} \dot{\theta}\underset{\approx}{q}\cdot\underset{\approx}{n}\,dA + \int_G \dot{\theta} r\,dV = -\frac{1}{\theta_o^2}\int_G (c\dot{\theta}^2 + \dot{\underset{\approx}{g}}\cdot K\underset{\approx}{g})\,dV . \qquad (17)$$

If we set

$$\phi(\underset{\approx}{g}) = -\tfrac{1}{2}\underset{\approx}{g}\cdot K\underset{\approx}{g} ,$$

the function ϕ is a conduction potential since K is positive semi-definite. In addition the symmetry of K tells us that

$$\dot{\overline{\phi(\underset{\approx}{g})}} = -\dot{\underset{\approx}{g}}\cdot K\underset{\approx}{g}$$

and this remark, when combined with (17) and the inequality $c \geq 0$, implies (16).

It should be remarked that the conduction potential ϕ

of Theorem 3 is unique. In fact, for a rigid heat conductor the production of entropy per unit volume is

$$\gamma = \underset{\sim}{g} \cdot K \underset{\sim}{g}$$

and so the conduction potential is

$$\phi = -\frac{1}{2}\gamma \ .$$

Furthermore, in view of the symmetry of K, the constitutive function for the heat flux can be written

$$\bar{S} = -\phi_{\underset{\sim}{g}} \ .$$

It should be remarked too that there is an interesting similarity in form between (16) and the Clausius-Duhem inequality which reads

$$\dot{\overline{\int_{S} s\,dV}} > -\int_{\partial V} G q \cdot n\,dA + \int_{V} G r\,dV \ ,$$

where s is the entropy.

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