NONLINEAR EIGENVALUE PROBLEMS AND

CRITICAL POINTS OF FUNCTIONS

James M. Greenberg

Report 68-39

1. Introduction

We here study the nonlinear eigenvalue problem

$$A\,\underset{\sim}{x} + \underset{\sim}{F}(\underset{\sim}{x}) = \lambda \underset{\sim}{x} \tag{1.1}$$

where $A : R^n \to R^n$ is self adjoint and linear and $\underset{\sim}{F}(\cdot)$ is the gradient of a potential Ψ; i.e.

$$\underset{\sim}{F}\,(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}} \Psi(\underset{\sim}{x}) \ . \tag{1.2}$$

It is well known that the nontrivial solutions of (1.1) of fixed amplitude r (i.e. $\underset{\sim}{x} \cdot \underset{\sim}{x} = r^2$) are the critical points of $\phi(\underset{\sim}{x}) \equiv \frac{1}{2} A\, \underset{\sim}{x} \cdot \underset{\sim}{x} + \Psi(\underset{\sim}{x})$ on $\underset{\sim}{x} \cdot \underset{\sim}{x} = r^2$. Moreover, if $\underset{\sim}{y}$ is such a critical point, then the eigenvalue $\lambda(\underset{\sim}{y})$ is given by

$$\lambda(\underset{\sim}{y}) = \frac{(A\underset{\sim}{y} + \nabla_{\underset{\sim}{x}} \Psi(\underset{\sim}{y})) \cdot \underset{\sim}{y}}{r^2} \ . \tag{1.3}$$

It is no loss in generality to assume that

$$\frac{1}{2} A\underset{\sim}{x} \cdot \underset{\sim}{x} = \sum_{i=1}^{n} \lambda_i x_i^2 \ ; \tag{1.4}$$

in this case ϕ takes the form

$$\phi(\underset{\sim}{x}) \equiv \sum_{i=1}^{n} \lambda_i\, x_i^2 + \Psi(\underset{\sim}{x}) \ . \tag{1.5}$$

Our interest is in showing that if appropriate conditions are met, then the nontrivial solutions of (1.1) (eigenvectors of $A + F$) may be parameterized smoothly by r.

We also obtain results about the maximal extension of a given branch of eigenvectors.

2. Statement of Results

Let ϕ be given by (1.4) and assume that the numbers $\{\lambda_j,\ j = 1,2,\ldots,n\}$ are distinct and indexed in decreasing order $(\lambda_j > \lambda_{j+1})$. In addition assume that the map $\underset{\sim}{x} \to \Psi(\underset{\sim}{x}) : R^n \to R$ is smooth (C^3 will suffice) and satisfies

$$(\sum_{|\alpha|=j} |D^\alpha \Psi(\underset{\sim}{x})|^2)^{1/2} \leq K\|\underset{\sim}{x}\|^{3-j}, \quad j = 0,1,2 . \tag{2.1}$$

In (2.1) D^α stands for any derivative of order j and $\|\cdot\|$ for the Euclidean norm.

The assumption that the λ_j's are distinct implies that the vectors

$$\pm r\, \underset{\sim}{e}_j, \quad \underset{\sim}{e}_j = (\overbrace{0,\ldots,0}^{j-1},\overset{j}{1},\overbrace{0,\ldots,0}^{n-j}), \quad j = 1,2,\ldots,n,$$

are the unique critical points of $\phi_o \equiv \sum_{i=1}^{n} \lambda_i x_i^2$ on $\underset{\sim}{x}\cdot\underset{\sim}{x} = r^2$.

For each $0 < \epsilon < 1$ we let

$$\eta_j^{+(-)}(1,\epsilon) = \left\{ \underset{\sim}{v} \;\middle|\; v_j = \underset{(-)}{+}\sqrt{1 - \sum_{\substack{k=1\\k\neq j}}^{n} v_k^2}\ ,\ \sum_{\substack{k=1\\k\neq j}}^{n} v_k^2 \leq \epsilon^2 \right\}.^{*} \tag{2.2}$$

For our purposes we will want two numbers $0 < \epsilon_1 < \epsilon_2 < 1$ such that

---

*or equivalently $\eta_j^{+(-)}(1,\epsilon) \equiv \left\{ \underset{\sim}{v} \;\middle|\; \|\underset{\sim}{v}\|^2 = 1\ ,\ \|\underset{\sim}{v} \underset{(+)}{-} \underset{\sim}{e}_j\|^2 \leq \epsilon^2 \right\}$

the neighborhoods $\{\eta_j^{+(-)}(1,\epsilon_1)\ ,\ j = 1,2,\dots,n\}$ are disjoint on the unit sphere while the neighborhoods $\{\eta_j^{+(-)}(1,\epsilon_2),\ j = 1,2,\dots,n\}$ cover the unit sphere.

Theorem 1 (Local Existence and Uniqueness Theorem)

There is an $r_o > o$ such that for any $r \in (o, r_o]$ the function ϕ has exactly $2n$ critical points on the sphere $\underset{\sim}{x} \cdot \underset{\sim}{x} = r^2$. These points may be labeled in pairs $(\underset{\sim}{x}_j^+(r),\ \underset{\sim}{x}_j^-(r)) \equiv r\,(\underset{\sim}{v}_j^+(r),\ \underset{\sim}{v}_j^-(r))$ according to the scheme

$$\underset{\sim}{v}_j^+(r) \in \eta_j^+(1,\epsilon_1) \quad \text{and} \quad \underset{\sim}{v}_j^-(r) \in \eta_j^-(1,\epsilon_1)\ . \tag{2.3}$$

The functions $r \to \underset{\sim}{v}_j^+(r)$ (respectively $r \to \underset{\sim}{v}_j^-(r)$) are $C^1(0 < r \leq r_o)$ and satisfy

$$\lim_{r\to o} \underset{\sim}{v}_j^+(r) = \underset{\sim}{e}_j \quad \text{and} \quad \lim_{r\to o} \underset{\sim}{v}_j^-(r) = -\underset{\sim}{e}_j\ ,\ j = 1,2,\dots,n. \tag{2.4}$$

In order to state the global existence theorem it is necessary to introduce some additional notation. For each $\underset{\sim}{v} \ni \|\underset{\sim}{v}\|^2 = 1$ we let $V(\underset{\sim}{v})$ be the $n-1$ dimensional vector space

$$V(\underset{\sim}{v}) \equiv \{\underset{\sim}{u} \in R^n \mid \underset{\sim}{u} \cdot \underset{\sim}{v} = 0\} \tag{2.5}$$

For each $r > 0$ and $\underset{\sim}{v} \ni \|\underset{\sim}{v}\|^2 = 1$ we define the symmetric bilinear form $B(r\underset{\sim}{v}\ ;\ \cdot,\cdot) : V(\underset{\sim}{v}) \times V(\underset{\sim}{v}) \to R$ by

$$B(r\underset{\sim}{v}\ ;\ \underset{\sim}{u},\underset{\sim}{w}) \equiv \frac{1}{r^2}\ \frac{\partial^2}{\partial s \partial t}\ \phi\,(r(\underset{\sim}{v} + s\underset{\sim}{u} + t\underset{\sim}{w}))\Big|_{s=t=0}$$

$$- \frac{1}{r}\,(\nabla_{\underset{\sim}{x}}\phi(r\underset{\sim}{v}) \cdot \underset{\sim}{v})\ (\underset{\sim}{u}\cdot\underset{\sim}{w}). \tag{2.6}$$

$\mathcal{B}(r\underset{\sim}{v}): V(\underset{\sim}{v}) \to V(\underset{\sim}{v})$ is the linear operator associated with the bilinear form $B(r\underset{\sim}{v}\ ;\ \cdot,\cdot)$.*

<u>Theorem 2</u> (Global Existence of a Given Branch of Critical Points)

For each j it is possible to extend the function $r \to \underset{\sim}{v}_j^+(r)$ from $[o,r_o]$ to some maximal interval $[o,R_j^+)$ in such a way that the function $\underset{\sim}{x}_j^+(r) \equiv r\, \underset{\sim}{v}_j^+(r)$ is a critical point of ϕ (or $\underset{\sim}{x}\cdot\underset{\sim}{x} = r^2$). The function $\underset{\sim}{v}_j^+(\cdot)$ is extended as the unique solution of the initial value problem:

$$\begin{aligned} &\mathcal{B}(r\underset{\sim}{v})\ \dot{\underset{\sim}{v}}(r) \equiv \underset{\sim}{F}(r,\underset{\sim}{v})\ ,\ r > r_o \\ &\qquad \underset{\sim}{v}(r_o) \equiv \underset{\sim}{v}_j^+(r_o) \end{aligned} \tag{2.7}$$

The initial data $r_o\, \underset{\sim}{v}_j^+(r_o)$ is the unique critical point of ϕ on $\underset{\sim}{x}\cdot\underset{\sim}{x} = r_o^2$ such that $\underset{\sim}{v}_j^+(r_o) \in \eta_j^+(1,\epsilon)$, and $\underset{\sim}{F}(r,\underset{\sim}{v}) \in V(\underset{\sim}{v})$ is defined by

$$F(r,\underset{\sim}{v}) \equiv \frac{\partial}{\partial r}\left\{\frac{1}{r}\left[\nabla_{\underset{\sim}{x}}\ \Psi(r\,\underset{\sim}{v}) - (\nabla_{\underset{\sim}{x}}\ \Psi(r\,\underset{\sim}{v}) \cdot \underset{\sim}{v})\,\underset{\sim}{v}\right]\right\}.$$

The number R_j^+ is characterized as the first $r > 0$ such that the quadratic form $B(r\underset{\sim}{v}_j^+(r);\ \underset{\sim}{u},\underset{\sim}{u})$ has zero as a critical value on the unit sphere $V(\underset{\sim}{v}_j^+(r))$. For all $r < R_j^+$ the quadratic has $j-1$ positive and $n-j$ negative critical values on the unit sphere $V(\underset{\sim}{v}_j^+(r))$.

The following example shows that Theorems 1 and 2 are the best that may be expected.

---

* For any $\underset{\sim}{u} \in V(\underset{\sim}{v})$ $\ B(r\underset{\sim}{v})\underset{\sim}{u} = [\nabla_{\underset{\sim}{x}}\nabla_{\underset{\sim}{x}}\phi(r\underset{\sim}{v})]\underset{\sim}{u} - (\underset{\sim}{u}\cdot[\nabla_{\underset{\sim}{x}}\nabla_{\underset{\sim}{x}}\phi(r\underset{\sim}{v})]\,\underset{\sim}{v})\underset{\sim}{v}$

$$- \frac{1}{r}\,(\nabla_{\underset{\sim}{x}}\phi(r\underset{\sim}{v})\cdot\underset{\sim}{v})\underset{\sim}{u}\ .$$

Let

$$\phi_o = x^2 + 2y^2 \quad ,$$

$$\Psi = \frac{2}{3} x^3 - (x^2 + y^2)(x^2 + 2y^2) \quad ,$$

and

$$\phi = \phi_o + \Psi \ .$$

If we introduce the polar coordinates

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta \ ,$$

then $\eta(r,\theta) \equiv \frac{\phi}{r^2} (r \cos \theta, r \sin\theta)$ takes the form

$$\eta(r,\theta) \equiv (1-r^2)\ (\cos^2 \theta + 2 \sin^2\theta) + \frac{2}{3} r \cos^3 \theta \ .$$

A simple computation shows that the critical points of η on $x^2 + y^2 = r^2$ are those numbers $\theta \in [0,2\pi)$ which satisfy

$$\frac{\partial\eta}{\partial\theta} (r,\theta) = \sin 2\theta \ (1- r^2 - r \cos \theta) = 0 \ .$$

It is clear that for all $r > 0$ the numbers $\theta = 0, \pi/2$, π, and $3\pi/2$ are critical points of η. For $\frac{\sqrt{5} - 1}{2} < r < \frac{\sqrt{5} + 1}{2}$ there are two additional critical points $\theta_1 \in (0,\pi)$ and $\theta_2 \in (\pi,2\pi)$ which satisfy

$$\cos \theta_i = \frac{1 - r^2}{r} \quad , \quad \frac{\sqrt{5} - 1}{2} < r < \frac{\sqrt{5} + 1}{2} \quad , \quad i = 1,2 \ .$$

Finally, for $r > \frac{\sqrt{5} + 1}{2}$ the numbers $\theta = 0, \frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$ are again the only critical points.

We now analyze these critical points in more detail. For $0 < r < \sqrt{\frac{5-1}{2}}$ the points $\theta = 0$ and π correspond to relative minima of η while $\theta = \pi/2$ and $3\pi/2$ correspond to relative maxima. At $r = \sqrt{\frac{5-1}{2}}$ (the point where $\left.\frac{\partial^2 \eta}{\partial\theta^2}\right|_{\theta = 0} = 0$) the point $\theta = 0$ becomes an inflection point, and for all $r > \sqrt{\frac{5+1}{2}}$ is a maximum.

For $\sqrt{\frac{5-1}{2}} < r < 1$ the points $\theta_1 \in (0, \pi/2)$ and $\theta_2 \in (3\pi/2, 2\pi)$ correspond to relative minima of η and the character of the points $\theta = \frac{\pi}{2}, \pi,$ and $3\pi/2$ is as before. At $r = 1$ (the point where

$\left.\frac{\partial^2 \eta}{\partial\theta^2}\right|_{\theta = \pi/2} = \left.\frac{\partial^2 \eta}{\partial\theta^2}\right|_{\theta = 3\pi/2} = 0$) the point θ_1 coalesces with $\pi/2$ and θ_2 with $3\pi/2$. For $1 < r < \sqrt{\frac{5+1}{2}}$ $\theta = \frac{\pi}{r}$ and $\frac{3\pi}{2}$ become relative minima, the points $\theta_1 \in (\pi, 3\pi/2)$ relative maxima. $\theta = \pi$ is still a relative minimum. At $r = \sqrt{\frac{5+1}{2}}$ (the place where

$\left.\frac{\partial^2 \eta}{\partial\theta^2}\right|_{\theta = \pi} = 0$) $\theta_1 = \theta_2 = \pi$ is an inflection point of η. For $r > \sqrt{\frac{5+1}{2}}$ the critical points $\theta = 0$ and π correspond to relative maxima of η while $\theta = \pi/2$ and $3\pi/2$ correspond to relative minima.

3. Proofs

To establish Theorem 1 we look at ϕ in neighborhoods of the critical points of $\phi_o \equiv \sum_{i=1}^{n} \lambda_i x_i^{\,2}$ on $\underset{\sim}{x} \cdot \underset{\sim}{x} = r^2$. Clearly, it suffices to look at ϕ in a neighborhoods of $r\underset{\sim}{e}_1$.

We now let $0 < \epsilon_1 < \epsilon_2 < 1$ be two numbers such that the neighborhoods $\eta_j^{+(-)}(1,\epsilon_1)$, $j = 1,2,\ldots,n$ are disjoint on $\underset{\sim}{v} \cdot \underset{\sim}{v} = 1$ while the neighborhoods $\eta_j^{+(-)}(1,\epsilon_2)$, $j = 1,2,\ldots,n$ cover $\underset{\sim}{v} \cdot \underset{\sim}{v} = 1$. We set $\underset{\sim}{x} = r\underset{\sim}{v}$, $\underset{\sim}{v} \equiv (v_1, v_2, \ldots, v_n)$ and introduce local coordinates:

$$v_1 = \sqrt{1 - \sum_{k=2}^{n} v_k^{\,2}}$$

where $\sum_{k=2}^{n} v_k^{\,2} \leq \epsilon_2^{\,2}$. (3.1)

The function ϕ becomes $r^2 \hat{\phi}\,(r,\underset{\sim}{v})$ where

$$\hat{\phi}\,(v_2, v_3, \ldots, v_n; r) \equiv [\lambda_1 + \sum_{k=2}^{n} \mu_k^1 v_k^{\,2}]$$

$$+ \frac{1}{r^2}\,\Psi(r\sqrt{1 - \sum_{k=2}^{n} v_k^2}\;, rv_2, \ldots, rv_n) \,, \qquad (3.2)$$

and $\mu_k^1 = \lambda_k - \lambda_1$.

To obtain Theorem 1 it suffices to show that for r sufficiently small ($\leq$ some r_o)

(A) There exists a unique n-1 tupple $(v_2^+, v_3^+, \ldots, v_n^+)$ with

$\sum_{k=2}^{n} v_k^2 \leq \epsilon_1^{\,2}$ such that

$$\frac{\partial\hat{\phi}}{\partial v_i}(v_2,v_3,\ldots,v_n;r)$$

$$= \left\{ 2\mu_i^1 - \frac{\Psi_{x_1}\left(r\sqrt{1-\sum_{k=2}^{n} v_k^2},\ rv_2,\ \ldots,\ rv_n\right)}{r\sqrt{1-\sum_{k=2}^{n} v_k^2}} \right\} v_i$$

$$+ \frac{1}{r}\ \Psi_{x_i}\left(r\sqrt{1-\sum_{k=2}^{n} v_k^2},\ rv_2,\ \ldots,\ rv_n\right) \equiv 0,\ i = 2,3,\ldots,n\ . \qquad (3.3)$$

(B) The (n-1) tupple $(v_2^+,v_3^+,\ldots,v_n^+)$ satisfying (3.3) is the only solution in the larger sphere $\sum_{k=1}^{n} v_k^2 \leq \epsilon_2^2$; and

(C) the map $r \to (v_2^+(r),v_3^+(r),\ldots,v_n^+(r))$ is $C^1(0 < r \leq r_o)$ and satisfies

$$\lim_{r\to o}\ (v_2^+(r),v_3^+(r),\ldots,v_n^+(r)) = (0,0,\ldots,0). \qquad (3.4)$$

The growth condition on Ψ implies that

$$\left| \frac{1}{r}\ \frac{\Psi_{x_1}\left(r\sqrt{1-\sum_{k=2}^{n} v_k^2},\ rv_2,\ \ldots,\ rv_n\right)}{\sqrt{1-\sum_{k=2}^{n} v_k^2}} \right| \leq \frac{K\ r}{\sqrt{1-\epsilon_2^2}} \qquad (3.5)$$

and

$$\left(\sum_{i=2}^{n} \left| \frac{\Psi_{x_i}\left(r\sqrt{1-\sum_{k=2}^{n} v_k^2},\ rv_2,\ \ldots, rv_n\right)}{r} \right|^2 \right)^{1/2} \leq K\, r \quad (3.6)$$

for all $(v_2, v_3, \ldots, v_n)$ satisfying

$$\sum_{k=2}^{n} v_k^{\,2} \leq \epsilon_2^{\,2} \quad . \qquad (3.7)$$

Equation (3.5) implies that if

$$r \leq r_1 \equiv \frac{\min_i |\mu_i^1| \sqrt{1-\epsilon_2^{\,2}}}{K} = \frac{|\mu_2^{\,1}| \sqrt{1-\epsilon_2^{\,2}}}{K} , \qquad (3.8)$$

and if (3.7) holds, then the operator

$$P \equiv \operatorname{diag}\,(p_1, p_2, \ldots, p_n)$$

with

$$p_i = 2\mu_i^1 - \frac{\Psi_{x_1}\left(r\sqrt{1-\sum_{k=2}^{n} v_k^{\,2}},\ rv_2, \ldots, rv_n\right)}{r}$$

is invertible and

$$\|P^{-1}\| \leq \frac{1}{|\mu_2^1|} \qquad (3.9)$$

It now follows that solving (3.3) is equivalent to solving

$$v_i \equiv T_i(v_2,v_3,\ldots,v_n;r),\ i = 2,3,\ldots,n \tag{3.10}$$

where

$$T_i\ (v_2,v_3,\ldots,v_n;r)$$

$$\equiv \frac{1}{2\mu_i^1 - \frac{\Psi_{x_1}}{r}} \Psi_{x_i}\left(r\sqrt{1 - \sum_{k=2}^{n} v_k^2}\ ,\ rv_2,\ldots,rv_n\right) \tag{3.11}$$

We now observe that for $r \leq r_1$ and $(v_2,v_3,\ldots,v_n)$ satisfying (3.7)

$$\left(\sum_{k=2}^{n} T_k^2\right)^{1/2} \leq \frac{K\ r}{|\mu_2^1|}\ . \tag{3.12}$$

Equation (3.12) implies that for

$$r \leq \min\left(r_1,\ \frac{|\mu_2^1|\ \epsilon_1}{K}\right) \equiv r_2 \tag{3.13}$$

$$T(\cdot,r)\ :\ \sum_{k=2}^{n} v_k^2 \leq \epsilon_2^2 \rightarrow \sum_{k=2}^{n} v_k^2 \leq \epsilon_1^2\ ; \tag{3.14}$$

hence for $r \leq r_2$ any fixed point of (3.10) in $\sum_{k=2}^{n} v_k^2 \leq \epsilon_2^2$ must be in $\sum_{k=2}^{n} v_k^2 \leq \epsilon_1^2$. The smoothness of Ψ implies that $T(\cdot,r)$ is C^1 in $(v_2,v_3,\ldots,v_n)$ and hence Browers Theorem guarantees (for all $r \leq r_2$) the existence of at least one solution of (3.10)(and hence (3.3)) in $\sum_{k=2}^{n} v_k^2 \leq \epsilon_2^2$.

To establish uniqueness, it suffices to show that for some $r_o \leq r_2$ and all r in $[o, r_o]$ the maps $T(\cdot, r)$ are contractions on $\sum_{k=2}^{n} v_k^2 \leq \epsilon_2^2$. This computation follows from (2.1).

The smoothness of $r \to \underset{\sim}{v}_1(r) \equiv \left(\sqrt{1 - \sum_{k=2}^{n} v_k^2(r)}, v_2(r), \ldots, v_n(r)\right)$ follows from the smoothness of $\underset{\sim}{x} \to \psi(\underset{\sim}{x})$. We find that for $o < r \leq r_o$ $\dot{v}_i(r) \equiv \frac{d}{dr} v_i(r)$ exists, is continuous, and satisfies

$$\sum_{j=2}^{n} B_{ij}(v_2, v_3, \ldots, v_n, r)\, \dot{v}_j(r) = F_i(v_2, v_3, \ldots, v_n; r), \quad i = 2,3,\ldots,n, \quad (3.15)$$

where

$$B_{ij}(v_2, v_3, \ldots, v_n; r)$$

$$= \left\{ 2\mu_i^1 - \frac{\psi_{x_1}}{r\sqrt{1 - \sum_{k=2}^{n} v_k^2}} \right\} \left(r\sqrt{1 - \sum_{k=2}^{n} v_k^2}, rv_2, \ldots, rv_n\right) \delta_{ij}$$

$$- \frac{\{\psi_{x_1 x_j} v_i + \psi_{x_1 x_i} v_j\}}{\sqrt{1 - \sum_{k=2}^{n} v_k^2}} \left(r\sqrt{1 - \sum_{k=2}^{n} v_k^2}, rv_2, \ldots, rv_n\right)$$

$$+ \frac{v_i v_j}{(1 - \sum_{k=2}^{n} v_k^2)^{3/2}} \left\{ \sqrt{1 - \sum_{k=2}^{n} v_k^2}\, \Psi_{x_1 x_1} - \frac{\Psi_{x_1}}{r} \right\} (r\sqrt{1 - \sum_{k=2}^{n} v_k^2},\ rv_2, \ldots, rv_n)$$

$$+ \Psi_{x_i x_j} (r\sqrt{1 - \sum_{k=2}^{n} v_k^2},\ rv_2, \ldots, rv_n) \ ;\ i,j = 2,3,\ldots,n\ , \qquad (3.16)$$

and

$$F_i(v_2, v_3, \ldots, v_n;\ r) \equiv$$

$$\frac{1}{r^2} \left\{ \Psi_{x_i} - \frac{\Psi_{x_1} v_i}{\sqrt{1 - \sum_{k=2}^{n} v_k^2}} \right\} (r\sqrt{1 - \sum_{k=2}^{n} v_k^2},\ rv_2, \ldots, rv_n)$$

$$\frac{v_i}{r} \left\{ \Psi_{x_1 x_1} + \frac{\sum_{j=2}^{n} \Psi_{x_1 x_j} v_j}{\sqrt{1 - \sum_{k=2}^{n} v_k^2}} \right\} (r\sqrt{1 - \sum_{k=2}^{n} v_k^2},\ rv_2, \ldots, rv_n)$$

$$+ \frac{1}{r} \left\{ \sqrt{1 - \sum_{k=2}^{n} v_k^2}\, \Psi_{x_1 x_i} + \sum_{j=2}^{n} \Psi_{x_i x_j} v_j \right\} (r\sqrt{1 - \sum_{k=2}^{n} v_k^2},\ rv_2, \ldots, rv_n),$$

$$i = 2,3,\ldots,n. \qquad (3.17)$$

To obtain the limiting relation (3.4) we simply make use of the estimate (3.12).

To establish theorem 2 it again suffices to work with a particular branch of critical points of ϕ . We shall extend the branch

$\underset{\sim}{x}_1^+(r) \equiv r\, \underset{\sim}{v}_1^+(r)$, $r \in [o, r_o]$. It is clear that if we extend $\underset{\sim}{v}_1^+(r)$ as a solution the initial value problem (2.7), then $\underset{\sim}{x}_1^+(r) \equiv r\underset{\sim}{v}_1^+(r)$ will be a critical point of ϕ on $\underset{\sim}{x}\cdot\underset{\sim}{x} = r^2$ It is also clear that the initial value problem (or any of its representations) has a (have) unique solution(s) provided the operator $\mathcal{B}(r\underset{\sim}{v}_1^+(r)) : V(\underset{\sim}{v}_1^+(r)) \to V(\underset{\sim}{v}_1^+(r))$ is invertible. The condition for the lack of invertibility $\mathcal{B}$ along $r\underset{\sim}{v}_1^+(r)$ is simply that the quadratic $B(r\underset{\sim}{v}_1^+(r)\ ;\ \underset{\sim}{u},\underset{\sim}{u})$ have 0 as a critical value on the unit sphere $V(\underset{\sim}{v}_1^+(r))$. That $B(r\underset{\sim}{v}_1^+(r); \underset{\sim}{u},\underset{\sim}{u})$ has no positive and $n-1$ negative critical values for $r < R_1$ follows from the fact that $\mathcal{B}(r\underset{\sim}{v}_1^+(r))$ is symmetric and invertible for $r < R_1$ and the fact that $\mathcal{B}_{o,1} \equiv \lim_{r\to o^+} \mathcal{B}(r\underset{\sim}{v}_1^+(r)) = \text{diag}\,(\mu_2^{\ 1}, \mu_3^{\ 1}, \ldots, \mu_n^{\ 1})$ maps $V(\underset{\sim}{e}_1) \to V(\underset{\sim}{e}_1)$ and has no positive and $n-1$ negative eigenvalues.