BOUNDS FOR THE GIRTH OF SPHERES

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Let X be a real normed linear space with norm || ||, and let Σ be its unit ball, with the boundary $\partial\Sigma$. Assume dim X $\geq$ 2. These notations and assumptions will be maintained throughout the paper. In [1] we defined the girth of Σ to be 2m(X), where m(X) = inf{δ(-p,p): p$\in\partial\Sigma$} and δ denotes the inner metric of $\partial\Sigma$ induced by the norm or, equivalently, m(X) = inf{L(c): c a rectifiable curve in $\partial\Sigma$ with antipodal endpoints}. If dim X < ∞, then these infima are attained, and m(X) $\geq$ 2 [1; Lemma 5.1, Theorem 5.5]. The purpose of this paper is to sharpen this inequality to m(X) $\geq 2(1+n^{-1})$, where n = dim X, and to remark that this bound is best possible when dim X is even.

In [4] the following property of a space X was defined, for a given positive integer n and a given real p, 0 < p < 1:

$(J_{n,p})$: There exist $x_k \in \Sigma$, k = 1,...,n such that $\|\sum_1^n \epsilon_k x_k\| > pn$ for every sequence (ϵ_k), $\epsilon_k = \pm 1$, k = 1,...,n, such that every -1, if any, precedes each +1, if any.

1. Lemma ([4; Theorem 3.2]). If $m(X) < 2/p^{-1}$, then X satisfies $(J_{n,p})$.

Proof. Under the assumption, there exists a rectifiable curve in $\partial\Sigma$ with endpoints, say, $-p_0, p_0$ and length $\ell < 2p^{-1}$. Let g: [0,1] $\to \partial\Sigma$ be its parametrization in terms of arc-length. For a given positive integer n, set $p_k = g(kn^{-1}\ell)$, k = 0,...,n so that $p_0+p_n = -p+p = 0$. Set $x_k = \ell^{-1}n(p_k-p_{k-1})$, k = 1,...,n. Then $\|x_k\| = \ell^{-1}n\|g(kn^{-1}\ell)-g((k-1)n^{-1}\ell)\| \leq 1$, and $\|-\sum_1^j x_i + \sum_{j+1}^n x_i\| = \ell^{-1}n\|-p_j+p_0+p_n-p_j\| = 2\ell^{-1}n\|p_j\| = 2\ell^{-1}n > pn$, j = 0,...,n, and

$(J_{n,p})$ holds.

2. <u>Theorem</u>. <u>If</u> $\dim X = n < \infty$, <u>then</u> $m \ge 2(1+n^{-1})$.

<u>Proof</u>. Assume that $m(X) < 2(1+n^{-1})$; by Lemma 1, X satisfies $(J_{n+1,\,n(n+1)^{-1}})$. Let then $x_k \in \Sigma$, $k = 0,\dots,n$, be such that

(1) $$\Big\|-\sum_0^{j-1} x_k + \sum_j^n x_k\Big\| > n(n+1)^{-1}(n+1) = n, \quad j = 0,\dots,n+1.$$

Since $\dim X = n$, there exist real numbers a_k, $k = 0,\dots,n$, not all 0, such that

(2) $$\sum_0^n a_k x_k = 0.$$

We may assume without loss that

(3) $$\max_k |a_k| = 1$$

and that, say, $|a_h| = 1$ for some h, $0 \le h \le n$. Then

$$\Big|-\sum_0^{h-1} a_k + \sum_h^n a_k\Big| + \Big|-\sum_0^{h} a_k + \sum_{h+1}^n a_k\Big| \ge \Big|\Big(-\sum_0^{h-1} a_k + \sum_h^n a_k\Big) - \Big(-\sum_0^{h} a_k + \sum_{h+1}^n a_k\Big)\Big| = 2|a_h| = 2.$$

Setting $j = h$ or $j = h+1$ and replacing <u>every</u> a_k by $-a_k$ if necessary, we may consequently assume, without invalidating (2), (3), that

(4) $$-\sum_0^{j-1} a_k + \sum_j^n a_k \ge 1 \quad \text{for some } j,\ 0 \le j \le n+1.$$

Combining (1) for that value of j with (2), (3), (4), we obtain

$$n < \Big\|-\sum_0^{j-1} x_k + \sum_j^n x_k\Big\| = \Big\|-\sum_0^{j-1} (1+a_k)x_k + \sum_j^n (1-a_k)x_k\Big\| \le \sum_0^{j-1}(1+a_k) + \sum_j^n (1-a_k) \le n+1-1 = n,$$

a contradiction.

In [2] it was shown that, if $\dim X = n$ and j is a parallelotope (i.e., X is congruent to ℓ_n^∞), then $m(X) = 2(1+(n-1)^{-1})$; it was further shown that, if n is odd, there is a subspace Y of co-dimension 1 such that $m(Y)$ is still $2(1+(n-1)^{-1})$. (We remark that exactly the same results obtain if X is taken to be congruent to ℓ_n^1 instead of to ℓ_n^∞, but we omit the proof.) Thus Theorem 2 yields the best lower bound for even dimension.

This conclusion is best stated in terms of $m_*(n) = \min\{m(X)\colon \dim X = n\}$, $n = 2,3,\dots$, a sequence of numbers introduced (and shown to exist) in [1].

3. *Theorem*. $m_*(n) = 2(1+n^{-1})$ *if* n *is even*, $2(1+n^{-1}) \le m_*(n) \le 2(1+(n-1)^{-1})$ *if* n *is odd*.

Proof. Theorem 2 and [2; Theorem 7].

This theorem confirms one-half of the conjecture at the end of [2]; the other half, asserting that $m_*(n) = 2(1+(n-1)^{-1})$ when n is odd, has so far only been confirmed for $n = 3$ (see [3]).

References.

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