NORM CONDITIONS FOR DISCONJUGACY OF COMPLEX DIFFERENTIAL SYSTEMS

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1. <u>Introduction</u>. The systems to be considered in this paper are of the form

$$(1) \qquad W'(z) = A(z)W(z),$$

where $W(z) = (w_{ik}(z))_1^n$ and $A(z) = (a_{ik}(z))_1^n$ are square matrices of analytic functions. We assume that the n^2 analytic functions $a_{ik}(z)$ are regular in a simply connected domain D not containing $z = \infty$; it follows that the same holds for the elements $w_{ik}(z)$ of any matrix solution $W(z)$ of (1). *The system (1) is called disconjugate in D if, for any fundamental solution $W(z) = (w_{ik}(z))_1^n$ (i.e., for any solution $W(z)$ for which the determinant $|w_{ik}(z)|_1^n \neq 0$ for all z of D), the determinant $|w_{ik}(z_i)|_1^n \neq 0$ for every choice of n (not necessarily distinct) points $z_1,\ldots,z_n$ of D.* It is easily seen that if this holds for one fundamental solution of (1), then it holds for all of them. Disconjugacy of the matrix differential equation (1) in D is equivalent to the assertion that, for every choice of n points $z_1,\ldots,z_n$ of D, the only solution $w(z) = [w_1(z),\ldots,w_n(z)]$ of the corresponding vector differential equation

$$(2) \qquad w'(z) = A(z)w(z),$$

satisfying $w_i(z_i) = 0$, $i = 1,\ldots,n$, is the trivial one [10, Theorem 3].

In section 2 we consider line integrals of the maximal row norm $\|A(z)\|_\infty$. We prove that if every point z of D can be connected with a given point z_0 of D by a path in D so that for all these paths

$$\int_{z_0}^{z} \|A(\zeta)\|_\infty |d\zeta| < \log 2, \tag{3}$$

then the system (1) is disconjugate in D (Theorem 1). We then restate a beautiful result of Kim [3, Lemma 2.2] in terms of the matrix equation (1), (Lemma 1) and show that every matrix norm $\|A(z)\|$ is a subharmonic function in D (Lemma 2). Using these results we obtain that if the line integral of $\|A(z)\|_\infty$ along the boundary C of D is not larger than $2 \log 2$, then (1) is disconjugate in D (Theorem 2 for the unit disk and Theorem 2' for any simply connected domain). We mention also a related result of Kim [3, Theorem 2.7]. We conclude this section with an analogous result for systems defined on an interval (Theorem 3) and compare this with a recent sharp result of Nehari [8, Theorem 3.3].

In section 3, we obtain conditions which imply z_0-absolute disconjugacy of (1) in D, $z_0 \in D$, and we thus start with the definition of this property [7]. We mention already now that z_0-absolute disconjugacy implies (ordinary) disconjugacy. In this section we work with arbitrary matrix norms, however not of $A(z)$, but of a real majorant matrix $P(x)$. Our result (Theorem 4) follows from [7, Theorem 1], and the present paper, especially section 3, is a continuation of this joint paper with D. London. We have, however, tried to make it reasonably self contained. We conclude this paper with applications of Theorem 4 to systems and to linear n-th order differential equations in the unit disk (Corollaries 1 and 2).

As indicated $|A| = |a_{ik}|_1^n$ denotes the determinant of the $n \times n$ matrix $A = (a_{ik})_1^n$. For completeness we bring here the definition of a matrix norm and also the basic properties used in the sequel [2,9]. A norm $\|A\|$ is a real valued function, defined for all $n \times n$ matrices A, satisfying

(I) $\quad A \neq 0$ implies $\|A\| > 0$,

(II) $\quad \|cA\| = |c|\,\|A\|$, $\quad c$ scalar,

(III) $\quad \|A + B\| \leq \|A\| + \|B\|$,

(IV) $\quad \|AB\| \leq \|A\|\,\|B\|$.

As we consider integrals of norms, we use the following consequence of (I) to (III):

(V) $\quad \|A\| = \|(a_{ik})_1^n\|$ is a continuous function of the elements a_{ik} of A.

(In our case, each element $a_{ik}(z)$ of $A(z)$ is a regular analytic function of z; it follows that $\|A(z)\|$ is a continuous function of z.) We denote the characteristic (proper) values of A by $\lambda_i(A)$, $i = 1,\ldots,n$. (I) to (IV) imply

(VI) $$\|\lambda_i(A)\| \leq \|A\|, \quad i = 1,\dots,n.$$

If A is a nonnegative matrix, $A \geq 0$ (i.e., $a_{ik} \geq 0, i,k = 1,\dots,n$), we denote the maximal characteristic value of A by $\lambda(A) = \lambda((a_{ik})_1^n)$. In this case (VI) can be replaced by

(VI') $$\lambda(A) \leq \|A\|, \qquad A \geq 0.$$

$\|A\|_\infty$ denotes the maximal row norm of $A = (a_{ik})_1^n$. This norm is defined by

(4) $$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{k=1}^{n} |a_{ik}|.$$

2. Disconjugacy.

Theorem 1. *Let* D *be a simply connected domain not containing* $z = \infty$ *and assume that the analytic functions* $a_{ik}(z)$, $i,k = 1,\dots,n$, *are regular in* D. *Denote the maximal row norm of the matrix* $A(z) = (a_{ik}(z))_1^n$ *by* $\|A(z)\|_\infty$, $z \in D$. *Let* $z_o \in D$ *and assume that for every* $z \in D$, $z \neq z_o$, *there exists a path* $C(z_o,z)$ *in* D, *from* z_o *to* z, *so that for all these paths*

(3) $$\int_{z_o}^{z} \|A(\zeta)\|_\infty \, |d\zeta| < \log 2.$$

Then the differential system

(1) $$W'(z) = A(z)W(z)$$

is disconjugate in D.

Proof. Let $W(z) = (w_{ik}(z))_1^n$ be the fundamental solution of (1) satisfying the initial condition

(5) $$W(z_o) = I,$$

$(I = (\delta_{ik})_1^n)$. By the Peano-Baker method of solution, we have for any $z \in D$

(6)
$$W(z) - I = \int_{z_o}^{z} A(\zeta)\,d\zeta + \int_{z_o}^{z} A(\zeta)\int_{z_o}^{\zeta} A(\zeta_1)\,d\zeta_1 d\zeta + $$
$$+ \int_{z_o}^{z} A(\zeta)\int_{z_o}^{\zeta} A(\zeta_1)\int_{z_o}^{\zeta_1} A(\zeta_2)\,d\zeta_2 d\zeta_1 d\zeta + \dots .$$

The integral from z_o to z is taken along $C(z_o,z)$; $\zeta, \zeta_1, \dots$ are on $C(z_o,z)$ and the inner integrals are taken along the corresponding parts of $C(z_o,z)$.

We use now properties (I) to (V) of the maximal row norm. (6) thus implies

(7)
$$\|W(z) - I\|_\infty \le \left\|\int_{z_o}^{z} A(\zeta)\,d\zeta\right\|_\infty + \left\|\int_{z_o}^{z} A(\zeta)\int_{z_o}^{\zeta} A(\zeta_1)\,d\zeta_1 d\zeta\right\|_\infty + \dots$$
$$\le \int_{z_o}^{z} \|A(\zeta)\|_\infty |d\zeta| + \int_{z_o}^{z} \|A(\zeta)\|_\infty \int_{z_o}^{\zeta} \|A(\zeta_1)\|_\infty |d\zeta_1|\,|d\zeta| + \dots$$

But
$$\int_{z_o}^{z} \|A(\zeta)\|_\infty \int_{z_o}^{\zeta} \|A(\zeta_1)\|_\infty |d\zeta_1|\,|d\zeta| = \frac{1}{2!}\left(\int_{z_o}^{z} \|A(\zeta)\|_\infty |d\zeta|\right)^2,$$

and similar equalities hold for the following terms of the last sum in (7). Using assumption (3) we obtain

(8) $$\|W(z) - I\|_\infty < \log 2 + \frac{\log^2 2}{2!} + \dots = 1,$$

for all $z \in D$. We choose now n, not necessarily distinct, points z_i in D. Definition (4) of the maximal row norm and (8) give

$$\sum_{k=1}^{n} |w_{ik}(z_i) - \delta_{ik}| \le \|W(z_i) - I\|_\infty < 1, \quad i = 1,\ldots,n. \tag{9}$$

Denoting $\hat{W} = (w_{ik}(z_i))_1^n$, and using again (4), we obtain

$$\|\hat{W} - I\|_\infty < 1. \tag{10}$$

Property (VI) of norms gives $|\lambda_i(\hat{W} - I)| < 1$, $i = 1,\ldots,n$. As $\lambda_i(\hat{W} - I) = \lambda_i(\hat{W}) - 1$, we obtain $|\lambda_i(\hat{W}) - 1| < 1$ which implies $\lambda_i(\hat{W}) \neq 0$, $i = 1,\ldots,n$. Hence, $|\hat{W}| = |w_{ik}(z_i)|_1^n \neq 0$. As the points $z_1,\ldots,z_n$ were arbitrary in D, we thus proved that the system (1) is disconjugate in D.

We do not know whether the constant $\log 2$ on the right-hand side of (3) is the best possible constant. However, $\log 2 = 0.693$ cannot be replaced by any number larger than $\pi/4 = 0.785$. This follows by considering the system (1) which corresponds to the differential equation $y^{(n)}(z) + y^{(n-2)}(z) = 0$. The matrix $A(z)$ of (1) is now the constant matrix $A = (a_{ik})_1^n$ with $a_{i\,i+1} = 1$, $i = 1,\ldots,n-1$, $a_{n\,n-1} = -1$ and $a_{ik} = 0$ for all other elements. It follows that $\|A\|_\infty = 1$, but (1) is not disconjugate in any domain D containing the two points $z = \pi/4$ and $z = -\pi/4$ [7,§4].

We remark that the assumption and the conclusion of Theorem 1 are invariant under conformal mapping. Indeed, let $z = \varphi(w)$ map the domain Δ of the w-plane onto the given domain D of the z-plane, so that $z_o = \varphi(w_o)$. (1) transforms into

(11) $$V'(w) = B(w)V(w),$$

where $V(w) = W(\varphi(w))$ and $B(w) = \varphi'(w)A(\varphi(w))$. The path $C(z_o,z)$ in D is mapped onto the path $\Gamma(w_o,w)$ in Δ and

(12) $$\int_{z_o}^{z} \|A(\zeta)\|_\infty |d\zeta| = \int_{w_o}^{w} \|B(\omega)\|_\infty |d\omega|.$$

Assumption (3) is thus invariant under this mapping. On the other hand, (1) and (11) are together disconjugate or not disconjugate in their domains.

Our next result on disconjugacy will first be proved for the unit disk (Theorem 2), and we then use this invariance under conformal mapping to obtain its validity for arbitrary simply connected domains (Theorem 2'). We now bring some lemmas needed for the proof of these theorems. The first lemma is a result of Kim [3, Lemma 2.2].

Lemma 1. *Let the analytic functions* $a_{ik}(z)$, $i,k = 1,\dots,n$, *be regular in* $|z| < 1$, *and assume that the differential system*

(1) $$W'(z) = A(z)W(z)$$

$(A(z) = (a_{ik}(z))_1^n)$ *is not disconjugate in* $|z| < 1$. *Then there exist a constant* K, $0 < K < 1$, *and* n *points* z_i *satisfying* $|z_1| = |z_2| = \dots = |z_n| = K$, *such that for every solution* $W(z) = (w_{ik}(z))_1^n$ *of* (1) *the determinant* $|w_{ik}(z_i)|_1^n$ *vanishes*.

Using the obvious definition for 'disconjugacy on a line' this lemma thus states that _if the system (1) is disconjugate on every circle $|z| = r$, $0 < r < 1$, then it is disconjugate in the unit disk_. Kim brought this result in terms of the vector differential equation (2).

For the proof of Theorems 2 and 2', it would be sufficient to state the following lemmas only for the maximal row norm. These lemmas do, however, hold for all matrix norms. Even more is true: we do not use property (IV) of norms in their proof and the lemmas hold therefore for Ostrowski's _generalized norms_. These generalized norms, which we again denote by $\|A\|$, are real valued functions, defined for all $n \times n$ matrices A, satisfying properties (I), (II) and (III). We already stated that this implies (V) (continuity) [9; 2, p.60].

Lemma 2. _Let the analytic functions_ $a_{ik}(z)$, $i,k = 1,\ldots,n$, _be regular in a domain_ D. _Every generalized norm_ $\|A(z)\|$ _of the matrix_ $A(z) = (a_{ik}(z))_1^n$ _is a continuous subharmonic function in_ D.

Continuity of $\|A(z)\|$ as function of z follows from (V). Let $|z - z_o| \leq r$, $0 < r < \infty$, be a disk contained in D. Cauchy's integral formula gives

$$A(z_o) = \frac{1}{2\pi} \int_o^{2\pi} A(z_o + re^{i\varphi})\, d\varphi.$$

This and (I), (II) and (III) imply

$$\|A(z_o)\| \le \frac{1}{2\pi}\int_o^{2\pi}\|A(z_o + re^{i\varphi})\|d\varphi,$$

which such holds for all z_o and r for which $|z - z_o| \le r$ belongs to D. $\|A(z)\|$ is thus subharmonic in D [6, § 21].

The integral mean of a subharmonic function over concentric circles is a nondecreasing function of the radius. Lemma 2 thus yields

Lemma 3. _Let the analytic functions_ $a_{ik}(z)$, $i,k = 1,\dots,n$, _be regular in_ $|z| < 1$ _and let_ $\|A(z)\|$ _be a generalized norm of the matrix_ $A(z) = (a_{ik}(z))_1^n$. _Then_

$$I(r) = \int_o^{2\pi}\|A(re^{i\varphi})\|d\varphi, \qquad 0 \le r < 1, \tag{13}$$

is a nondecreasing function of r.

Having prepared everything for the proof of Theorem 2, we bring now a few remarks in order to obtain a concise statement of this theorem. We shall there assume that the elements $a_{ik}(z)$ of $A(z)$ are of class H_1 in $|z| < 1$; this means each analytic function $a_{ik}(z)$ is regular in $|z| < 1$ and the nondecreasing function $I_{ik}(r)$, defined by

$$I_{ik}(r) = \int_o^{2\pi}|a_{ik}(re^{i\varphi})|d\varphi, \qquad 0 \le r < 1, \tag{14}$$

is bounded as $r \to 1$, $i,k = 1,\dots,n$. $\lim_{r\to 1} I_{ik}(r)$ is usually denoted by $\int_o^{2\pi}|a_{ik}(e^{i\varphi})|d\varphi$. This, indeed, is more than a notation, but all we need is that if $a_{ik}(z)$ is regular in $|z| < 1$ and continuous in $|z| \le 1$, then $\lim_{r\to 1} I_{ik}(r) = I_{ik}(1)$ and $\int_o^{2\pi}|a_{ik}(e^{i\varphi})|d\varphi$

is then the Riemann integral of the continuous function $|a_{ik}(e^{i\varphi})|$. _We call_ $A(z) = (a_{ik}(z))_1^n$ _of class_ H_1 _in_ $|z| < 1$ _if each_ $a_{ik}(z)$ _is of class_ H_1 _in_ $|z| < 1$. In this case we use the analogous notation for the limit of the integral $I(r)$ of any generalized norm:

$$(15) \qquad \lim_{r \to 1} I(r) = \int_0^{2\pi} \|A(e^{i\varphi})\| d\varphi.$$

If each $a_{ik}(z)$ is continuous in $|z| \leq 1$, then $\lim_{r \to 1} I(r) = I(1)$ and the right-hand side of (15) is again a Riemann integral. To justify the notation (15) in the general case, it seems necessary to add the following statement. _For any generalized norm,_ $\lim_{r \to 1} I(r)$ _is finite if, and only if,_ $A(z)$ _is of class_ H_1 _in_ $|z| < 1$. We show this first for the maximal row norm. The inequalities

$$|a_{ik}(z)| \leq \|A(z)\|_\infty, \quad i,k = 1,\ldots,n, \quad \|A(z)\|_\infty \leq \sum_{i,k=1}^{n} |a_{ik}(z)|, \quad |z| < 1,$$

imply that

$$\lim_{r \to 1} I_\infty(r) = \lim_{r \to 1} \int_0^{2\pi} \|A(re^{i\varphi})\|_\infty d\varphi$$

is finite if, and only if, $A(z)$ is of class H_1 in $|z| < 1$. But the validity of the italicized statement for one generalized norm implies its validity for all generalized norms as the quotient of two generalized norms of the same matrix A lies between two positive constants which are independent of A [9, Satz IV; 2, p. 61].

After all these preparations (and digressions) we state now our next theorem.

Theorem 2. *Let* $A(z) = (a_{ik}(z))_1^n$ *be of class* H_1 *in* $|z| < 1$ *and assume that*

$$\int_0^{2\pi} \|A(e^{i\varphi})\|_\infty d\varphi \leq 2 \log 2. \tag{16}$$

Then the differential system

$$W'(z) = A(z)W(z) \tag{1}$$

is disconjugate in $|z| < 1$.

Proof. By the preceding remarks (16) is equivalent to

$$\lim_{r\to 1} \int_0^{2\pi} \|A(re^{i\varphi})\|_\infty d\varphi \leq 2 \log 2.$$

Lemma 3 implies that for each r, $0 < r < 1$,

$$\int_0^{2\pi} \|A(re^{i\varphi})\|_\infty r d\varphi = \int_{C_r} \|A(\zeta)\|_\infty |d\zeta| < 2 \log 2.$$

Here C_r denotes the circle $|z| = r$, $0 < r < 1$. We divide now C_r into two arcs C_r' and C_r'' both starting e.g. at $z_o = r$ and ending at $re^{i\psi}$, where $\psi = \psi(r)$, $0 < \psi < 2\pi$, is so chosen that

$$\int_{C_r'} \|A(\zeta)\|_\infty |d\zeta| < \log 2, \quad \int_{C_r''} \|A(\zeta)\|_\infty |d\zeta| < \log 2. \tag{3'}$$

Let $W(z) = (w_{ik}(z))_1^n$ be the fundamental solution of (1) satisfying the initial condition $W(z_o) = W(r) = I$. We choose now n, not necessarily distinct, points z_i on C_r. As each z_i, $i = 1,\ldots,n$, lies either on C_r' or on C_r'', we obtain, by using the series (6) for $W(z_i) - I$, that

$$\sum_{k=1}^n |w_{ik}(z_i) - \delta_{ik}| < 1, \quad |z_i| = r, \quad i = 1,\ldots,n.$$

This implies $|w_{ik}(z_i)|_1^n \neq 0$, which thus holds for any n points on the circle $|z| = r$. As this holds for each r, $0 < r < 1$, it follows from Lemma 1 that the system (1) is disconjugate in $|z| < 1$.

Using the invariance of the assumption and conclusion of this theorem under conformal mapping (see (11),(12)), we can state it for an arbitrary simply connected domain.

Theorem 2'. *Let the analytic functions* $a_{ik}(z)$, $i,k = 1,\ldots,n$, *be regular in the simply connected domain* D *not containing* $z = \infty$. *Let* C *be the boundary of* D *and* $A(z) = (a_{ik}(z))_1^n$. *If*

$$(16') \qquad \int_C \|A(\zeta)\|_\infty |d\zeta| \leq 2 \log 2,$$

then the system $W'(z) = A(z)W(z)$ *is disconjugate in* D.

If each $a_{ik}(z)$ is continuous in $\bar{D}$ and if C is piecewise smooth, then the integral on the left-hand side of (16') is a Riemann integral. If not, then this integral has to be interpreted as the limit, for $r \to 1$, of integrals taken along the level lines Γ_r, $0 < r < 1$, of the function $w = \varphi(z)$ which maps D onto $|w| < 1$; and it is thus assumed that the limit of this increasing function of r is not larger than $2 \log 2$.

We add a remark about the relation of Theorem 2' to Theorem 1. If the constant $2 \log 2$ on the right-hand side of (16') is replaced by $(2 \log 2)/n$, then the corresponding weaker assertion is a consequence of Theorem 1. This follows from a result of

Fejér and Riesz [1], stating that for every function $a(z)$ of class H_1 in $|z| < 1$, $a(z) \not\equiv 0$,

(17) $$\int_{-1}^{+1} |a(re^{i\theta})|\,dr < \frac{1}{2}\int_0^{2\pi} |a(e^{i\varphi})|\,d\varphi, \quad 0 \le \theta < \pi.$$

For a matrix $A(z)$ of class H_1 in $|z| < 1$, $A(z) \not\equiv 0$, it thus follows that

$$\int_0^1 \|A(re^{i\theta})\|_\infty dr \le \int_0^1 \sum_{i,k=1}^{n} |a_{ik}(re^{i\theta})|\,dr < \frac{1}{2}\int_0^{2\pi} \sum_{i,k=1}^{n} |a_{ik}(e^{i\varphi})|\,d\varphi$$

$$\le \frac{n}{2}\int_0^{2\pi} \|A(e^{i\varphi})\|_\infty d\varphi, \quad 0 \le \theta < 2\pi.$$

Hence, if we replace the assumption (16) of Theorem 2 by the more restringent assumption

(16'') $$\int_0^{2\pi} \|A(e^{i\varphi})\|_\infty d\varphi \le \frac{2}{n}\log 2,$$

then the assumption (3) of Theorem 1 is satisfied for the unit disk ($z_0 = 0$ and $C(0,z)$ is now the segment from 0 to z, $|z| < 1$). Using the invariance of the line integrals, it follows that the similarly weakened version of Theorem 2' is a consequence of Theorem 1. If the analogue of the Fejér-Riesz inequality (17) holds for the maximal row norm of matrices $A(z)$ of class H_1 in $|z| < 1$, or at least if

$$\int_0^1 \|A(re^{i\theta})\|_\infty dr < \frac{1}{2}\int_0^{2\pi} \|A(e^{i\varphi})\|_\infty d\varphi, \quad 0 \le \theta < 2\pi,$$

is true, then Theorem 2' is a consequence of Theorem 1.

In [3] Kim obtained sufficient conditions for disconjugacy by using the spectral norm $\|A(z)\|_2$. ($\|A\|_2 = \sup_{w\neq 0} \|Aw\|_2/\|w\|_2$, where $\|w\|_2$ is the Euclidean norm of the vector $w = [w_1,\ldots,w_n]$.) For the unit disk he obtained that *if, for all* r, $0 < r < 1$,

$$r \int_0^{2\pi} \|A(re^{i\varphi})\|_2 d\varphi < 1,$$

then the system (1) *is disconjugate in* $|z| < 1$. [3, Theorem 2.7]. By Lemma 3, this condition can be simplified to

$$\int_0^{2\pi} \|A(e^{i\varphi})\|_2 d\varphi \leq 1. \tag{18}$$

It follows that

$$\int_C \|A(\zeta)\|_2 |d\zeta| \leq 1 \tag{18'}$$

implies the disconjugacy of (1) in the simply connected domain D with boundary C. As for $n \times n$ matrices

$$\frac{1}{\sqrt{n}} \|A\|_\infty \leq \|A\|_2 \leq \sqrt{n}\ \|A\|_\infty, \tag{19}$$

these results neither imply Theorems 2 and 2' nor are they implied by them.

For systems defined on an interval the method of this section yields the following result.

Theorem 3. *Let the complex valued functions* $a_{ik}(x), i,k=1,\ldots,n$, *be continuous on* (a,b), $-\infty \leq a < b \leq \infty$. *Let* $W(x) = (w_{ik}(x))_1^n$ *be a fundamental solution of the differential system*

(1') $$W'(x) = A(x)W(x),$$

$(A(x) = (a_{ik}(x))_1^n)$, $a < x < b$. _If_

(20) $$\int_a^b \|A(x)\|_\infty dx < 2 \log 2,$$

then $|w_{ik}(x_i)|_1^n \neq 0$ _for every choice of_ n _points_ x_i _in_ (a,b).

Following a recent remark of Nehari [8], we avoid in this case, of systems on an interval of the real line, the term disconjugacy. For the proof, we remark that (20) implies the existence of a point x_o in (a,b) such that

$$\int_a^{x_o} \|A(x)\|_\infty dx < \log 2, \quad \int_{x_o}^b \|A(x)\|_\infty dx < \log 2.$$

For the solution $W(x) = (w_{ik}(x))_1^n$ of (1'), satisfying $W(x_o) = I$, we obtain, as before, $|w_{ik}(x_i)|_1^n \neq 0$ for every choice of n points x_i in (a,b). Theorem 3 should be compared with [3, Theorem 2.1] and with a recent sharp result of Nehari [8, Theorem 3.3], implying that the condition

$$\int_a^b \|A(x)\|_2 dx < \frac{\pi}{2}$$

yields the assertion of Theorem 3. We do not know whether Theorem 3 is sharp; the example brought after the proof of Theorem 1 shows only that $2 \log 2$ on the right-hand side of (20) cannot be replaced by any number larger than $\pi/2$.

3. z_o-_absolute disconjugacy_. This notion was defined as follows [7]: _The system_ (1) _is called_ z_o-_absolute disconjugate_

in D *if there exists a point* $z_o \epsilon D$ *such that the solution* $W(z) = (w_{ik}(z))_1^n$, *determined by*

(5) $$W(z_o) = I,$$

satisfies

(21) $$\lambda((|w_{ik}(z_{ik}) - \delta_{ik}|)_1^n) < 1,$$

for every choice of n^2 (*not necessarily distinct*) *points* z_{ik} *in* D. (The left-hand side of (21) is our notation for the maximal characteristic value of the nonnegative matrix $(|w_{ik}(z_{ik}) - \delta_{ik}|)_1^n$.) As already mentioned, z_o-absolute disconjugacy of (1) in D implies its ordinary disconjugacy there; the converse is, in general, not true. In the following theorem $\|P(x)\|$ will be an arbitrary norm of the matrix $P(x)$, satisfying conditions (I) to (IV).

Theorem 4. *Let the bounded domain* D *be starlike with respect to its point* z_o, *and assume that the analytic functions* $a_{ik}(z)$, $i,k = 1,\ldots,n$, *are regular in* D. *Let* $r = \sup|z - z_o|$, $z \epsilon D$. *For each* x, $0 \le x < r$, *denote by* $C(x)$ *the intersection of the circle* $|z - z_o| = x$ *and* D, *and assume that*

(22) $$m_{ik}(x) = \sup_{z \epsilon C(x)} |a_{ik}(z)| < \infty, \quad i,k = 1,\ldots,n, \quad 0 \le x < r.$$

Let the nonnegative functions $p_{ik}(x)$, $i,k = 1,\ldots,n$, *be continuous in* $0 \le x < r$, *and satisfy*

(23) $$p_{ik}(x) \ge m_{ik}(x), \quad i,k = 1,\ldots,n, \quad 0 \le x < r.$$

Let $\|P(x)\|$ *be any norm of the matrix* $P(x) = (p_{ik}(x))_1^n$, $0 \leq x < r$. *If*

(24) $$\int_0^r \|P(x)\| dx \leq \log 2,$$

then the differential system

(1) $$W'(z) = A(z)W(z)$$

$(A(z) = (a_{ik}(z))_1^n)$ *is* z_0*-absolute disconjugate in* D.

Proof. By property (V) $\|P(x)\|$ is a continuous function of x, $0 \leq x < r$. We may disregard the trivial system $W'(z) = 0$, and it then follows, by (22), (23) and (I), that $\|P(x)\| > 0$ for $0 < x < r$. The integral in (24) may have been an improper one, but (24) is now equivalent to

(24') $$\int_0^\rho \|P(x)\| dx < \log 2, \quad 0 \leq \rho < r,$$

and $\|P(x)\|$ is continuous on $[0,\rho]$. We consider the fundamental solution $U(x) = (u_{ik}(x))_1^n$, $0 \leq x < r$, of the real differential system

(25) $$U'(x) = P(x)U(x), \quad 0 \leq x < r,$$

satisfying the initial condition

(26) $$U(0) = I.$$

The Peano-Baker series for this solution is

(27) $$U(\rho) - I = \int_0^\rho P(\xi)d\xi + \int_0^\rho P(\xi) \int_0^\xi P(\xi_1)d\xi_1 d\xi + \ldots, \quad 0 \leq \rho < r.$$

Using properties (I) to (IV) of the norm, we obtain (cf. formulas (7) to (8) in the proof of Theorem 1)

$$\|U(\rho) - I\| \le \int_0^\rho \|P(\xi)\|d\xi + \int_0^\rho \|P(\xi)\| \int_0^\xi \|P(\xi_1)\|d\xi_1 d\xi + \dots$$
$$= \exp(\int_0^\rho \|P(\xi)\|d\xi) - 1, \qquad 0 \le \rho < r.$$

This and (24') imply

$$(28) \qquad \|U(\rho) - I\| < 1, \qquad 0 \le \rho < r.$$

By (27) the matrix $U(\rho) - I$, $0 \le \rho < r$, is nonnegative; (28) and (VI') give

$$(29) \qquad \lambda(U(\rho) - I) < 1, \qquad 0 \le \rho < r.$$

For any $z \in D$ we set $x = |z - z_o|$ and compare, term by term, the series (6) for $W(z) - I$, where now all the integrals are taken along segments, with the corresponding series (cf. (27)) for $U(x) - I$. Using (22) and (23) we obtain

$$(30) \qquad |w_{ik}(z) - \delta_{ik}| \le u_{ik}(x) - \delta_{ik}, \quad i,k = 1,\dots,n.$$

We choose now n^2 points z_{ik} in D, set $x_{ik} = |z_{ik} - z_o|$, $i,k = 1,\dots,n$, and denote $\rho = \max_{i,k=1,\dots,n} x_{ik}$. Using (30) and the fact, following from (27), that each element $u_{ik}(x) - \delta_{ik}$ of $U(x) - I$ is a nondecreasing function of x, we obtain

$$(31) \qquad |w_{ik}(z_{ik}) - \delta_{ik}| \le u_{ik}(\rho) - \delta_{ik}, \quad i,k = 1,\dots,n.$$

As for nonnegative matrices $A = (a_{ik})_1^n$ and $B = (b_{ik})_1^n$ the inequalities $a_{ik} \le b_{ik}$, $i,k = 1,\dots,n$, imply $\lambda(A) \le \lambda(B)$, it follows that (29) and (31) imply (21) and Theorem 4 is thus proved.

We remark that only the first part of this proof (up to formula (29)) is new. If we replace in the statement of this theorem the assumption (24) by assuming (29), or the equivalent inequality.

$$\lambda(U(\rho)) < 2, \quad 0 \leq \rho < r, \tag{29'}$$

then we obtain a minor modification of [7,Theorem 1]. Due to the variety of easily computed matrix norms, the verification of (24), involving only the majorant matrix $P(x)$ and not the solution $U(x)$ of (25), is a convenient way to establish (29').

We apply now Theorem 4 to systems in the unit disk.

Corollary 1. *Let the analytic functions* $a_{ik}(z)$, $i,k = 1,\ldots,n$, *be regular in* $|z| < 1$. *Let the nonnegative function* $h(x)$ *be continuous in* $0 \leq x < 1$ *and assume that*

$$\int_0^1 h(x)\,dx = \frac{1}{\beta} < \infty. \tag{32}$$

Assume that there exist constants b_{ik}, $i,k = 1,\ldots,n$, *such that*

$$|a_{ik}(z)| \leq b_{ik} h(|z|), \quad i,k = 1,\ldots,n, \quad |z| < 1. \tag{33}$$

Let $\lambda(B)$ *be the maximal characteristic value of the nonnegative matrix* $B = (b_{ik})_1^n$. *If*

(34) $$\lambda(B) < \beta \log 2,$$

<u>then the system</u> $W'(z) = A(z)W(z)$, $(A(z) = (a_{ik}(z))_1^n)$, <u>is</u> 0-<u>absolute disconjugate in</u> $|z| < 1$.

<u>Proof</u>. The sets $C(x)$ of Theorem 4 are now the circles $|z| = x$, $0 \leq x < 1$, and, by (33) we may use as elements of the majorant matrix $P(x)$ the functions

(35) $$p_{ik}(x) = b_{ik}h(x), \quad i,k = 1,\ldots,n, \quad 0 \leq x < 1.$$

To assure the 0-absolute disconjugacy of (1) in $|z| < 1$ we have to show that the present assumptions imply the existence of a norm such that the inequality

(24'') $$\int_0^1 \|P(x)\| dx \leq \log 2$$

holds. Property (II) and (35) imply that for every norm

(36) $$\|P(x)\| = h(x)\|B\|, \quad 0 \leq x < 1,$$

and we are thus looking for norms whose value for the fixed argument B is as small as possible. Property (VI') states that $\lambda(B)$ is a lower bound for the set of values of all norms at B. It is, however, known that this is the greatest lower bound: for the given matrix B, $B \geq 0$, and any $\epsilon > 0$, there always exists a particular norm such that $\|B\| \leq \lambda(B) + \epsilon$, [2,p. 46]. This and (34) give the existence of a norm such that

(37) $$\|B\| \leq \beta \log 2$$

holds. (32), (36) and (37) imply now the validity of (24'') for this particular norm and we thus proved Corollary 1.

Choosing

$$h(x) = \frac{1}{(1-x)^{1-\beta}}, \quad 0 \le x < 1, \qquad 0 < \beta \le 1, \tag{38}$$

we obtain that the uniform growth condition

$$|a_{ik}(z)| \le \frac{b_{ik}}{(1-|z|)^{1-\beta}}, \quad i,k = 1,\dots,n, \quad |z| < 1, \quad 0 < \beta \le 1, \tag{39}$$

and the inequality (34) imply 0-absolute disconjugacy of the system (1) in $|z| < 1$. We do not claim that for fixed β, $0 < \beta < 1$, condition (34) for the coefficients b_{ik} appearing in (39) could not be improved. However, these sufficient conditions for 0-absolute disconjugacy are approximately of the right order of growth. Indeed, no condition of the form

$$|a_{ik}(z)| \le \frac{b_{ik}}{(1-|z|)^{1+\epsilon}}, \quad i,k = 1,\dots,n, \quad |z| < 1, \quad \epsilon > 0,$$

can possibly imply ordinary disconjugacy of (1) in $|z| < 1$, however small the coefficients b_{ik} may be (if there exist two distinct indices such that $b_{ik}b_{ki} > 0$). This follows from a result of M. Lavie [4, Theorem 5] stating that

$$|a_{ik}(z)\,a_{ki}(z)| \le \frac{1}{(1-|z|^2)^2}, \quad i,k = 1,\dots,n, \ i \ne k, \ |z| < 1,$$

is a necessary condition for disconjugacy of (1) in $|z| < 1$. We also remark that the case $\beta = 1$ of (38), i.e., $h(x) = 1$, was known previously [7, Theorem 2]. The discussion there shows also that the constant $\log 2$ appearing on the right-hand side of (34), and hence also of (24), is sharp (for z_o-absolute disconjugacy). For ordinary disconjugacy the former example shows again that $\log 2$ cannot be replaced by any number larger than $\pi/4$.

Relying on former results [7, §3], we conclude with an application to n-th order differential equations. We repeat here that <u>strong disconjugacy</u> of the equation

$$y^{(n)}(z) + a_{n-1}(z)y^{(n-1)}(z) + \dots + a_o(z)y(z) = 0 \tag{40}$$

in a domain D implies both disconjugacy and disfocality of (40) in D. Using that 0-absolute disconjugacy of the system (1), corresponding to the equation (40), in the unit disk implies strong disconjugacy of (40) there, we obtain from Corollary 1 the following result.

<u>Corollary</u> 2. <u>Let the analytic functions</u> $a_\ell(z)$, $\ell = 0,\dots,n-1$, <u>be regular in</u> $|z| < 1$. <u>Let the function</u> $h(x)$ <u>be continuous in</u> $0 \le x < 1$ <u>and assume that</u>

$$h(x) \ge 1, \qquad 0 \le x < 1, \tag{41}$$

<u>and that</u>

$$\int_o^1 h(x)\,dx = \frac{1}{\beta} < \infty . \tag{32}$$

<u>Assume that there exist constants</u> b_ℓ, $\ell = 0,\dots,n-1$, <u>such that</u>

$$|a_\ell(z)| \le b_\ell h(|z|), \qquad \ell = 0,\dots,n-1, \quad |z| < 1. \tag{33'}$$

<u>If</u>

(34') $b_{n-1}(\beta \log 2)^{n-1} + b_{n-2}(\beta \log 2)^{n-2} + \ldots + b_o < (\beta \log 2)^n$,

then the differential equation (40) *is strong disconjugate in* $|z| < 1$.

Using again (38), we obtain that

(39') $$|a_\ell(z)| \leq \frac{b_\ell}{(1 - |z|)^{1-\beta}}, \quad \ell = 0, \ldots, n-1, \ |z| < 1, \quad 0 < \beta \leq 1,$$

and the inequality (34') imply strong disconjugacy of the equation (40) in $|z| < 1$. These uniform growth conditions are probably too restringent. In view of necessary conditions, recently obtained by Lavie, both for disconjugacy [5, Theorem 2] and for disfocality [4, Theorem 7] of the equation (40) in $|z| < 1$, better sufficient conditions, perhaps of the form

$$|a_\ell(z)| \leq \frac{c_\ell}{(1 - |z|)^{n-\ell-\beta}}, \ \ell = 0, \ldots, n-1, \ |z| < 1, \ 0 < \beta,$$

may be expected to hold.

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