ON THE TIME DERIVATIVES OF EQUILIBRATED RESPONSE FUNCTIONS

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I. Introduction.

In this paper we discuss a class of materials with memory whose response functions[1] are both differentiable and 'invariant under static continuations'. The condition of invariance requires that the response be constant in time when the determining properties are held fixed. This condition is satisfied by hypoelastic and, in general, rate-independent materials but has not in itself been regarded as a defining property for a class of materials. We call a function which is both differentiable and invariant under static continuations an equilibrated function. Thus, the response functions for hyperelastic materials are equilibrated functions, while the response functions for viscoelastic materials are not (such functions are not invariant under static continuations).

Our aim here is to show that the time derivative of an equilibrated function satisfies a relation of the same form as is satisfied by the response functions of elastic, hyperelastic, and, more generally, hypoelastic functions. For example, suppose the stress in a material is given in terms of the history of the strain. Our main result may be stated as follows: If the stress function is equilibrated, then the stress rate is a linear function of

[1] We use the term 'function' in place of the often used term 'functional'.

the current value of the strain rate; the linear operator which maps the current strain rate into the stress rate depends upon the history of strain but not upon the history of strain-rate. These results are given in Theorem 1 and the proposition which follows it, both of which appear in Section III.

We note first that materials with equilibrated response behave, from the standpoint of time rates, very much like elastic materials; this is shown in particular when we examine the thermodynamics of such materials in Section V. These materials obey a generalized stress relation and, more significantly, exhibit no internal dissipation, exactly like elastic materials. In fact, if the materials are not only equilibrated but also rate-independent, we call them generalized elastic materials since they share many properties of hypoelastic materials. Beyond this, our results give us a new way of describing hypoelastic materials by making assumptions on the response function rather than on the time derivative of the response function. More specifically, we show that a smooth rate-independent function is hypoelastic if and only if its instantaneous response is defined and is determined solely by the value of the function.

All of our results are phrased in terms of elementary concepts from the theory of differentiable functions defined on a subset of a Banach space. We wish to emphasize that the Banach spaces considered here need not contain <u>any</u> histories which correspond to functions with discontinuities. Thus, our theory differs from the theories of fading memory due to COLEMAN [1],[2] and COLEMAN and MIZEL [1],[2],[3]. In both, the expressions for time derivatives of response functions contain terms which are linear in the present values of the time derivative of the independent variable. In our theory the singular dependence on present values arises through invariance properties of the response function, while in the theories of fading memory, this dependence arises through properties of the Banach space itself. These remarks are illustrated by the examples given in Section VI.

The ideas of the present paper can be applied to obtain representations for the time derivatives of response functions which are smooth but not equilibrated. One obtains formulae of the same form as obtained by COLEMAN [1], MIZEL and WANG, COLEMAN and MIZEL [2], while assuming no singular dependence of the norm upon the present value of the history. Such considerations lead to a better understanding of the notion of 'instantaneous elasticity' and will be studied in a future paper.

II. <u>Equilibrated Functions; Rate-Independent Functions</u>.

In this section we introduce the concepts underlying our theory. For the discussion of rate-independent materials we follow the main lines of the development given in our earlier paper. Let $\mathfrak{X}$ denote some finite-dimensional normed vector space. We consider a set of functions $\mathfrak{F}$ each of whose elements has domain the interval $[0,\infty)$ and range a subset of $\mathfrak{X}$. We suppose that $\mathfrak{F}$ is a subset of a Banach space $\tilde{\mathfrak{F}}$ with norm $\|\cdot\|$ and that

$\mathfrak{F}$1) each $f\in\mathfrak{F}$ is absolutely continuous on bounded subintervals of $[0,\infty)$, and $\dot{f}$ is defined near 0 and is continuous at 0,

$\mathfrak{F}$2) if $f\in\mathfrak{F}$ then $f_\sigma\in\mathfrak{F}$ for all sufficiently small $\sigma > 0$; f_σ, the σ-section of f, is defined by the relations

$$f_\sigma(s) = f(s + \sigma), \qquad s\in[0,\infty),$$

$\mathfrak{F}$3) if $f\in\mathfrak{F}$ and $\delta > 0$ then there exists $\sigma(f,\delta) > 0$ such that for all $\sigma < \sigma(f,\delta)$ the linear perturbation of f

$$s \mapsto \begin{cases} f(s) & \sigma \le s < \infty \\ f(s) + (s - \sigma)a & 0 \le s < \sigma \end{cases}$$

is in $\mathfrak{F}$ for each $a\in\mathfrak{X}$ with $\|a\|_{\mathfrak{X}} < \delta$.

$\mathfrak{F}$4) if $f\in\mathfrak{F}$ and $\varphi: [0,\infty) \to [0,\infty)$ is

i) absolutely continuous on bounded subintervals,

ii) monotone non-decreasing,

iii) such that $\dot{\varphi}$ is defined near 0 and is continuous at 0,

iv) such that $\varphi[0,\infty)$ includes the essential support of

then $f \circ \varphi\in\mathfrak{F}$.

If one thinks of the functions in $\mathfrak{F}$ as histories, the above requirements have the following interpretations: $\mathfrak{F}$1) no 'singular histories' (histories with zero derivative almost everywhere but with positive total variation) are admitted, $\mathfrak{F}$2) for each history corresponding 'earlier histories' also are admitted in $\mathfrak{F}$, $\mathfrak{F}$3) linear perturbations of each history are admitted, $\mathfrak{F}$4) reparameterizations of each history belong to $\mathfrak{F}$. Requirements similar to $\mathfrak{F}$2) and $\mathfrak{F}$3) are common to the theories of materials with memory considered by COLEMAN and MIZEL. The restrictions $\mathfrak{F}$1) and $\mathfrak{F}$4) are significant in theories of rate-independent materials; in fact, the existing definitions of the concept of rate-independence rest on such requirements.

The functions φ satisfying the conditions in $\mathfrak{F}$4) are called *rescaling functions for the function* f, and the set of all rescaling functions for f is denoted by Φ_f. Of special interest are the rescalings ξ^σ defined by

$$\xi^\sigma(s) = \begin{cases} 0, & 0 \leq s \leq \sigma \\ s - \sigma, & \sigma < s < \infty \end{cases}$$

Here, σ is any non-negative, real number. We note that for every $\sigma \geq 0$ and $f \in \mathfrak{F}$, $\xi^\sigma \in \Phi_f$. The function $f^\sigma = f \circ \xi^\sigma$ is called the *static continuation of* f *by amount* σ. A function $\pi: \mathfrak{F} \to \mathfrak{V}$, where $\mathfrak{V}$ is some normed vector space, is said to be *invariant under static continuations* if

$$\pi(f^\sigma) = \pi(f)$$

for every $f \in \mathfrak{F}$ and every $\sigma > 0$. If we regard π as a response function for a mterial, the above condition provides that <u>the response of the material does not change during time intervals on which f is constant</u>.

The property of invariance under static continuations is implied by a more stringent requirement: π is said to be *rate-independent* if

$$\pi(f \circ \varphi) = \pi(f)$$

for all $f \in \mathfrak{F}$ and $\varphi \in \Phi_f$ (cf. TRUESDELL and NOLL, OWEN and WILLIAMS). In this case, the values of π depend upon the values of f but not upon the rate at which these values are assumed.

Our theory deals with functions $\pi : \mathfrak{F} \to \mathfrak{Y}$ which are *smooth*, i.e., for each $f \in \mathfrak{F}$ there exists a continuous linear function $\delta\pi(f\,|\,\cdot): \tilde{\mathfrak{F}} \to \mathfrak{Y}$ satisfying the following conditions:

(i) the map $f \mapsto \delta\pi(f\,|\,\cdot)$ is continuous;

(ii) for each $f \in \mathfrak{F}$, there exists a function $r(f,\cdot): \tilde{\mathfrak{F}} \to \mathfrak{Y}$ such that

$$\pi(f + h) = \pi(f) + \delta\pi(f|h) + r(f,h)$$

whenever $f + h \in \mathfrak{F}$, and $r(f,h) = o(h)$, i.e. given $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|r(f,h)\|_{\mathfrak{Y}} < \epsilon\|h\|$$

whenever $\|h\| < \delta$.

We say that a function π is *equilibrated* if π is both smooth and invariant under static continuations. The set of all

π which are equilibrated includes many non-trivial functions. In fact, the set of rate-independent, equilibrated functions on $\mathfrak{F}$ may be expected to contain infinitely many non-trivial functions. (In the final section, we indicate how this assertion may be proved in certain special cases.)

In the case when the value of π represents the value of a response at time t computed from $g^t \in \mathfrak{F}$, a history up to time t, of a function $g : \mathbb{R} \to \mathcal{X}$ we are interested in the rate of change of the values of π as t varies. More precisely, consider

$$\frac{d}{dt}\pi(g^t) = \lim_{\sigma \downarrow 0} \frac{1}{\sigma}\{\pi(g^t) - \pi(g^{t-\sigma})\}$$

$$= \lim_{\sigma \downarrow 0} \frac{1}{\sigma}\{\pi(g^t) - \pi(g^t_\sigma)\}.$$

With this in mind we define a map $\dot{\pi}$ by

$$\dot{\pi}(f) = \lim_{\sigma \downarrow 0} \frac{1}{\sigma}\{\pi(f) - \pi(f_\sigma)\},$$

whenever the limit exists, and call $\dot{\pi}(f)$ the *rate of change of π at the function* f.

We can prove with no further restrictions on π one property which is indicative of our general results and which shall prove of use in Section V.

Proposition: <u>If π is rate-independent and $f \in \mathfrak{F}$ such that $\dot{\pi}(f)$ exists, then for any $\varphi \in \Phi_f$ for which $\dot{\varphi}$ is continuous in a neighborhood of the origin $\dot{\pi}(f \circ \varphi)$ exists and</u>

$$\dot{\pi}(f \circ \varphi) = \dot{\pi}(f)\dot{\varphi}(0).$$

Proof: Note that

$$(f \circ \varphi)_\sigma = f_{\varphi(\sigma)-\varphi(0)} \circ \mu$$

where $\mu(s) = \varphi(s+\sigma) - \varphi(\sigma) + \varphi(0)$. Clearly $\mu \epsilon \Phi_{f_{\varphi(\sigma)-\varphi(0)}}$ by the restriction on φ. Hence

$$\frac{1}{\sigma}\{\pi(f\circ\varphi) - \pi((f\circ\varphi)_\sigma)\} = \frac{1}{\sigma}\{\pi(f\circ\varphi) - \pi(f_{\varphi(\sigma)-\varphi(0)}\circ\mu)\}$$
$$= \frac{\varphi(\sigma) - \varphi(0)}{\sigma}\,\frac{1}{\varphi(\sigma) - \varphi(0)}\{\pi(f) - \pi(f_{\varphi(\sigma)-\varphi(0)})\}$$

and the proposition follows.

III. Representation for $\dot{\pi}$.

In this section we present our main result: the rate of change of an equilibrated function at f is a linear function of $\dot{f}(0)$. In order to obtain this result, we shall have to place restrictions on the norm $\|\cdot\|$ for $\mathfrak{F}$. We begin by presenting restrictions which suffice for our purpose, but which are in fact stronger then necessary. The proof of the representation theorem suggests weaker restrictions under which the conclusion of the theorem remains valid.

We let $f \epsilon \mathfrak{F}$ and let $d(f) = \inf\{s \mid f(s') = 0 \text{ a.e. } s' \geq s\}$. Our assumption on the norm can be stated in the following form: Let $f_1, f_2, \ldots,$ be a sequence of functions in $\mathfrak{F}$ which satisfy the condition: $\lim_{n\to\infty} d_n = 0$; $d_n = d(f_n)$. We assume that if $\int_0^{d_n} \|\dot{f}_n(s)\|_{\mathfrak{X}} ds$ is respectively of order $\mathcal{O}(d_n)$, $o(d_n)$ as $n \to \infty$, then $\|f_n\|$ is respectively of order $\mathcal{O}(d_n), o(d_n)$, as $n \to \infty$.

This assumption is satisfied in both the examples considered in Section VI. These examples indicate that our assumption on $\|\cdot\|$ neither excludes nor necessitates a singular dependence of $\|\cdot\|$ on the value $f(0)$. In anticipation of the singular role that $\dot{f}(0)$ plays in the expression for $\dot{\pi}(f)$, we remark that *invariance properties of π rather than any singular dependence of $\|\cdot\|$ on $f(0)$ determine the singular role of $\dot{f}(0)$ in the expression for $\dot{\pi}(f)$.*

Let $\pi : \mathfrak{F} \to \mathfrak{U}$ be given. We present our proof of the representation for $\dot{\pi}$ in a series of lemmas. The essential observation is contained in

Lemma 1. *If π is invariant under static continuations then for every sufficiently small $\sigma > 0$*

$$\pi(f_\sigma) = \pi(f_\sigma^\sigma)$$

where $f_\sigma^\sigma = f_\sigma \circ \xi^\sigma$ is the static continuation of f_σ by amount σ.

Proof. By $\mathfrak{F}2)$, $\mathfrak{F}3)$, and $\mathfrak{F}4)$, $f_\sigma^\sigma \in \mathfrak{F}$ for every σ sufficiently small. Since f_σ^σ is a static continuation of f_σ, the result follows immediately.

Lemma 2. *If $f \in \mathfrak{F}$ then $\|f - f_\sigma^\sigma\| = \mathcal{O}(\sigma)$ as $\sigma \to 0$ and $\|f - f_\sigma^\sigma - L_\sigma \dot{f}(0)\| = o(\sigma)$ as $\sigma \to 0$, where*

$$L_\sigma(s) = \begin{cases} 0 & s > \sigma \\ s - \sigma & 0 \le s \le \sigma. \end{cases}$$

Proof. We rely on the assumption on $\|\cdot\|$ given above. First, we observe that

$$(f - f_\sigma^\sigma)(s) = \begin{cases} 0 & s > \sigma \\ f(s) - f(\sigma) & 0 \le s \le \sigma \end{cases}$$

and

$$\overline{(f - f_\sigma^\sigma)}^{\bullet}(s) = \begin{cases} 0 & s > \sigma \\ \dot{f}(s) & 0 \le s < \sigma \end{cases}$$

for σ sufficiently small. Note that $d_\sigma = d(f - f_\sigma^\sigma) \le \sigma$ so that $d_\sigma \to 0$ as $\sigma \to 0$. Furthermore,

$$\int_0^{d_\sigma} \|\overline{(f - f_\sigma^\sigma)}^{\bullet}(s)\|_* ds = \int_0^{d_\sigma} \|\dot{f}(s)\|_* ds = \mathcal{O}(d_\sigma)$$

since $\dot{f}(0+)$ exists. It follows that $\|f - f_\sigma^\sigma\| = \mathcal{O}(d_\sigma)$: moreover, since $d_\sigma \le \sigma$, $\|f - f_\sigma^\sigma\| = \mathcal{O}(\sigma)$. This verifies the first assertion in Lemma 2. If $\Delta_\sigma \overset{\text{def}}{=} f - f_\sigma^\sigma - L_\sigma f(0)$ then

$$\Delta_\sigma(s) = \begin{cases} 0 & s > \sigma \\ f(s) - f(\sigma) - (s - \sigma)\dot{f}(0), & 0 \le s \le \sigma \end{cases}$$

and

$$\dot{\Delta}_\sigma(s) = \begin{cases} 0 & s > \sigma \\ \dot{f}(s) - \dot{f}(0), & 0 \le s \le \sigma. \end{cases}$$

The function Δ_σ is in $\tilde{\mathfrak{F}}$ for small σ since $f - L_\sigma \dot{f}(0)$ and f_σ^σ are elements of $\mathfrak{F}$ for every σ sufficiently small. Again, $d_\sigma^* \overset{\text{def}}{=} d(\Delta_\sigma) \le \sigma$; so that $d_\sigma^* \to 0$ as $\sigma \to 0$. From the expression for $\dot{\Delta}_\sigma$, we have

$$\int_0^{d^*_\sigma} \|\dot{\Delta}_\sigma(s)\|_{\mathfrak{X}} ds = \int_0^{d^*_\sigma} \|\dot{f}(s) - \dot{f}(0)\|_{\mathfrak{X}} ds$$

$$= \{\frac{1}{d^*_\sigma} \int_0^{d^*_\sigma} \|\dot{f}(s) - \dot{f}(0)\|_{\mathfrak{X}} ds\} d^*_\sigma$$

$$= o(d^*_\sigma)$$

since $\dot{f}(s) \to \dot{f}(0)$ as $s \to 0$. In case d^*_σ vanishes for values of σ arbitrarily close to 0, it follows that $d^*_\sigma \equiv 0$ for σ near zero. Thus $\int_0^{d^*_\sigma} \|\dot{\Delta}_\sigma(s)\|_{\mathfrak{X}} ds \equiv 0$ for σ near zero and trivially, $\int_0^{d^*_\sigma} \|\dot{\Delta}_\sigma(s)\|_{\mathfrak{X}} ds = o(d^*_\sigma)$. These considerations show that $\|f - f^\sigma_\sigma - L_\sigma \dot{f}(0)\| = o(\sigma)$ as $\sigma \to 0$, which completes the proof.

Lemma 1 was concerned with a function π which is invariant under static continuations. For our main theorem we shall suppose π is smooth and define in terms of $\delta\pi(f \mid \cdot)$ a function $\delta\bar{\pi}(f \mid \cdot)$ which operates on a particular class of scalar valued functions $\lambda : [0,\infty) \to \mathbb{R}$. Recall that range $(\mathfrak{F}) \subset \mathfrak{X}$; we define $\mathfrak{F}_{\mathbb{R}}$ to consist of those functions $\lambda : [0,\infty) \to \mathbb{R}$ such that $\lambda a \in \tilde{\mathfrak{F}}$ for every $a \in \mathfrak{X}$.

Lemma 3: <u>The set $\tilde{\mathfrak{F}}_{\mathbb{R}}$ is non-empty. In fact, for σ sufficiently small $L_\sigma \in \mathfrak{F}_{\mathbb{R}}$. Moreover, $\mathfrak{F}_{\mathbb{R}}$ is closed under scalar multiplication.</u>

Proof: Let $f \in \mathfrak{F}$. $\mathfrak{F}$3) guarantees that $f + L_\sigma a \in \mathfrak{F}$ for any a within some ball about $0 \in \mathfrak{X}$ whenever σ is sufficiently small. Since $\mathfrak{F}$ is a vector space then $(f + L_\sigma a) - f = L_\sigma a \in \mathfrak{F}$. But since $\tilde{\mathfrak{F}}$ is closed under scalar multiplication we may remove the restriction on $\|a\|_{\mathfrak{X}}$ and conclude for the selected σ that $L_\sigma a \in \tilde{\mathfrak{F}}$ for all $a \in \mathfrak{X}$. Thus, $L_\sigma \in \tilde{\mathfrak{F}}_{\mathbb{R}}$ for σ sufficiently small. That $\tilde{\mathfrak{F}}_{\mathbb{R}}$ is closed under scalar multiplication follows from the corresponding closure property of $\tilde{\mathfrak{F}}$, which completes the proof.

Let $\pi : \mathfrak{F} \to \mathfrak{Y}$ be a smooth function with differential $\delta\pi(f \mid \cdot)$ at $f \in \mathfrak{F}$. We define $\delta\bar{\pi}(f \mid \cdot\,)$ as a function on $\tilde{\mathfrak{F}}_{\mathbb{R}}$ whose values $\delta\bar{\pi}(f \mid \lambda)$ are linear functions from $\mathfrak{X}$ into $\mathfrak{Y}$ given by

$$\delta\bar{\pi}(f \mid \lambda)\, a = \delta\pi(f \mid \lambda a)$$

for each $a \in \mathfrak{X}$. We define a_f, *the instantaneous modulus of* π *at* f, to be

$$a_f = \lim_{\sigma \to 0} \frac{1}{\sigma}\, \delta\bar{\pi}(f \mid L_\sigma)$$

whenever that limit exists. Of course $a_f \in \mathfrak{L}(\mathfrak{X},\mathfrak{Y})$, the set of linear mappings of $\mathfrak{X}$ into $\mathfrak{Y}$.

We are now prepared to prove

Theorem 1 (Representation of $\dot{\pi}$) If π is equilibrated and if the instantaneous modulus of π is defined at $f \in \mathfrak{F}$, then the time derivative of π at f exists and

$$\dot{\pi}(f) = a_f \dot{f}(0).$$

Proof: From Lemmas 1 and 2

$$\begin{aligned} \pi(f) - \pi(f_\sigma) &= \pi(f) - \pi(f_\sigma^\sigma) \\ &= \delta\pi(f \mid f - f_\sigma^\sigma) + o(\|f - f_\sigma^\sigma\|) \\ &= \delta\pi(f \mid L_\sigma \dot{f}(0)) + \delta\pi(f \mid \Delta_\sigma) \\ &\quad + o(\|f - f_\sigma^\sigma\|) \end{aligned}$$

where $\Delta_\sigma = f - f_\sigma^\sigma - L_\sigma \dot{f}(0)$. Lemma 2 implies that $\delta\pi(f|\Delta_\sigma) = o(\sigma)$ and $o(\|f - f_\sigma^\sigma\|) = o(\sigma)$ so

$$\pi(f) - \pi(f_\sigma) = \delta\bar{\pi}(f|L_\sigma)\dot{f}(0) + o(\sigma)$$

when σ is small enough that $L_\sigma \in \tilde{\mathfrak{F}}_{\mathbb{R}}$ (Lemma 3). Hence

$$\frac{1}{\sigma}(\pi(f) - \pi(f_\sigma)) = \frac{1}{\sigma}\delta\bar{\pi}(f|L_\sigma)\dot{f}(0) + \frac{1}{\sigma}o(\sigma)$$

and since a_f exists, we obtain as $\sigma \to 0$

$$\dot{\pi}(f) = a_f \dot{f}(0).$$

Remark: From the proof of Theorem 1 it is clear that if $\dot{\pi}(f)$ exists for a given f, we can assert that

$$\lim_{\sigma\to 0} \frac{1}{\sigma}\, \delta\bar{\pi}(f|L_\sigma)\dot{f}(0)$$

exists, but not that a_f exists. The second assertion can be seen to follow under the further assumption on $\mathfrak{F}$ given in Section V.

The functions $\frac{1}{\sigma}L_\sigma$ are shown in Figure 1. Pointwise, this sequence of functions tends to the function λ_0, which is -1 at $s = 0$ and zero elsewhere. The function λ_0 is not necessarily in the set $\tilde{\mathfrak{F}}_{\mathbb{R}}$. In fact, if the space $\tilde{\mathfrak{F}}$ contains only continuous functions, then λ_0 cannot be an element of $\tilde{\mathfrak{F}}_{\mathbb{R}}$. Hence, one cannot expect to write the instantaneous modulus in the form $\delta\bar{\pi}(f|\lambda_0)$. We note in addition that the representation $\dot{\pi}(f) = a_f\dot{f}(0)$ might be trivial if no further properties of a_f could be deduced. For example, if $\mathfrak{X} = \mathfrak{Y} = \mathbb{R}$, $\tilde{a}_f \overset{\text{def}}{=} \pi(f)/\dot{f}(0)$ always yields a representation of the form given above. In the

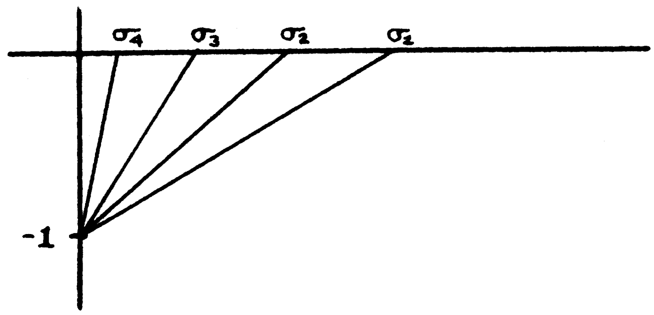

Figure 1

following proposition, we use the fact that $f \mapsto \delta\pi(f \mid \cdot)$ is continuous to show that $f \mapsto a_f$ is continuous wherever it is defined. This rules out the trivial case for many possible topologies.

Proposition: The map $f \mapsto a_f$ is continuous wherever it is defined.

Proof: First we note that $\|L_\sigma a\| = \mathcal{O}(\sigma)$ since $d(L_\sigma a) = \sigma$ and $\int_0^\sigma \|\dot{\overline{(L_\sigma a)}}(s)\|_{\mathfrak{X}} ds = \sigma \|a\|_{\mathfrak{X}}$. Next, let f and g be elements of $\mathfrak{F}$ for which a_f and a_g exist. For σ sufficiently small $L_\sigma \in \tilde{\mathfrak{F}}_{\mathbb{R}}$ and

$$\|a_f - a_g\|_{\mathcal{L}(\mathfrak{X},\mathfrak{Y})} \leq \|a_f - \frac{1}{\sigma}\delta\bar{\pi}(f|L_\sigma)\|_{\mathcal{L}(\mathfrak{X},\mathfrak{Y})}$$
$$+ \|a_g - \frac{1}{\sigma}\delta\bar{\pi}(g|L_\sigma)\|_{\mathcal{L}(\mathfrak{X},\mathfrak{Y})}$$
$$+ \|\frac{1}{\sigma}\delta\bar{\pi}(g|L_\sigma) - \frac{1}{\sigma}\delta\bar{\pi}(f|L_\sigma)\|_{\mathcal{L}(\mathfrak{X},\mathfrak{Y})}.$$

The last term is

$$\|\tfrac{1}{\sigma}\delta\bar{\pi}(f|L_\sigma) - \tfrac{1}{\sigma}\delta\bar{\pi}(g|L_\sigma)\|_{\mathcal{L}(\mathfrak{X},\mathfrak{Y})} =$$

$$= \sup_{\|a\|_{\mathfrak{X}}=1} \|\tfrac{1}{\sigma}\delta\bar{\pi}(f|L_\sigma)a - \tfrac{1}{\sigma}\delta\bar{\pi}(g|L_\sigma)a\|_{\mathfrak{Y}}$$

$$= \sup_{\|a\|_{\mathfrak{X}}=1} \|\tfrac{1}{\sigma}\delta\pi(f|L_\sigma a) - \tfrac{1}{\sigma}\delta\pi(g|L_\sigma a)\|_{\mathfrak{Y}}$$

$$\leq (\sup_{\|a\|_{\mathfrak{X}}=1} \|\tfrac{1}{\sigma}L_\sigma a\|)\,\|\delta\pi(f\,|\,\cdot\,) - \delta\pi(g\,|\,\cdot\,)\|_{\mathcal{L}(\tilde{\mathfrak{F}},\mathfrak{Y})}$$

But for any $a \in \mathfrak{X}$ we have $\|L_\sigma a\| = \mathcal{O}(\sigma)$. Hence

$$\|a_f - a_g\|_{\mathcal{L}(\mathfrak{X},\mathfrak{Y})} \leq \|a_f - \tfrac{1}{\sigma}\delta\bar{\pi}(f|L_\sigma)\|_{\mathcal{L}(\mathfrak{X},\mathfrak{Y})}$$
$$+ \|a_g - \tfrac{1}{\sigma}\delta\bar{\pi}(g|L_\sigma)\|_{\mathcal{L}(\mathfrak{X},\mathfrak{Y})}$$
$$+ \frac{\mathcal{O}(\sigma)}{\sigma}\,\|\delta\pi(f\,|\,\cdot\,) - \delta\pi(g\,|\,\cdot\,)\|_{\mathcal{L}(\tilde{\mathfrak{F}},\mathfrak{Y})}$$

and the result follows from the definition of a_f and the continuity of $f \mapsto \delta\pi(f\,|\,\cdot\,)$; for σ sufficiently small $\frac{\mathcal{O}(\sigma)}{\sigma}$ is bounded.

The relation $\dot{\pi}(f) = a_f\,\dot{f}(0)$ can yield a representation for π under certain additional assumptions. Let us suppose that f is such that $f_\sigma \in \mathfrak{F}$ and a_{f_σ} exists for each $\sigma \geq 0$. Then

$$\frac{d}{d\sigma}\pi(f_\sigma) = -\dot{\pi}(f_\sigma) = -a_{f_\sigma}\,\dot{f}(\sigma).$$

If $\sigma \mapsto \dot{\pi}(f_\sigma)$ is bounded and integrable on $[0,\infty)$ we may define

$$\pi_\infty(f) = \lim_{\sigma\to\infty} \pi(f_\sigma)$$

and hence obtain the

Corollary to Theorem 1: Let π be equilibrated, $f \in \mathfrak{F}$. If

i) for each $\sigma \in [0,\infty)$, $f_\sigma \in \mathfrak{F}$, and a_{f_σ} exists,

ii) $\sigma \mapsto \dot{\pi}(f_\sigma)$ is bounded and integrable,

then

$$\pi(f) = \pi_\infty(f) + \int_0^\infty a_{f_\sigma} \dot{f}(\sigma)\, d\sigma.$$

IV. Smooth Rate-Independent Functions; Hypoelasticity.

If π is assumed to be rate-independent and smooth (and hence equilibrated) we will show that the mapping $f \mapsto a_f$ is also rate-independent.

Theorem 2: Let π be rate-independent and smooth. If a_f exists and $\varphi \in \Phi_f$ is such that $\varphi(0) = 0$, $\dot{\varphi}(0) \neq 0$ then $a_{f\circ\varphi}$ exists and

$$a_{f\ \varphi} = a_f.$$

In any case if a_f exists for all $f \in \mathfrak{F}$ this relation holds for all $f \in \mathfrak{F}$, $\varphi \in \Phi_f$.

We prove this in a series of lemmas.

Lemma 4: Let π be rate-independent and smooth. If $f \in \mathfrak{F}$ and $\varphi \in \Phi_f$ with $\varphi(0) = 0$ then

$$\delta\bar{\pi}(f|L_\sigma) = \delta\bar{\pi}(f\circ\varphi|L_\sigma\circ\varphi)$$

for all sufficiently small σ.

Proof: Let σ be sufficiently small to guarantee $f + L_\sigma a \in \mathfrak{F}$ for any $a \in \mathfrak{X}$ with $\|a\|_{\mathfrak{X}} < 1$ ($\mathfrak{F}3$)). We can take σ smaller if necessary to guarantee $\varphi \in \Phi_{f+L_\sigma a}$ ($\mathfrak{F}4$)) for all such a. Then let $0 \leq \epsilon \leq 1$. Since $\varphi \in \Phi_{f+\epsilon L_\sigma a}$ and π is rate-independent

$$\pi(f \circ \varphi + \epsilon(L_\sigma \circ \varphi)a) - \pi(f \circ \varphi) = \pi(f + \epsilon L_\sigma a) - \pi(f).$$

Thus

$$\delta\pi(f \circ \varphi | \epsilon(L_\sigma \circ \varphi)a) = \delta\pi(f|\epsilon L_\sigma a) + o(\epsilon)$$

or

$$\delta\bar{\pi}(f \circ \varphi | L_\sigma \circ \varphi)a = \delta\bar{\pi}(f|L_\sigma)a + o(1)$$

as $\epsilon \to 0$. Hence

$$\delta\bar{\pi}(f \circ \varphi | L_\sigma \circ \varphi) = \delta\bar{\pi}(f|L_\sigma).$$

Lemma 5: Let $\varphi(0) = 0$, $\dot{\varphi}(0) \neq 0$. Then for each $a \in \mathfrak{X}$

$$\|(L_\sigma \circ \varphi - \dot{\varphi}(0)L_{\sigma/\dot{\varphi}(0)})a\| = o(\sigma).$$

Proof: Let $\sigma^* = \sigma/\dot{\varphi}(0)$. For σ sufficiently small both $L_\sigma \circ \varphi$ and L_{σ^*} are in $\mathfrak{F}_{\mathbb{R}}$. Since $\dot{\varphi}(0) \neq 0$, φ is invertible in some neighborhood of zero; letting σ lie within this neighborhood we can write

$$d(L_\sigma \circ \varphi) = \overset{-1}{\varphi}(\sigma)$$

$$d(L_{\sigma^*}) = \sigma^*$$

(recall that $d(f)$ is the largest bound on supp f). Note that

$$\overset{-1}{\varphi}(\sigma) = \frac{1}{\dot{\varphi}(0)}\sigma + o(\sigma) = \sigma^* + o(\sigma).$$

Hence both $\overset{-1}{\varphi}(\sigma)$ and σ^* are $\mathcal{O}(\sigma)$ as $\sigma \to 0$; their difference is $o(\sigma)$. Finally, we define

$$\sigma_{max} = \max(\overset{-1}{\varphi}(\sigma), \sigma^*)$$
$$\sigma_{min} = \min(\overset{-1}{\varphi}(\sigma), \sigma^*).$$

If $a \in \mathfrak{X}$ is given we define $g_\sigma = (L_\sigma \circ \varphi)a - \dot{\varphi}(0)L_{\sigma^*}a$ and note

$$\int_0^\infty \|\dot{g}_\sigma(s)\|_{\mathfrak{X}} ds = \int_0^{\sigma_{max}} \|\dot{\varphi}(s)a - \dot{\varphi}(0)a\|_{\mathfrak{X}} ds$$
$$\leq \int_0^{\sigma_{min}} \|\dot{\varphi}(s)a - \dot{\varphi}(0)a\|_{\mathfrak{X}} ds + \int_{\sigma_{min}}^{\sigma_{max}} [\|\dot{\varphi}(0)a\|_{\mathfrak{X}} + \|\dot{\varphi}(s)a\|_{\mathfrak{X}}] ds.$$

Of course

$$\frac{1}{\sigma}\int_0^{\sigma_{min}} \|\dot{\varphi}(s)a - \dot{\varphi}(0)a\|_{\mathfrak{X}} ds = \frac{\sigma_{min}}{\sigma}\frac{1}{\sigma_{min}} \int_0^{\sigma_{min}} \|\dot{\varphi}(s)a - \dot{\varphi}(0)a\|_{\mathfrak{X}} ds$$

and this tends to zero as $\sigma \to 0$. Similarly we note that $\dot{\varphi}$ must be bounded in a neighborhood of zero—say by k—and thus as σ becomes small

$$\frac{1}{\sigma}\int_{\sigma_{min}}^{\sigma_{max}} [\|\dot{\varphi}(0)a\|_{\mathfrak{X}} + \|\dot{\varphi}(s)a\|_{\mathfrak{X}}] ds \leq 2k\|a\|_{\mathfrak{X}} \frac{\sigma_{max} - \sigma_{min}}{\sigma}.$$

But since $\overset{-1}{\varphi}(\sigma) - \sigma^*$ is $o(\sigma)$ this term tends to zero with σ and hence the arc-length of g is $o(\sigma)$; this implies the desired result.

We now can prove the first part of the theorem.

Lemma 6: Let π be rate-independent and smooth. If a_f exists, $\varphi \in \Phi_f$ with $\varphi(0) = 0$, $\dot{\varphi}(0) \neq 0$ then $a_{f\circ\varphi}$ exists and

$$a_{f\circ\varphi} = a_f.$$

Proof: By Lemma 4

$$a_f = \lim_{\sigma\to 0} \frac{1}{\sigma} \delta\bar{\pi}(f|L_\sigma) = \lim_{\sigma\to 0} \frac{1}{\sigma} \delta\bar{\pi}(f \circ \varphi | L_\sigma \circ \varphi) .$$

But $\frac{1}{\sigma} \delta\bar{\pi}(f \circ \varphi | L_\sigma \circ \varphi) = \frac{1}{\sigma} \delta\bar{\pi}(f \circ \varphi | \dot{\varphi}(0) L_{\sigma*})$

$$+ \frac{1}{\sigma} \delta\bar{\pi}(f \circ \varphi | L_\sigma \circ \varphi - \dot{\varphi}(0) L_{\sigma*})$$

where $\sigma*$ is defined as in the previous proof. By Lemma 5

$$\lim_{\sigma\to 0} \delta\bar{\pi}(f \circ \varphi | \tfrac{1}{\sigma}(L_\sigma \circ \varphi - \dot{\varphi}(0) L_{\sigma*}))a$$
$$= \lim_{\sigma\to 0} \delta\pi(f \circ \varphi | \tfrac{1}{\sigma}(L_\sigma \circ \varphi - \dot{\varphi}(0) L_{\sigma*})a) = 0$$

for any $a \in \mathfrak{X}$. Thus we know the limit

$$\lim_{\sigma\to 0} \frac{1}{\sigma} \delta\bar{\pi}(f \circ \varphi | \dot{\varphi}(0) L_{\sigma*})$$
$$= \lim_{\sigma\to 0} \frac{1}{\sigma*} \delta\bar{\pi}(f \circ \varphi | L_{\sigma*}) = a_{f\circ\varphi}$$

exists; hence $a_{f\circ\varphi}$ exists and equals a_f.

Lemma 6 is now used to prove the remainder of the theorem. Let us first take the case $\varphi(0) \neq 0$.

Lemma 7: Let π be rate-independent and smooth. If $a_f, a_{f\circ\varphi}$ exist, for $\varphi(0) \neq 0$, then

$$a_{f\circ\varphi} = a_f .$$

Proof: In light of 34) f must be constant in some interval $[0,s]$; let $[0,\hat{s}]$ be the largest such interval. Then if $\bar{s} = \varphi(0)$ we have $0 < \bar{s} \leq \hat{s}$. We consider two cases.

First, if $\bar{s} < \hat{s}$ we define $s^* = \min\{s \mid \varphi(s) = \hat{s}\}$ and let

$$\varphi^*(s) = \begin{cases} \frac{\hat{s}}{s^*}s & 0 \leq s \leq s^* \\ \varphi(s) & s^* \leq s < \infty \end{cases}$$

The function φ^* falls under the assumptions of Lemma 6, so $a_{f\circ\varphi^*}$ exists. However, it is clear that $f \circ \varphi^* = f \circ \varphi$ and thus $a_{f\circ\varphi}$ exists and $a_{f\circ\varphi} = a_{f\circ\varphi^*} = a_f$.

Next, suppose $\bar{s} = \hat{s}$. Note first that if φ is identically equal to $\bar{s}$ in some neighborhood of zero we can define φ^* in much the same way as above, concluding $f \circ \varphi = f \circ \varphi^*$, and thus that $a_{f\circ\varphi}$ exists and equals a_f. Otherwise we let

$$s_n = \min\{s \mid \varphi(s) - \bar{s} = \tfrac{1}{n}\}$$

and note that $\lim_{n\to\infty} s_n = 0$. Then, consider the sequence $\{\varphi_n\}$ defined by

$$\varphi_n(s) = \begin{cases} (\bar{s} - \frac{1}{n}) + \frac{2s}{ns_n} & 0 \leq s \leq s_n \\ \varphi(s) & s_n \leq s < \infty \end{cases}$$

Then $\varphi_n(0) = \bar{s} - \frac{1}{n} < \bar{s} = \hat{s}$ and thus φ_n falls into the previous category. This means that $a_{f\circ\varphi_n}$ exists for each n and is equal to a_f. We now show that $f \circ \varphi_n \to f \circ \varphi$ so that $a_{f\circ\varphi_n} \to a_{f\circ\varphi}$ as $n \to \infty$; this of course gives the desired result. We note first that

$$\lim_{n\to\infty} \frac{\varphi(s_n) - \varphi(0) - \dot{\varphi}(0)s_n}{s_n} = 0$$

so $\frac{1}{ns_n} \to \dot{\varphi}(0)$ as $n \to \infty$.

Now define $g_n = f \circ \varphi - f \circ \varphi_n$. Then $\operatorname{supp} g_n \subset [0, s_n]$ and on $[0, s_n]$

$$\dot{g}_n(s) = \dot{f}(\varphi(s))\dot{\varphi}(s) - \dot{f}(\varphi_n(s))\frac{2}{ns_n} .$$

Within $[0,s_n]$ both φ and φ_n are bounded by $\bar{s} + \frac{1}{n}$ so for n sufficiently large $\dot{f} \circ \varphi$ and $\dot{f} \circ \varphi_n$ are both uniformly bounded on this interval; similarly one can establish a uniform bound for $\dot{\varphi}$ and $\frac{2}{ns_n}$ for n sufficiently large. Thus, the arc-length of g_n, $\int_0^{s_n}\|\dot{g}_n(s)\|_{\mathfrak{X}}ds$, must be $\mathcal{O}(d(g_n))$ and thus $\|g_n\| = \mathcal{O}(d(g_n))$, which means $\|g_n\| \to 0$, since $d(g_n) < s_n$. (One can show that under our assumptions $d(g_n) \neq 0$ for each n, although this is of no significance. If $d(g_m)$ were zero then $f \circ \varphi = f \circ \varphi_m$ and the conclusion $a_{f\circ\varphi} = a_f$ would be immediate.)

Finally, we establish the case $\varphi(0) = 0$, $\dot{\varphi}(0) = 0$.

Lemma 8: *Let a_f exists for all $f \in \mathfrak{F}$. If $\varphi \in \Phi_f$ is such that $\varphi(0) = 0$, $\dot{\varphi}(0) = 0$, then $a_{f\circ\varphi} = a_f$.*

Proof: We proceed in a manner similar to the second part of Lemma 7. Thus, define

$$s_n = \min\{s \mid \varphi(s) = \tfrac{1}{n}\},$$

$$s_\infty = \lim_{n\to\infty} s_n .$$

We consider two cases. First, let $s_\infty = 0$. We can show exactly as above that

$$\lim_{n\to\infty} \frac{1}{ns_n} = 0.$$

Define a sequence $\{\varphi_n\}$ as follows:

$$\varphi_n(s) = \begin{cases} \dfrac{s}{ns_n} & 0 \leq s \leq s_n \\ \varphi(s) & s_n \leq s \leq \infty . \end{cases}$$

Then define $g_n = f \circ \varphi - f \circ \varphi_n$; we will show that $\|g_n\| = \mathcal{O}(d(g_n))$ and thus that $f \circ \varphi_n \to f \circ \varphi$. To dispose first of a pathology: if f is constant in some neighborhood of zero then eventually $f \circ \varphi_n = f \circ \varphi$. Otherwise, we obtain

$$\dot{g}_n(s) = \dot{f}(\varphi(s))\dot{\varphi}(s) - \dot{f}(\varphi_n(s))\frac{1}{ns_n}$$

within $[0,s_n]$, which includes the support of g_n. Inside $[0,s_n]$ the functions φ and φ_n are both bounded by $\frac{1}{n}$ and hence for n sufficiently large $\dot{f} \circ \varphi$ and $\dot{f} \circ \varphi_n$ are both bounded. Similarly $\dot{\varphi}(s)$ and $\frac{1}{ns_n}$ are bounded so that the arc-length of g_n must be $\mathcal{O}(d(g_n))$. Hence $f \circ \varphi_n \to f \circ \varphi$; since each φ_n is of the type considered in Lemma 6, $a_{f\circ\varphi_n}$ exists and is equal a_f. Thus $a_{f\circ\varphi} = a_f$.

Finally, let $s_\infty \neq 0$. Recall that ξ^σ is the static continuator. First suppose $\varphi = \xi^\sigma$, and consider $\hat{\varphi}\epsilon\Phi_{f\circ\varphi}$ given by $\hat{\varphi}(s) = s + \sigma$. $\hat{\varphi}$ is of the type covered in Lemma 7 and thus since a_{f^σ}, a_f exist and $f^\sigma \circ \hat{\varphi} = f$ we have $a_f = a_{f^\sigma}$. If φ is not a static continuator we have

$$\varphi = \lambda \circ \xi^{s_\infty}$$

where $\lambda\epsilon\Phi_f$, λ not identically zero near 0. Hence since $a_{f\circ\lambda}$ exists by assumption,

$$a_{f\circ\varphi} = a_{f\circ\lambda\circ\xi^s{}_{\infty}} = a_{f\circ\lambda} = a_f$$

using previous results.

Theorem 2 shows that $f \mapsto a_f$ is a rate-independent function if π is rate-independent. By placing further restrictions on the instantaneous modulus a_f, we can give sufficient conditions that π be hypoelastic. In fact, π <u>is a hypoelastic function if</u>

1) π <u>is rate-independent and smooth</u>,
2) a_f <u>exists for every</u> $f \in \mathfrak{F}$,
3) $a_f = a_g$ <u>whenever</u> $\pi(f) = \pi(g)$.

Conditions 1) and 2) along with Theorem 1 guarantee that $\dot{\pi}(f)$ exists and equals $a_f\,\dot{f}(0)$. Then, using 3) we can deduce the existence of a mapping $A : \mathcal{Y} \to \mathcal{L}(\mathcal{X},\mathcal{Y})$ such that $a_f = A(\pi(f))$ so that we obtain the defining equation of hypoelasticity (TRUESDELL and NOLL , pp. 402f.):

$$\dot{\pi}(f) = A(\pi(f))\,\dot{f}(0).$$

V. <u>Applications to Thermodynamics</u>.

In this section we discuss the thermodynamics of simple materials with memory whose constitutive functions are of the sort previously considered. We follow the technique of COLEMAN and NOLL to deduce restrictions which are placed on constitutive functions by the Clausius-Duham inequality. We omit all physical motivation and discussion—the reader is referred to COLEMAN and NOLL , COLEMAN and MIZEL [4],[5], or TRUESDELL and NOLL for such considerations.

Let $\mathcal{V}$ denote a three-dimensional vector space, $\mathcal{L}$ the set of endomorphisms on $\mathcal{V}$, $\mathcal{L}^+$ the elements of $\mathcal{L}$ with positive determinant, and $\mathbb{R}^+$ the (strictly) positive reals. We consider the collection $\mathfrak{F}$ of generalized strain histories. Each element $\Gamma \in \mathfrak{F}$ maps $[0,\infty)$ into $\mathcal{L}^+ \times \mathbb{R}^+$; we suppose $\mathfrak{F}$ obeys the conditions of Section II. In addition to the condition in Section III, we will add a further assumption on the norm associated with $\mathfrak{F}$: if $\Gamma \in \mathfrak{F}$, $a \in \mathcal{L} \times \mathbb{R}$, and the linear continuation $\Gamma(\sigma, a) \in \mathfrak{F}$ is defined by[1]

$$\Gamma(\sigma,a)(s) = \begin{cases} \Gamma(s-\sigma) & \sigma \leq s < \infty \\ \Gamma(0) + (s-\sigma)a & 0 \leq s < \sigma \end{cases}$$

then

$$\lim_{\sigma \to 0} \Gamma(\sigma,a) = \Gamma.$$

We now define a function π mapping $\mathfrak{F}$ into some normed vector space $\mathcal{U}$ to be a generalized elastic function if there exists a continuous rate-independent function $A : \Gamma \mapsto A_\Gamma \in \mathcal{L}(\mathcal{L} \times \mathbb{R}, \mathcal{U})$ such that for each $\Gamma \in \mathfrak{F}$

$$\dot{\pi}(\Gamma) = A_\Gamma \dot{\Gamma}(0).$$

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[1] If σ is sufficiently small then for any a, $\Gamma(\sigma,a) \in \mathfrak{F}$. This follows from $\mathfrak{F}$3) and $\mathfrak{F}$4), for $\Gamma(\sigma,a)$ is a linear perturbation of a static continuation of Γ.

Hypoelastic functions are also generalized elastic; Theorems 1 and 2 give conditions under which an equilibrated function is generalized elastic.

Finally, we introduce $\mathcal{G}$ as the collection of all continuous piecewise continuously differentiable mappings of $[0,\infty)$ into $\mathcal{V}$. We call elements of $\mathcal{G}$ *temperature gradient histories*.

We suppose that there exist functions

$$\tilde{\Psi} : \mathfrak{F}\times\mathcal{V} \to \mathbb{R}$$
$$\tilde{\Sigma} : \mathfrak{F}\times\mathcal{V} \to \mathcal{L} \times \mathbb{R}$$
$$\tilde{q} : \mathfrak{F}\times\mathcal{V} \to \mathcal{V}$$

which, given any generalized strain history and temperature gradient history, yield the Helmholtz free energy $\Psi : [0,\sigma) \to \mathbb{R}$, generalized stress $\Sigma: [0,\sigma) \to \mathcal{L} \times \mathbb{R}$, and heat flux $q: [0,\sigma) \to \mathcal{V}$:

$$\Psi(s) = \tilde{\Psi}(\Gamma_s, g(s))$$
$$\Sigma(s) = \tilde{\Sigma}(\Gamma_s, g(s))$$
$$q(s) = \tilde{q}(\Gamma_s, g(s))$$

for some positive σ.

Let us denote right-hand derivatives by superposed dots. We require that for all Γ, g the functions $\tilde{\Psi}, \tilde{\Sigma}, \tilde{q}$ be such that the Clausius-Duhem inequality is satisfied:

$$-\dot{\Psi}(0) + \Sigma(0)\cdot\dot{\Gamma}(0) + \frac{1}{\rho\theta}q(0)\cdot g(0) \leq 0.$$

Here, the inner-product of two elements $(A,s),(B,t)\in\mathcal{L} \times \mathbb{R}$ is $\mathrm{tr}(AB^T) + st$, θ denotes the second entry of $\Gamma(0)$ (the temperature), and ρ is the density, $\rho = \det(\Gamma_1(0))^{-1}$ where $\Gamma_1(0)$ is the first entry of $\Gamma(0)$.

We take two different approaches to the problem of determining restrictions imposed on $\tilde{\Psi}, \tilde{\Sigma}, \tilde{q}$ by the Clausius-Duhem inequality. Let us first suppose $\tilde{\Sigma}, \tilde{q}$ are (jointly) continuous and $\tilde{\Psi}$ (jointly) smooth. We suppose that for each $v \in \mathcal{V}$ the function $\tilde{\Psi}(\cdot, v)$ is equilibrated and has an instantaneous modulus $-A(\cdot, v)$. Then for any $(\Gamma, g) \in \mathfrak{F} \times \mathcal{G}$

$$\dot{\Psi}(0) = A(\Gamma, g(0))\dot{\Gamma}(0) + B(\Gamma, g(0)) \cdot \dot{g}(0)$$

where $B(\Gamma, g(0)) = \delta_2 \tilde{\Psi}(\Gamma, g(0)) \in \mathcal{V}$. By assumption the mappings $(\Gamma, v) \mapsto \delta_1 \tilde{\Psi}(\Gamma, v)$, $(\Gamma, v) \mapsto B(\Gamma, v)$ are continuous The proof given in Section III that $f \mapsto a_f$ is a continuous map establishes the

Lemma: A is continuous on $\mathfrak{F} \times \mathcal{V}$.

Suppose now the functions Γ, g are specified. The linear continuations $\hat{\Gamma} = \Gamma(\tau, a)$, $\hat{g} = g(\tau, v)$

iii) $\tilde{q}(\Gamma,g(0)) \cdot g(0) \leq 0$, <u>all</u> Γ,g.

<u>Proof</u>: From the Clausius-Duhem inequality, it is clear that $B(\Gamma,g(0))$ must be zero, and thus that $\tilde{\Psi}$ is independent of its second argument. Hence $A(\Gamma,g(0))$ must be independent of $g(0)$, and of course $\tilde{\Sigma}(\Gamma,g(0)) = A(\Gamma)$.

Remark: These results are similar to those obtained by COLEMAN [1],[2], WANG and BOWEN , GURTIN , and GREEN and LAWS. In fact it is clear that our system obeys (sufficiently closely) postulates (A_1), (A_2), (A_3) specified by GURTIN, and hence Theorem 3 is a consequence of his result (p. 43). In fact we have simply repeated his proof. Property ii) is usually called the *generalized stress relation*.

Corollary: <u>If $\tilde{\Psi},\tilde{\Sigma},\tilde{q}$ are as in Theorem 3 and in addition $\tilde{\Psi}$ is rate-independent—and hence generalized-elastic—then $\tilde{\Sigma}$ is rate-independent</u>.

This Corollary is a consequence of Theorem 2 and the result $\tilde{\Sigma}(\Gamma) = A(\Gamma)$.

As a second approach to the thermodynamic theory we suppose that $\tilde{\Psi},\tilde{\Sigma},\tilde{q}$ are all rate-independent. Given any $(\Gamma,g) \in \mathfrak{F} \times \mathcal{G}$ and any rescaling function $\varphi \in \Phi_\Gamma$ for which $\dot{\varphi}$ is continuous in a neighborhood of zero we shall write the Clausius-Duhem inequality both for the functions

$$\Psi(s) = \tilde{\Psi}(\Gamma_s,g(s)),\ \Sigma(s) = \tilde{\Sigma}(\Gamma_s,g(s)),\ q(s) = \tilde{q}(\Gamma_s,g(s))$$

and the corresponding quantities obtained from $(\Gamma \circ \varphi, g)$. These are

$$-\dot{\Psi}(0) + \Sigma(0) \cdot \dot{\Gamma}(0) + \frac{1}{\rho\theta} q(0) \cdot g(0) \leq 0$$

and

$$-(\dot{\Psi}(0) - \Sigma(0)\cdot\dot{\Gamma}(0))\dot{\varphi}(0) + \frac{1}{\rho\Theta} q(0)\cdot g(0) \leq 0$$

since $\tilde{\Psi}, \tilde{\Sigma}, \tilde{q}$ are rate independent and since we can apply the proposition in Section II to evaluate the derivative of $\tilde{\Psi}(\Gamma\circ\varphi, g(0))$. It is clear that we can choose such φ with $\dot{\varphi}(0)$ of arbitrary (positive) value. Hence the

Lemma: If $\tilde{\Psi}, \tilde{\Sigma}, \tilde{q}$ are rate-independent then the Clausius-Duhem inequality is equivalent to

i) the *dissipation inequality*

$$\sigma \overset{\text{def}}{=} +\dot{\Psi}(0) - \Sigma(0)\cdot\dot{\Gamma}(0) \geq 0$$

and ii) the *heat-conduction inequality*

$$q(0)\cdot g(0) \leq 0.$$

If the functions $\tilde{\Psi}, \tilde{\Sigma}$ are such that $\sigma = 0$ for all $(\Gamma, g) \in \mathfrak{F} \times \mathcal{G}$ we say they are *non-dissipative*. The proof of Theorem 3 can be used to show that if $\tilde{\Psi}$ is generalized elastic then $\tilde{\Psi}, \tilde{\Sigma}$ are non-dissipative. A partial converse is provided by

Theorem 4: Let $\tilde{\Psi}, \tilde{\Sigma}, \tilde{q}$ be rate-independent and $\tilde{\Sigma}$ be continuous. Then if $\tilde{\Psi}$ and $\tilde{\Sigma}$ are non-dissipative, they are both independent of the second argument and Ψ is generalized elastic.

Proof: If $\tilde{\Psi}, \tilde{\Sigma}$ are non-dissipative, $\sigma = 0$, then

$$\dot{\tilde{\Psi}}(\Gamma, v) = \tilde{\Sigma}(\Gamma, v)\cdot\dot{\Gamma}(0)$$

and $\tilde{\Sigma}$ replaces A in the condition of generalized elasticity, provided that we can remove the second argument in each function.

Next we show that $\tilde{\Psi}$ is in fact independent of $g(0)$. Let $(\Gamma,g)\in\mathfrak{F}\times\mathcal{G}$ be given. For any $v\in\mathcal{V}$ we construct the linear continuation $\hat{g} = g(\tau,v)$.

Now consider the map $\sigma \mapsto \tilde{\Psi}(\Gamma^{\sigma},\hat{g}(\sigma))$. Since $(\Gamma^{\sigma},\hat{g}(\sigma))\in\mathfrak{F}\times\mathcal{G}$ for each $\sigma\in[0,\tau]$ we know that this function has at each $\sigma\in[0,\tau]$ a right-hand derivative; on $(0,\tau]$ this must be equal to zero. But this suffices (e.g. HOBSON , p. 366) to guarantee that $\sigma \mapsto \tilde{\Psi}(\Gamma^{\sigma},g(\sigma))$ is constant on $[0,\tau]$ (the map is continuous by the assumption on $\tilde{\Psi}$ and the property assumed for the norm applied to linear continuations). Hence in particular its values at 0 and τ are equal:

$$\tilde{\Psi}(\Gamma^{\tau},g(0) + v) = \tilde{\Psi}(\Gamma,g(0)).$$

Since $\tilde{\Psi}$ is rate-independent in its first argument this means $\tilde{\Psi}(\Gamma,g(0) + v) = \tilde{\Psi}(\Gamma,g(0)) = \tilde{\Psi}(\Gamma,0)$.

Next, we note that $\dot{\tilde{\Psi}}(\Gamma)$ is also necessarily independent of $g(0)$. Since

$$\dot{\tilde{\Psi}}(\Gamma) = \tilde{\Sigma}(\Gamma,g(0)) \cdot \dot{\Gamma}(0),$$

it is clear that for any $v,u\in\mathcal{V}$

$$\tilde{\Sigma}(\Gamma,u) \cdot \dot{\Gamma}(0) = \tilde{\Sigma}(\Gamma,v) \cdot \dot{\Gamma}(0).$$

However, we can now apply the technique of linear continuations to this equation (exactly as in the proof of the previous theorem) to conclude that $\tilde{\Sigma}(\Gamma,u) = \tilde{\Sigma}(\Gamma,v)$.

VI. Two Examples.

In this section, we obtain expressions for the instantaneous modulus a_f for two choices of $\tilde{\mathfrak{F}}$. In addition, we will give an example of a smooth rate-independent function which is not an elastic function.

First, we let $\tilde{\mathfrak{F}}$ be the set of all absolutely continuous functions from $[0,\infty)$ into $\mathbb{R}^n$, $\mathfrak{F}$ all functions in $\tilde{\mathfrak{F}}$ smooth near the origin. In this case, we take

$$\|f\| = \int_0^\infty |\dot{f}(s)|\,ds + |f(0)|.$$

If π is a smooth function mapping $\mathfrak{F}$ into $\mathbb{R}^m$, we have a representation for the continuous linear function $\delta\pi(f\,|\,\cdot\,)$ (DUNFORD and SCHWARTZ p. 343)

$$\delta\pi(f|g) = b_f(0)g(0) + \int_0^\infty b_f(s)\dot{g}(s)\,ds$$

where $b_f : [0,\infty) \to \mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)$ is essentially bounded. Hence

$$\delta\bar{\pi}(f|\lambda) = \lambda(0)b_f(0) + \int_0^\infty \dot{\lambda}(s)b_f(s)\,ds$$

for any absolutely continuous $\lambda : [0,\infty) \to \mathbb{R}$. Thus

$$\frac{1}{\sigma}\delta\bar{\pi}(f|L_\sigma) = -b_f(0) + \frac{1}{\sigma}\int_0^\sigma b_f(s)\,ds.$$

Consequently, a_f exists if and only if $\lim_{\sigma\to 0}\frac{1}{\sigma}\int_0^\sigma b_f(s)\,ds$ exists, in which case

$$a_f = \lim_{\sigma\to 0} \frac{1}{\sigma}\int_0^\sigma b_f(s)\,ds - b_f(0).$$

If $b_f(0+)$ exists

$$a_f = b_f(0+) - b_f(0).$$

As a second example, we let $\tilde{\mathfrak{F}}$ consist of all bounded continuous functions on $[0,\infty)$ with values in $\mathbb{R}^n$, $\mathfrak{F}$ be all functions in $\tilde{\mathfrak{F}}$ which are absolutely continuous on finite subintervals of $[0,\infty)$ and smooth near the origin. We take

$$\|f\| = \sup_{s\in[0,\infty)} |f(s)|.$$

The differential $\delta\pi(f\,|\,\cdot)$ of a smooth function $\pi : \mathfrak{F} \to \mathbb{R}^m$ takes the form (DUNFORD and SCHWARTZ p. 344)

$$\delta\pi(f|g) = \int_0^\infty (d\alpha_f(s))\,g(s)$$

where α_f is a function of bounded variation on $[0,\infty)$ with values $\alpha_f(s)$ which are linear functions from $\mathbb{R}^n$ into $\mathbb{R}^m$. In this case, $\delta\bar{\pi}(f|\lambda) = \int_0^\infty \lambda(s)\,d\alpha_f(s)$ and

$$\begin{aligned}\frac{1}{\sigma}\delta\bar{\pi}(f|L_\sigma) &= \frac{1}{\sigma}\int_0^\sigma (s-\sigma)\,d\alpha_f(s)\\ &= \frac{1}{\sigma}\,\alpha_f(s)\,(s-\sigma)\Big|_0^\sigma - \frac{1}{\sigma}\int_0^\sigma \alpha_f(s)\,ds\\ &= \alpha_f(0) - \frac{1}{\sigma}\int_0^\sigma \alpha_f(s)\,ds.\end{aligned}$$

In contrast to our first example, we see that since α_f is of bounded variation, the function a_f necessarily exists:

$$a_f = \alpha_f(0) - \alpha_f(0+).$$

As a final note, we mention that for the set $\mathfrak{F}$ in the first example, there are many equilibrated functions. Although we give only one example here, it is possible to generate many such functions by integrating any given one. For example, if we take $\mathfrak{X} = \mathfrak{Y} = \mathcal{L}(\mathbb{R}^3,\mathbb{R}^3)$, $\mathfrak{F}$ the set of absolutely continuous functions, then

$$\pi_0(f) = \int_0^\infty \mathrm{tr}[f(s)]\dot{f}(s)\,ds$$

is an equilibrated function. Furthermore, it is not difficult to show that

$$\pi_1(f) = \int_0^\infty \pi_0(f_\sigma)\cdot\dot{f}(\sigma)\,d\sigma$$

is equilibrated; one can continue in this fashion to generate many other such functions.

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